Exercise 1. The Klein-Gordon and Dirac Equations

Goal: The Klein-Gordon and Dirac equations are based on the relativistic energy-momentum relation, $E^2 = p^2c^2 + m^2c^4$. We'll understand how the Dirac equation is related to the Klein-Gordon equation, and see how we obtain negative energy solutions, which one interprets as antiparticles.

(a) Using the correspondance principle $E \to i\hbar\partial_t$ and $\vec{p} \to -i\hbar\nabla$ in the same way as for the non-relativistic Schrödinger equation, derive the Klein-Gordon equation using the relativistic energy-momentum relation $E^2 = p^2c^2 + m^2c^4$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi\left(\vec{x}, t\right) = 0 \ . \tag{1}$$

In natural units ($c = 1$, $\hbar = 1$), which we will use for the rest of the exercise sheet, the equation simply becomes $\left(\Box + m^2\right)\phi = 0$, with $\Box = \partial_\mu\partial^\mu$. We'll use the convention $g_{\mu\nu} = \mathrm{diag}\left(+,-,-,-\right)$ for the metric.

If one starts from the square root relativistic energy-momentum relation $E = \sqrt{p^2c^2 + m^2c^4}$ instead, one arrives to the Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi\left(\vec{x}, t\right) = 0 \ , \tag{2}$$

where γ^μ are a set of matrices obeying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

(b) Show that if ψ satisfies the Dirac equation, then each of its components individually satisfies the Klein-Gordon equation:

$$\left(\Box + m^2\right)\psi_\alpha = 0 \ . \tag{3}$$

Hint: Apply $\gamma^\nu\partial_\nu$ onto (2).

(c) Show that, with the Dirac-Pauli representation of the γ^μ matrices and going to momentum space, the Dirac equation (2) becomes

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix} \ , \tag{4}$$

where u_A and u_B are two-component spinors.

(d) Show that $u\left(\vec{p}, s\right)$ and $v\left(-\vec{p}, -s\right)$ as defined in (25.39) in the lecture notes are solutions of (4). Which values of E are allowed?

Exercise 2. The Need for Antiparticles

Goal: We'll see here how, in order to unify quantum mechanics with special relativity, we necessarily have to introduce the concept of antiparticles, along with pair creation and annihilation.

Consider a free particle which is in the state $|\vec{x}_1\rangle$ at time t_1. The amplitude to find this particle at time t_2 at a position $\vec{x}_2$ is given by

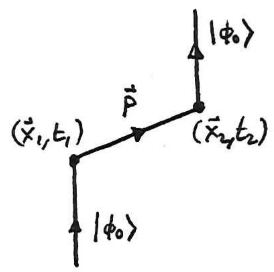
$$A_{\vec{x}_1 \to \vec{x}_2} = \langle \vec{x}_2 | U(t_2 - t_1) | \vec{x}_1 \rangle \ , \tag{5}$$

where $U(t_2 - t_1) = \exp(-iH_0(t_2 - t_1))$ is the unitary time evolution corresponding to a free particle with the relativistic dispersion relation $\omega_p = \sqrt{p^2 + m^2}$.

(a) Assuming that the particle may only have *positive* energies $\omega_p = \sqrt{p^2 + m^2}$, show that the amplitude (5) cannot vanish completely in any finite region of space-time. In particular, there is a nonzero transition amplitude for points $(\vec{x}_2, t_2)$ which lie outside of the light cone of $(\vec{x}_1, t_1)$.

Hints: Write out the amplitude $A_{\vec{x}_1 \to \vec{x}_2}$ using an integration over $\int \frac{d^3\vec{p}}{(2\pi)^3}$ in order to express $U(t_2 - t_1)$, and go to spherical coordinates. Eventually, you may invoke the following theorem from Fourier analysis: if a function $f(t)$ can be written as a superposition of only positive frequencies, i.e. $f(t) = \int_0^\infty d\omega\, F(\omega)\, e^{-i\omega t}$, then $f(t)$ cannot be zero for any finite range of t, unless f is zero everywhere.

Consider now a particle in an initial state $|\phi_0\rangle$. If we want to compute the amplitude to find this particle at a later time again in the same state $|\phi_0\rangle$, under some perturbation V, we may resort to a perturbative expansion of the evolution unitary (cf. time-dependent perturbation theory). The leading order corresponds to nothing happening: $\langle \phi_0 | \phi_0 \rangle = 1$. The next nonzero term corresponds to a disturbance $\langle x_1 | V | \phi_0 \rangle$ happening at $(\vec{x}_1, t_1)$, producing a intermediate particle of momentum $\vec{p}$, followed by another disturbance $\langle \phi_0 | V | x_2 \rangle$ at $(\vec{x}_2, t_2)$ which brings the particle back into the state $|\phi_0\rangle$. The diagram corresponding to these events is shown on the right (time runs upwards).



(b) Suppose $(\vec{x}_2, t_2)$ lies outside of the light cone of $(\vec{x}_1, t_1)$. As we have seen in point (a), the amplitude corresponding to this situation is typically nonzero (which might let us wonder whether we still have causality). Draw the diagram corresponding to an observer which is boosted such that t'_2 happens before t'_1. How does the intermediate particle look like for the boosted observer?

Explain why this intermediate particle should be interpreted as an antiparticle, and describe the sequence of events as the boosted observer would see them.

How is the energy, momentum, and charge of the antiparticle observed by the moving observer related to the energy, momentum and charge of the intermediate particle described by the stationary observer?