Exercise 1. Ideal gases

Goal: The goal of this exercise is to calculate the partition function of the ideal gas and derive the thermodynamic relations from it.

The simplest system of N identical particles in a volume V is composed of noninteracting members. These systems are called ideal gases. In nature there are two types of ideal gases: the ideal Fermi gas and the ideal Bose gas. In addition to these two types of ideal systems one can also define a useful mathematical model called Boltzmann gas or classical ideal gas. The state of an ideal gas can be specified by a set of occupation numbers $\{n_{\boldsymbol{p}}\}_N$ which sum up to the total number of particles: $N = \sum_{\boldsymbol{p}} n_{\boldsymbol{p}}$. The total energy of the system is given by $E = \sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\varepsilon_p$, where ε_p is the single particle energy level: $\varepsilon_p = p^2/2m$. In the framework of the canonical ensemble the partition function is

$$Z_N = \sum_{\{n_{\boldsymbol{p}}\}_N} g_{\{n_{\boldsymbol{p}}\}_N} \cdot e^{-\beta \sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\varepsilon_p}, \quad (\beta = 1/k_{\mathrm{B}}T) \tag{1}$$

where $g_{\{n_{\boldsymbol{p}}\}_N}$ is the number of states of the system corresponding to the given set of occupation numbers $\{n_{\boldsymbol{p}}\}_N$.

(a) Consider the Boltzmann gas, which is composed of independent and *distinguishable* particles. In this model it is assumed that each state contribute to the partition function only by weight $1/N!$. This assumption corresponds to the "correct Boltzmann counting" and it is not related to any physical property of the particles. It is just a rule that defines the model. Using the last assumption determine $g_{\{n_{\boldsymbol{p}}\}_N}$ and calculate the partition function Z_N. In the thermodynamic limit derive the equation of state and calculate the entropy.

Hint: Use the multinomial theorem. In the thermodynamic limit replace sum by integration $\sum_{\boldsymbol{p}} \to \frac{V}{(2\pi\hbar)^3}\int d^3p$.

(b) It is more convenient to consider the ideal quantum gases in the framework of grand canonical ensemble. The grand partition function is defied as

$$\mathcal{Z}(V,T,z) = \sum_{N=0}^{\infty} z^N Z_N(V,T), \tag{2}$$

where $z = e^{\beta\mu}$ is the fugacity. Calculate the grand partition function for the ideal Bose gas and Fermi gas. Calculate the average total number of particles and the average occupation number.

Exercise 2. Bose-Einstein condensation in $d = 2$?

Goal: In this exercise we want to see how Bose-Einstein condensation (BEC) depends on the dimensionality of the problem. Namely, we want to check if there is a BEC of ideal Bose gas in $d = 2$.

Write down the grand partition function $\mathcal{Z}(V,T,z)$ (see the previous problem) for the $d-$dimensional ideal Bose Gas. From the grand partition function calculate the mean particle density $n = \langle N\rangle/V$ as a function of the fugacity z and the temperature T in d-dimensions . Show that the ideal Bose gas at $d = 2$ does not condense.