Electrodynamics

Babis Anastasiou
Institute for Theoretical Physics, ETH Zurich,
8093 Zurich, Switzerland
E-mail: *babis@phys.ethz.ch*

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Abstract

The subject of the course is classical electrodynamics. The following topics are discussed:

- Electrostatics
- Magnetostatics
- Laws of Electrodynamics
- Electromagnetic waves
- Retarded Potentials
- Special Relativity
- Radiation
- Electrodynamics in Matter

Contents

1 Electromagnetic force

The electromagnetic force has a simple qualitative description. Electrically charged matter is divided into two types of positive and negative charges. Charges of the same type repel each other while charges of different types attract each other. The force among two charges grows when they are brought closer together.

Matter contains enormous numbers of positive (protons) and negative (electrons) charges. The forces exerted by these charges are enormous. However, protons and electrons are found to be very close together in equal numbers and these forces cancel each other almost entirely.

Exercise: *Calculate the force exerted to a single electron from only the protons in a gram of sugar placed in a distance of one meter. How does this force compare to the weight of a typical European man?*

While positive and negative charges are perfectly balanced at a macroscopic level, charge imbalances are evident at an atomic level. The electric force is responsible for protons and electrons binding into atoms as well as molecular chemical bonds.

One realizes easily that the classical qualitative description of the electric force as an attraction or repulsion of electric charges does not work when applied to atoms. Protons and electrons should be collapsing on top of each other due to their attraction. While in orbit, electrons are getting accelerated and according to classical electrodynamics they should lose energy in the form of radiation, thus falling to lower and lower orbits. To explain the stability of atoms, we shall need to combine the laws of classical electrodynamics with the laws of quantum mechanics. The latter, impose a minimum energy for electrons orbiting around protons which cannot be reduced any further. In this course, we will not apply electrodynamics to atomic systems. Nevertheless, there are many interesting macroscopic phenomena which we can understand within the classical theory, without using quantum laws.

The force acting on an electric charge q at a position $\vec{x}$ depends on the relative position and relative motion of all other electric charges. We can sum up the effects of all other charges into two vectors:

- $\vec{E}(\vec{x})$, the electric field and
- $\vec{B}(\vec{x})$, the magnetic field.

The force is then given by:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{1}$$

where $\vec{v}$ is the velocity of the charge q. The electric and magnetic fields are determined from the **equations of Maxwell**:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{4}$$

$$\vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c^2 \epsilon_0} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t.} \tag{5}$$

where $\rho(\vec{x})$ is the electric charge density and $\vec{j}$ is the electric current density. ϵ_0 is a constant, the so-called vacuum permittivity, and has the value

$$\epsilon_0 = 8.854187817\ldots 10^{12} \frac{A \cdot s}{\text{Volt} \cdot m}. \tag{6}$$

c is the speed of light

$$c = 2.99792458\ldots 10^{8} \frac{m}{s}. \tag{7}$$

We remind the definition of the differential operator:

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \tag{8}$$

The inner product of two vectors is:

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i \tag{9}$$

while the components of the outer product of two vectors are

$$\left(\vec{A} \times \vec{B}\right)_i \equiv \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k \tag{10}$$

In the above, ϵ_{ijk} is the fully antisymmetric tensor with $\epsilon_{123} = +1$.

The solutions of Maxwell equations describe all macroscopic electromagnetic phenomena. First, we will explore these solutions in mathematically simple situations, assuming steady currents, fixed charge distributions and static electric and magnetic fields. In such setups, the electric and magnetic fields decouple from each other in Maxwell equations. After gaining experience with solving the equations in electrostatics and magnetostatics, we will examine phenomena such as induction and electromagnetic radiation, where the fields vary with time. Finally, we will examine the structure of the Maxwell equations from a pure aesthetic or better said theoretic point of view. We will compare the equations in different inertial frames of reference and will discover how they transform under relativistic transformations. We will also explore another symmetry of Maxwell equations, the so-called gauge symmetry. Strikingly, this symmetry governs the laws of physics at the smallest distances ($10^{-16}m$) that we have been able to explore experimentally.

2 Electrostatics

Consider fixed charge distributions and static electric fields. In this situation, the first two of Maxwell equations take the form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss Law} \tag{11}$$

$$\vec{\nabla} \times \vec{E} = 0. \tag{12}$$

Notice that these equations imply the superposition principle. If two charge distributions ρ_1 and ρ_2 the electric fields yield electric fields $\vec{E_1}$ and $\vec{E_2}$ respectively, a charge distribution $\rho = \rho_1 + \rho_2$ produces an electric field $\vec{E} = \vec{E_1} + \vec{E_2}$.

2.1 Coulomb's law

These equations are equivalent to the very familiar **Coulomb's law**. It states that for two charges q and q_1 at positions $\vec{x}$ and $\vec{y_1}$ respectively, the force acting on the charge q is

$$\vec{F} = \frac{q}{4\pi\epsilon_0} \frac{q_1}{|\vec{x} - \vec{y_1}|^3} (\vec{x} - \vec{y_1}). \tag{13}$$

According to Coulomb's law, the electric field due to the charge q_1 at a point $\vec{x}$ is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{x} - \vec{y_1}|^3} (\vec{x} - \vec{y_1}). \tag{14}$$

In order to compute the electric field at a point $\vec{x}$ due to a distribution of N charges q_i at positions $\vec{y_i}$, we invoke the superposition principle [1]:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{x} - \vec{y_i}|^3} (\vec{x} - \vec{y_i}). \tag{15}$$

It is extremely cumbersome to keep track of the numerous charges which are contained in very small volumes of matter. It is then appropriate to define

[1] We know that the superposition principle is valid down to very small distances. However, at subatomic distances where quantum physics is also at play, we know that it breaks down. The electric field of one charge interacts with the electric field of another charge and the combined electric field is different than the sum. The effect ("light by light scattering") is tiny but very well measured in precision experiments.

macroscopic continuous charge distributions $\rho(\vec{y})$. The charge dq in a small volume $d^3\vec{y}$ is given by

$$dq = \rho(\vec{y})d^3\vec{y}. \tag{16}$$

The electric field is then given by an integral over the charge density:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y}). \tag{17}$$

2.1.1 Mathematical interlude: the delta-function

Is there an elegant way to transition from the continuous integral of Eq. 17 to the discrete sum of Eq. 15 in the case of a distribution of point-like well separated charges? Assume for simplicity the case of only one point-like charge q at a point $\vec{y}$. The charge distribution should be zero at every point in space except $\vec{y}$. On the contrary, at the point $\vec{y}$ where the charge stands, the distribution is infinite, since the volume of a single point is zero ($\rho = q/0$). Dirac introduced a function, the so-called, $\delta-$function which is:

$$\delta(x-y) = \begin{cases} 0, & x \neq y, \\ \infty, & x = y. \end{cases} \tag{18}$$

In addition, the integrals with a delta function kernel are defined as

$$\int_{-\infty}^{\infty} dx f(x)\delta(x-y) = f(y). \tag{19}$$

In many (D) dimensions, the delta-function is defined as

$$\delta(\vec{x}-\vec{y}) = \prod_{i=1}^{D}\delta(x_i - y_i). \tag{20}$$

With the help of the delta-function, the charge density of a single charge q at a position $\vec{y}$ is expressed as:

$$\rho(\vec{x}) = q\delta(\vec{x}-\vec{y}). \tag{21}$$

The volume integral over this charge distribution gives correctly:

$$\int d^3\vec{x}\rho(\vec{x}) = \int d^3\vec{x} q\delta(\vec{x}-\vec{y}) = q. \tag{22}$$

The charge distribution of many point-like charges q_i at positions $\vec{y_i}$ is

$$\rho(\vec{x}) = \sum_i q_i \delta(\vec{x} - \vec{y_i}). \tag{23}$$

Substituting the charge density of Eq. 23 into Eq. 17 we recover the expression of Eq. 15.

Integrating with an infinite kernel may be tricky. A $\delta-$function is a distribution, mapping test well-behaved functions into real numbers. Their properties can be derived by a limiting procedure on suitable representation functions. For example, we can think of a delta function as the limit:

$$\delta(x-y) = \lim_{a \to 0} \delta_a(x-y), \tag{24}$$

where

$$\delta_a(x-y) = \begin{cases} \frac{1}{a}, & x \in \left[y - \frac{1}{a}, y + \frac{1}{a}\right], \\ 0, & x \notin \left[y - \frac{1}{a}, y + \frac{1}{a}\right]. \end{cases} \tag{25}$$

Exercise: What is the meaning of

- the derivative of a δ function?
- $\delta(g(x))$?

Exercise: Prove that the derivative of a step function is a delta function.

2.2 Gauss' law from Coulomb's law

In this section we will show that Gauss' law can be derived from Coulomb's law. First we make the mathematical observation that

$$\vec{\nabla}_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|} = -\frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}. \tag{26}$$

The subscript in the nabla operator denotes that it acts on the components of the vector $\vec{x}$, i.e. $\vec{\nabla}_{\vec{x}} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. One can easily prove the above performing the differentiations of the left hand side. Therefore, the electric field is the gradient of a scalar function:

$$\vec{E} = -\vec{\nabla}\Phi(\vec{x}) \tag{27}$$

the, so called, scalar potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}. \tag{28}$$

We can now compute the divergence of the electric field:

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \Phi = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y}\, \rho(\vec{y}) \left(-\nabla^2_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|} \right). \tag{29}$$

We remind the definition of the Laplacian operator

$$\nabla^2 \equiv \vec{\nabla} \cdot \vec{\nabla} = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}. \tag{30}$$

We will prove shortly that:

$$\nabla^2_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|} = -4\pi\delta(\vec{x} - \vec{y}). \tag{31}$$

Thus, we have that

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \Phi = \frac{\rho}{\epsilon_0}. \tag{32}$$

This is the differential form of Gauss' law. **Additional Reading:** Derivation of Gauss' law from Coulomb's law in Feynman Lectures Vol. 2, 4-5,4-6

2.2.1 Integral form of Gauss' law

Consider a volume $V(S)$ bounded from a surface $S(V)$. We can integrate both sides of Gauss'law in this volume:

$$\int_{V(S)} d^3\vec{x} \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \int_{V(S)} d^3\vec{x} \rho(\vec{x}) \tag{33}$$

The numerator of the rhs is the total charge $Q_{S(V)}$ enclosed in the surface $S(V)$. The lhs can be written as a surface integral, by applying the divergence theorem, which states that the flux of a vector $\vec{A}$ through a closed surface $S(V)$ is equal to the divergence of the vector integrated over the volume $V(S)$ enclosed by the surface.

$$\int_{S(V)} d\vec{S} \cdot \vec{A} = \int_{V(S)} d^3\vec{x}\, \vec{\nabla} \cdot \vec{A}. \tag{34}$$

Combining the above, we obtain Gauss' law in an integral form, stating that *the flux of the electric field through any closed surface $S(V)$ is proportional to the electric charge*:

$$\int_{S(V)} d\vec{S} \cdot \vec{E} = \frac{1}{\epsilon_0} \int_{V(S)} d^3\vec{x} \rho(\vec{x}). \tag{35}$$

Given that this is valid for any closed surface, all the steps that we have made in our derivation are reversible. Therefore, the differential and integral form of Gauss' law are equivalent.

2.2.2 Mathematical interlude: The Laplacian of inverse distance

It is now time to prove the identity of Eq. 31 which we used to derive Gauss' law from Coulomb's law. For reasons to become soon apparent, we shall assume a generic number N of space dimensions. At the end, we will specialize to $N = 3$, but keeping the dimension as a generic parameter will elucidate mathematically how to proceed in intermediate steps of our derivation. With a direct differentiation we find that

$$\nabla^2 \frac{1}{|\vec{x} - \vec{y}|} = \frac{3 - N}{|\vec{x} - \vec{y}|^3}. \tag{36}$$

In $N = 3$ and for $\vec{x} \neq \vec{y}$, we find that the rhs is zero. For $\vec{x} = \vec{y}$ and arbitrary N the result is infinity. What is the result for $N = 3$? It will still be an infinity, as we would expect for a delta function. But let's resolve carefully this $\frac{0}{0}$ conundrum.

Consider the integral in N dimensions:

$$I_N[f] \equiv \int d^N\vec{y}\, f(\vec{y}) \nabla^2_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|}. \tag{37}$$

f is an arbitrary function. Let's now split the integration volume into two regions:

- V_{in} is a sphere (in N dimensions) drawn around the point $\vec{x}$ with a very small radius R. Inside the sphere, we can approximate:

$$f(\vec{y}) \approx f(\vec{x}), \quad \forall\, \vec{y} \text{ in } V_{in}. \tag{38}$$

- V_{out} is the rest of the integration volume, not included in V_{in}.

We first compute the integral $I_N[f]$ inside V_{in}. We have

$$\begin{aligned}
I_{N,in}[f] &= \int_{V_{in}} d^N\vec{y}\, f(\vec{y})\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} \\
&= f(\vec{x})\int_{V_{in}} d^N\vec{y}\,\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} + \mathcal{O}(\mathcal{R}) \\
&= (3-N)f(\vec{x})\int_{V_{in}} d^N\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|^3} + \mathcal{O}(\mathcal{R}) \\
&= (3-N)\Omega_N f(\vec{x})\int_0^R dr\, r^{N-1}\frac{1}{r^3} + \mathcal{O}(\mathcal{R}) \\
&= (3-N)\Omega_N f(\vec{x})\frac{R^{N-3}-0^{N-3}}{N-3} + \mathcal{O}(\mathcal{R}) \\
&= -\Omega_N f(\vec{x})R^{N-3} + \mathcal{O}(\mathcal{R}). \qquad (39)
\end{aligned}$$

Ω_N is the solid angle in N dimensions. In the above, we have used spherical coordinates in a system centered at $\vec{x}$. For $N=3$ and setting the radius of V_{in} to zero, $R\to 0$, we obtain:

$$I_{3,in}[f] = -4\pi f(\vec{x}). \qquad (40)$$

The integral in the rest of the integration volume is:

$$\begin{aligned}
I_{N,out}[f] &= \int_{V_{out}} d^N\vec{y}\, f(\vec{y})\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} \\
&= (3-N)\int_{V_{out}} d^N\vec{y}\,\frac{f(\vec{y})}{|\vec{x}-\vec{y}|^3} \qquad (41)
\end{aligned}$$

The integrand has no singularity, since outside the sphere V_{in} we always guarantee that $\vec{x}\neq\vec{y}$ and it is converged for the typical smooth functions $f(\vec{x})$ that we are interested in. For $N=3$, we then have:

$$I_{3,out}[f] = 0. \qquad (42)$$

We have then found that

$$I_3[f] = I_{3,in}[f] + I_{3,out}[f] = -4\pi f(\vec{x}), \qquad (43)$$

or, explicitly,

$$\int d^3\vec{y}\, f(\vec{y})\nabla^2_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|} = -4\pi f(\vec{x}). \qquad (44)$$

The above is valid for every smooth function $f(\vec{x})$. We have therefore proven the desired identity:

$$\nabla^2_{\vec{x}} \frac{1}{|\vec{x}-\vec{y}|} = -4\pi\delta(\vec{y}-\vec{x}). \tag{45}$$

2.3 Scalar potential

We have found earlier (Eq. 27) that the electric field can be derived from the gradient of the effective potential. We conclude immediately that the curl of a static electric field is zero:

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla}\Phi = \sum_{jk} \epsilon_{ijk} \frac{\partial^2 \Phi}{\partial x_j \partial x_k} = \frac{1}{2} \sum_{jk} \left(\epsilon_{ijk} + \epsilon_{ikj}\right) \frac{\partial^2 \Phi}{\partial x_j \partial x_k} = 0. \tag{46}$$

We have then proven that also the second law of electrostatics,

$$\vec{\nabla} \times \vec{E} = 0. \tag{47}$$

Using Stokes' theorem,

$$\int_S d\vec{S} \cdot \vec{\nabla} \times \vec{A} = \oint_{\partial S} d\vec{l} \cdot \vec{A}, \tag{48}$$

we find that the circulation of the electrostatic field in any closed loop Γ is zero:

$$\oint_\Gamma d\vec{l} \cdot \vec{E} = 0. \tag{49}$$

The scalar potential has an immediate physics interpretation. Consider an electric field $\vec{E}$ and compute the work needed to transport a test charge from a position $\vec{x}_A$ to a position $\vec{x}_B$. The field exerts a force $\vec{F} = q\vec{E} = -q\vec{\nabla}\Phi$ which we need to counteract with an external force $\vec{F}_{\text{ext}} = -\vec{F} = q\vec{\nabla}\Phi$ while transferring the charge from the starting to the finishing point over a given path. The work done is

$$W_{A\to B} = q \int_{\vec{x}_A}^{\vec{x}_B} d\vec{l} \cdot \vec{\nabla}\Phi = q\left(\Phi(\vec{x}_B) - \Phi(\vec{x}_A)\right). \tag{50}$$

Nicely, the work done is independent of the path chosen and depends through the potential only on the starting and ending points.

2.4 Potential energy of a charge distribution

In this section, we will compute the energy stored in a charge distribution. Assume N charges q_i at positions $\vec{x}_i$. The potential energy of the system is equal to the energy required to bring the charges one by one to their positions from an infinite distance (where the electric field vanishes).

$$W = \sum_{i=1}^{N} W_i, \tag{51}$$

where W_i is the work needed to bring the charge q_i at $\vec{x}_i$. It costs no energy to bring the first charge q_1 at $\vec{x}_1$. Once at $\vec{y}_1$, q_1 creates a potential

$$\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{x} - \vec{x}_1|} \tag{52}$$

To bring q_2 at $\vec{x}_2$ requires work

$$W_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|}. \tag{53}$$

Consequently, the new potential is

$$\Phi_{12} = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|} \right] \tag{54}$$

and the work needed to bring the charge q_3 at $\vec{x}_3$ is

$$W_3 = \frac{q_3}{4\pi\epsilon_0} \left[\frac{q_1}{|\vec{x}_3 - \vec{x}_1|} + \frac{q_2}{|\vec{x}_3 - \vec{x}_2|} \right] \tag{55}$$

For the $i-$th charge the work needed is:

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j<i} \frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{56}$$

The potential energy is then

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=2}^{N} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}. \tag{57}$$

Alternatively, we can write a more symmetric form

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}. \tag{58}$$

Additional Reading: Electrostatic energy of a salt crystal in:
Feynman Lectures Vol. 2, 8-3

2.4.1 Potential energy of a continuous charge distribution

The expression of Eq. 58 in the continous charge distribution limit becomes:

$$W = \frac{1}{8\pi\epsilon_0}\int d^3\vec{x}d^3\vec{y}\,\frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}. \tag{59}$$

This can also be written as

$$W = \frac{1}{2}\int d^3\vec{x}\,\rho(\vec{x})\Phi(\vec{x}), \tag{60}$$

where we have recognised in the kernel of the $\vec{x}$ integral the scalar potential. Using Gauss' law

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{61}$$

(which in this form is known as the Poisson equation), we have

$$W = -\frac{\epsilon_0}{2}\int d^3\vec{x}\,\Phi(\vec{x})\nabla^2\Phi(\vec{x}). \tag{62}$$

We now use integration by parts identity (**exercise**)

$$\Phi\nabla^2\Phi = \vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right) - \left|\vec{\nabla}\Phi\right|^2 = \vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right) - \left|\vec{E}\right|^2. \tag{63}$$

The first term gives a surface integral (using the divergence theorem) and vanishes if we take the boundary at infinity, where the electric field vanishes. We are then left with the result

$$W = \frac{\epsilon_0}{2}\int d^3\vec{x}\,\left|\vec{E}\right|^2. \tag{64}$$

which expresses the electrostatic energy as an integral with a kernel the squared magnitude of the electric field. We can interpret the positive definite quantity

$$w = \frac{\epsilon_0}{2}\left|\vec{E}\right|^2 \tag{65}$$

as the energy density of the electric field. This is also intuitive, since we associate more energy with regions of space where the electric field is stronger.

2.4.2 Self-energy

There is a problem with the expression of Eq. 59 when applying to discrete charge distributions with a density

$$\rho(\vec{x}) = \sum_i q_i \delta(\vec{x} - \vec{x}_i). \tag{66}$$

We obtain a potential energy

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i,j=1}^{N} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{67}$$

where in the above summation we do not exclude the cases of $i = j$. These terms yield an infinity. The problem is not overcome either when we apply Eq. 64 to discrete cases. For example, if we consider only one point-like charge q the energy density is

$$w = \frac{\epsilon_0}{2} \left|\vec{E}\right|^2 = \frac{q^2}{32\pi^2\epsilon_0} \frac{1}{r^4}, \tag{68}$$

which becomes infinite for $r = 0$. This infinity is not integrable. The total energy associated with the existence of a single charge is

$$W = 4\pi \int_0^\infty r^2 dr\, w = -\frac{q}{4\pi\epsilon_0} \int_0^\infty d\left(\frac{1}{r}\right) = \infty. \tag{69}$$

Obviously, our classical electrodynamics results are not valid when we try to compute the energy associated with point-like particles. The physics laws need to be modified. The problem persists also in quantum electrodynamics, although there we have a technical way to remove the singularities due to the self-energy of point-like charges and absorb then in a renormalisation of the electric charge unit.

Exercise: Calculate the electrostatic energy of a uniformly charged sphere in a couple of ways. What happens when you take the limit of a zero radius? *Check out your answer here:* Feynman Lectures Vol. 2, 8-1.

2.5 Charged conductors

Conductors are materials which allow electrons to move freely within their mass. Commonly, conductors are neutral containing the same amount of

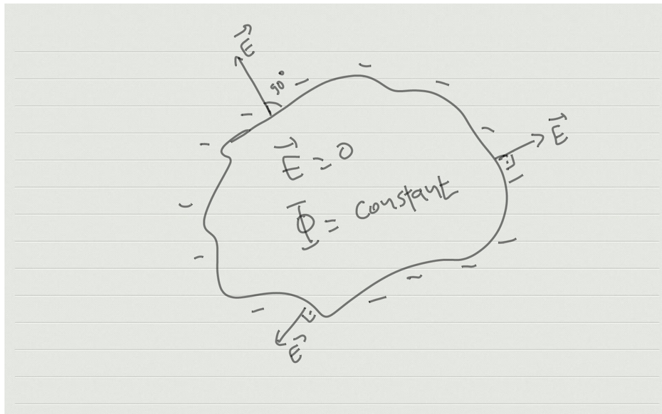


Figure 1: A conductor charged with some excess charge. The excess charge distributes itself at the surface. The electric field vanishes in the interior and the potential is a constant.

positive and negative charges. However, it is possible to charge them bringing some excess ions on it (adding or removing electrons). After we stop charging the conductor, the excess charges will settle in such a way so that the electrostatic energy is minimum. This happens when their relative distances are maximal. Therefore, the excess charges distribute themselves on the surface of the conductor. Inside a settled charged conductor the electric field vanishes.

$$\vec{E}_{\text{inside}} = 0. \tag{70}$$

If it did not, it would push and set in motion the free electrons in the inside (which are balanced by positive ions in the crystal structure) contrary to the requirement that the charges have settled to fix positions. Consequently, the scalar potential is everywhere the same in the conductor.

$$\vec{E}_{\text{inside}} = 0 \rightsquigarrow \vec{\nabla}\Phi = 0 \rightsquigarrow \Phi = \text{constant}. \tag{71}$$

The electric field is non-zero on the surface of the conductor and it is perpendicular to the surface. Indeed, a parallel component would set in motion the charges on the surface of the conductor in contrast again of the electrostatic assumption.

We can compute easily the magnitude of the electric field on the surface by applying Gauss's law in the integral form for the surface of Fig. 2. The

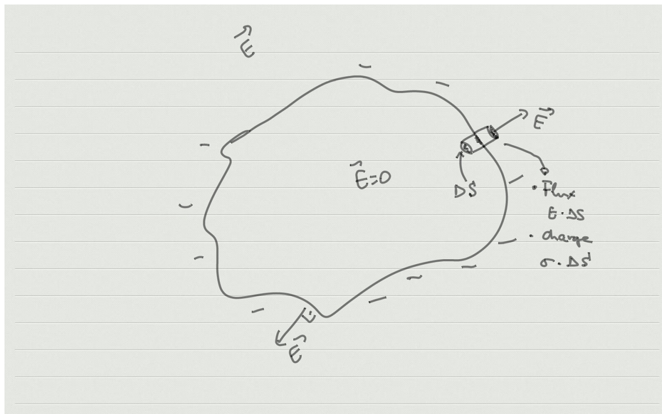


Figure 2: Using the integral form of Gauss' law in order to compute the electric field on the surface of the conductor.

flux of the electric field through the Gauss surface is:

$$\text{Flux} = \int d\vec{S} \cdot \vec{E} = E \Delta S \tag{72}$$

and the charge enclosed is:

$$\text{Charge} = \sigma(\vec{x}) \Delta S, \tag{73}$$

where $\sigma(\vec{x})$ is the charge surface density (charge per unit surface) of the conductor. The electric field on the surface is then:

$$E = \frac{\sigma(\vec{x})}{\epsilon_0}. \tag{74}$$

The energy density on the surface of the conductor is:

$$w = \frac{\epsilon_0}{2} \left|\vec{E}\right|^2 = \frac{\sigma^2(\vec{x})}{2\epsilon_0}. \tag{75}$$

The excess charges on the surface of a conductor exert a pressure (which we can calculate). If the charges on the surface of the conductor were not constrained and were allowed to increase further their separation distances the electrostatic energy would reduce. Consider the original conductor and

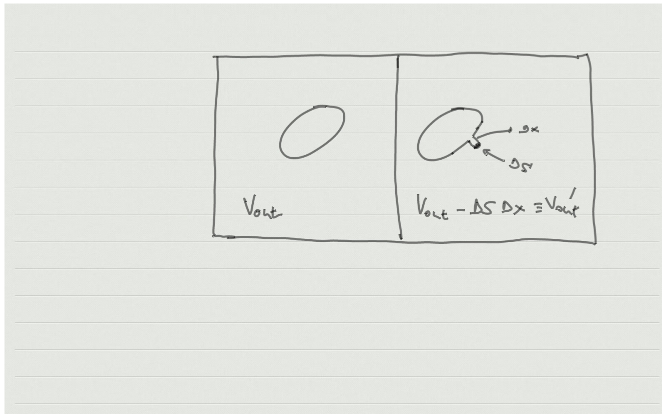


Figure 3: Two almost identical conductors, with an infinitesimal $\Delta x \Delta S$ volume deformation. The conductors have the same excess charge and charge density. The electrostatic energy density of the two systems is also identical.

an otherwise identical one but a small deformation $\Delta x \Delta S$ in its volume, as in Fig. 3. The difference in electrostatic energy for the two configurations is

$$\Delta W = \int_{\text{Vout}} d^3\vec{x} w - \int_{\text{Vout}-\Delta\text{x}\Delta\text{S}} d^3\vec{x} w = -\Delta S \Delta x \frac{\sigma^2}{2\epsilon_0}. \tag{76}$$

The force which is needed to undo such a deformation should provide the same amount of work:

$$\Delta W = F \Delta x. \tag{77}$$

Thus, the pressure on the surface of the conductor is

$$\frac{|F|}{\Delta S} = \frac{|\Delta W|}{\Delta x \Delta S} = \frac{\sigma^2}{2\epsilon_0}. \tag{78}$$

END OF WEEK 1

3 Boundary condition problems in electrostatics

We can claim that we have solved an electrostatics physics problem if we are able to compute the scalar potential. From it, we can deduce the electric field, the forces on electric charges and everything else we may need to know. The electrostatic potential is given by an integral over all space:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int\int\int_{-\infty}^{\infty} d^3\vec{y}\, \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}. \tag{79}$$

This equation may be unpractical for computing the potential in a plethora of situations. It requires an integration over an infinite volume in which we are supposed to have full knowledge of the charge density. It is more likely, however, that we are able to have partial information about the charge density, restricted within the boundaries of a smaller finite volume V (such as the boundaries of a laboratory or an electric device). Is it possible to account for the missing information on the charge density outside the small volume that we control with something equivalent obtained/measured on the boundaries of our volume?

We have expressed Gauss' law in a form of a differential equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}. \tag{80}$$

Instead of using the infinite integration in order to compute the potential within the volume V, we can solve the differential equation. Differential equations yield multiple solutions. To obtain a unique solution, we will need to supply additional information, in lieu of the charge density outside V. What boundary conditions guarantee a unique solution?

3.1 Dirichlet and Neumann boundary conditions

Assume that we know two different solutions Φ_1 and Φ_2 of Poisson's equation 80 inside a volume V with boundary $S(V)$. The difference $U = \Phi_1 - \Phi_2$ will satisfy a Laplace equation:

$$\nabla^2 U = 0 \tag{81}$$

Therefore

$$\begin{aligned}
U\nabla^2 U &= 0 \\
\rightsquigarrow \vec{\nabla}\left(U\vec{\nabla}U\right) - \left|\vec{\nabla}U\right|^2 &= 0 \\
\rightsquigarrow \int_V d^3\vec{x}\,\left|\vec{\nabla}U\right|^2 &= \int_V d^3\vec{x}\,\vec{\nabla}\cdot(U\vec{\nabla}U) \\
\rightsquigarrow \int_V d^3\vec{x}\,\left|\vec{\nabla}U\right|^2 &= \int_{S(V)} d\vec{S}\cdot(U\vec{\nabla}U) \qquad (82)
\end{aligned}$$

The first integral is over the entire volume V in which we want to compute the potential. The second integral is over the surface of the boundary.

Let's assume that the potential is fully known, or it can be measured, at the boundary (but not inside the volume):

$$\Phi_1(\vec{x}) = \Phi_2(\vec{x}) = \Phi(\vec{x}) \quad \forall\vec{x}\in S(V). \tag{83}$$

Then the rhs of Eq. 82 vanishes. The lhs is a volume integrand with a positive definite integrand. In order for the lhs integral to vanish, the integrand must vanish at all points inside V. We conclude that

$$\vec{\nabla}U = 0 \rightsquigarrow \vec{\nabla}(\Phi_1 - \Phi_2) = 0 \rightsquigarrow \vec{E}_1(\vec{x}) = \vec{E}_2(\vec{x}), \quad \forall\vec{x}\in V. \tag{84}$$

We conclude that if we know the scalar potential on the boundary $S(V)$ (Dirichlet boundary condition) the solution of the Poisson equation is unique for the electric field inside the volume V. The potential $\Phi(x)$ inside the volume V.

Let us assume now that the information that we have on the boundary is the value of the electric field $\vec{E} = -\vec{\nabla}\Phi$ (Neumann boundary condition), thus

$$\vec{\nabla}\Phi_1(\vec{x}) = \vec{\nabla}\Phi_2(\vec{x}) = \vec{\nabla}\Phi(\vec{x}) \quad \forall\vec{x}\in S(V). \tag{85}$$

As for a Dirichlet boundary condition, we conclude that $\nabla\vec{U} = 0$ for all points inside the volume V and the solution for the electric field $\vec{E}$ inside V is unique.

A unique solution for $\vec{E}$ inside V is also guaranteed due to Eq. 82 for a mixed boundary condition, where for part of the surface $S(V)$ we know the potential Φ and for the rest of the surface we know the electric field $\vec{E}$.

We note that while the three types of boundary conditions (Dirichlet, Neumann, mixed) determine uniquely the solution of Poisson's equation for the electric field $-\vec{\nabla}\Phi$, the scalar potential is determined only up to a physically unimportant constant.

3.2 Green's functions

From the divergence theorem,

$$\int_V d^3\vec{x}\,\vec{\nabla}\cdot\vec{F} = \int_{S(V)} d\vec{S}\cdot\vec{F} \tag{86}$$

and by choosing

$$\vec{F} = \phi\vec{\nabla}\psi \tag{87}$$

we obtain

$$\int_V d^3\vec{x}\,\vec{\nabla}\cdot\phi\vec{\nabla}\psi = \int_{S(V)} d\vec{S}\cdot\phi\vec{\nabla}\psi \tag{88}$$

Interchanging ϕ and ψ in the above and taking the difference, we have:

$$\int_V d^3\vec{x}\,\vec{\nabla}\cdot\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right) = \int_{S(V)} d\vec{S}\cdot\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right) \tag{89}$$

and performing the differentiations on the lhs we obtain:

$$\int_V d^3\vec{x}\,\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int_{S(V)} d\vec{S}\cdot\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right) \tag{90}$$

Equation 90 will be our main tool for solving Poisson's differential equation with Dirichlet or Neumann boundary conditions.

Let ϕ in Eq. 90 be a potential $\phi = \Phi(\vec{x})$ satisfying Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}.$$

and ψ a, so called, Green's function $G(\vec{x},\vec{y})$ defined to satisfy

$$\nabla^2_{\vec{x}}G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y}). \tag{91}$$

It is of the form,

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} + F(\vec{x},\vec{y}) \tag{92}$$

where the function F satisfies the Laplace equation:

$$\nabla^2_{\vec{x}}F(\vec{x},\vec{y}) = 0. \tag{93}$$

Substituting for ϕ and ψ into Eq. 90, we obtain:

$$\begin{aligned}\int_V d^3\vec{x}\left[\Phi(\vec{x})\nabla^2_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\nabla^2\Phi(\vec{x})\right] &= \int_{S(V)} d\vec{S}\cdot\left[\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\vec{\nabla}\Phi(\vec{x})\right]\\ \rightsquigarrow \int_V d^3\vec{x}\left[-4\pi\delta(\vec{x}-\vec{y})\Phi(\vec{x}) + \frac{\rho(\vec{x})}{\epsilon_0}G(\vec{x},\vec{y})\right] &= \int_{S(V)} d\vec{S}\cdot\left[\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\vec{\nabla}\Phi(\vec{x})\right],\end{aligned} \tag{94}$$

Performing the integration over the $\delta-$function we finally obtain,

$$\begin{aligned}\Phi(\vec{y}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\, G(\vec{x},\vec{y})\rho(\vec{x})\\ &\quad -\frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot\left[\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\vec{\nabla}\Phi(\vec{x})\right].\end{aligned} \tag{95}$$

This expression is valid for every point $\vec{y}\in V$.

3.2.1 Dirichlet boundary conditions

For Dirichlet boundary conditions, we search for a Green's function $G_D(\vec{x},\vec{y})$ which vanishes on the surface $S(V)$:

$$G_D(\vec{x},\vec{y}) = 0, \quad \forall\,\vec{x} \text{ on } S(V), \tag{96}$$

such as that the second term of the surface integral of Eq. 95. Then we have:

$$\begin{aligned}\Phi(\vec{y}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\, G_D(\vec{x},\vec{y})\rho(\vec{x})\\ &\quad -\frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G_D(\vec{x},\vec{y}).\end{aligned} \tag{97}$$

3.2.2 Neumann boundary conditions

With Neumann boundary conditions, one is tempted to use a Green's function with

$$\vec{\nabla}G(\vec{x},\vec{y})\cdot\hat{n} = 0,$$

where $\hat{n}$ is a unit vector perpendicular to $S(V)$. However, such a Green's function cannot exist. From the definition of a Green's function we expect:

$$\nabla^2 G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y})$$

$$\begin{aligned}
&\rightsquigarrow \int_V d^3\vec{x}\,\nabla^2 G(\vec{x},\vec{y}) = -4\pi \int_V d^3\vec{x}\,\delta(\vec{x}-\vec{y}) \\
&\rightsquigarrow \int_{S(V)} d\vec{S}\cdot\vec{\nabla}G(\vec{x},\vec{y}) = -4\pi \neq 0 \\
&\rightsquigarrow \vec{\nabla}G(\vec{x},\vec{y})\cdot\hat{n} \neq 0.
\end{aligned} \tag{98}$$

We can choose, however, a Green's function which has a constant component for its gradient in the direction perpendicular to the surface, such as

$$\vec{\nabla}G_N(\vec{x},\vec{y}) = -\frac{4\pi}{S}\hat{n}, \tag{99}$$

where S is the total surface of the boundary $S(V)$:

$$S = \int_{S(V)} d\vec{S}\cdot\hat{n}. \tag{100}$$

Then, Eq. 95 yields:

$$\begin{aligned}
\Phi(\vec{y}) \;=\;& \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\,G_N(\vec{x},\vec{y})\rho(\vec{x}) \\
&+\frac{1}{4\pi}\int_{S(V)} d\vec{S}\cdot G_N(\vec{x},\vec{y})\vec{\nabla}_{\vec{x}}\Phi(\vec{x}) + \langle\Phi\rangle_{S(V)},
\end{aligned} \tag{101}$$

where the last term is the average of the potential on the boundary

$$\langle\Phi\rangle_{S(V)} = \frac{\int_{S(V)} d\vec{S}\cdot\Phi\hat{n}}{S} \tag{102}$$

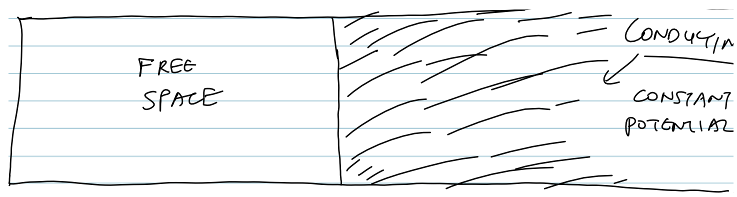
and it is an unimportant constant for physical observables.

3.3 Explicit solutions of boundary condition problems

We have identified required properties of Green's functions for Dirichlet and Neumann boundary conditions. Finding these Green's functions is not easy. In the following, we will consider some simple examples and develop techniques for such a purpose.

3.3.1 Example 1: A conductor filling half of space

Suppose that we fill half of space with a conducting material as in the following figure:



We live in the other half of space where we can have a charge density $\rho(\vec{x})$. The potential of the conductor is constant throughout its mass. The boundary condition for this problem is of Dirichlet type. We require a Green's function which vanishes on the surface S separating the conductor from the half-space we live in:

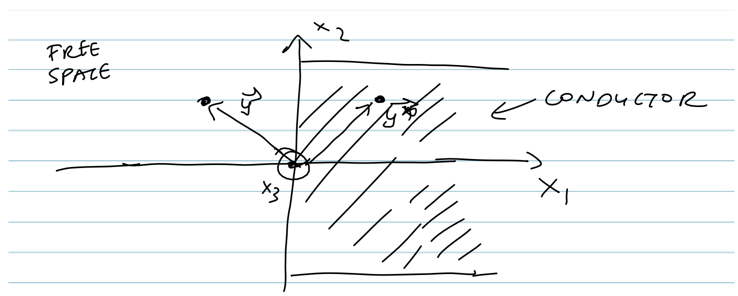
$$G(\vec{x},\vec{y}) = 0 \quad \forall\, \vec{x} \text{ on } S. \tag{103}$$

$G(\vec{x},\vec{y})$ has the form

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} + F(\vec{x},\vec{y}) \tag{104}$$

where

$$\nabla^2 F(\vec{x},\vec{y}) = 0. \tag{105}$$



Let's choose a coordinate system as in the figure above. For each vector $\vec{y} = (y_1, y_2, y_3)$, we define a dual vector $\vec{y}^* = (-y_1, y_2, y_3)$. We take $\vec{y}$ to belong to the free-space, which makes $\vec{y}^*$ to belong to the space occupied by the conductor. The two vectors $\vec{y}$ and $\vec{y}^*$ are identical for $y_1 = 0$, on the surface S separating the free space and the conductor. With some guesswork, we find that

$$F(\vec{x},\vec{y}) = -\frac{1}{|\vec{x}-\vec{y}^*|} \tag{106}$$

satisfies all the criteria we have required for our Green's function. Indeed,

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}^*|} \tag{107}$$

vanishes on the boundary $\vec{y} = (0, y_2, y_3)$. In addition,

$$\nabla^2 F(\vec{x}, \vec{y}) = -\nabla^2 \frac{1}{|\vec{x} - \vec{y}^*|} = 4\pi\delta(\vec{x} - \vec{y}^*) = 0. \tag{108}$$

The delta function above is always zero since $\vec{x}$ belongs to the free half-space, while $\vec{y}^*$ belongs always to the half-space occupied by the conductor.

Eq. 97 yields for the potential:

$$\begin{aligned} \Phi(\vec{y}) &= \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{y}|} \\ &\quad - \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{y}^*|} \\ &\quad - \frac{1}{4\pi} \int_S d\vec{S} \cdot \Phi(\vec{x}) \vec{\nabla} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x} - \vec{y}^*|} \right]. \end{aligned} \tag{109}$$

We now make the observation that

$$|\vec{x} - \vec{y}^*| = \sqrt{(x_1 + y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = |\vec{y} - \vec{x}^*|, \tag{110}$$

where $\vec{x}^* = (-x_1, x_2, x_3)$. We can then write:

$$\begin{aligned} \Phi(\vec{y}) &= \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{y}|} \\ &\quad + \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{(-\rho(\vec{x}))}{|\vec{x}^* - \vec{y}|} \\ &\quad - \frac{1}{4\pi} \int_S d\vec{S} \cdot \Phi(\vec{x}) \vec{\nabla} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x}^* - \vec{y}|} \right]. \end{aligned} \tag{111}$$

The potential on the surface S is a constant $\Phi(\vec{x}) = V$. It is left as an exercise to calculate the surface integral in the last term. One finds,

$$-\frac{1}{4\pi} \int_S d\vec{S} \cdot \Phi(\vec{x}) \vec{\nabla} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x}^* - \vec{y}|} \right] = V. \tag{112}$$

and we have

$$\begin{aligned} \Phi(\vec{y}) &= V + \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{y}|} \\ &\quad + \frac{1}{4\pi\epsilon_0} \int_{\text{free-space}} d^3\vec{x} \, \frac{(-\rho(\vec{x}))}{|\vec{x}^* - \vec{y}|} \end{aligned} \tag{113}$$

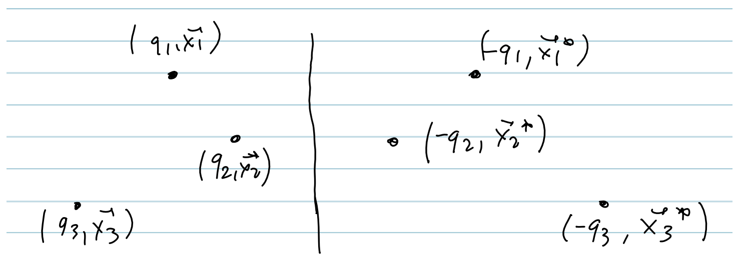


Figure 4: The potential for discrete charges q_i and an infinite conducting half-space is the same in the free half-space as the potential of the charges q_i and their mirror charges $-q_i$.

Eq. 113 can be used to calculate the potential for arbitrary charge distributions for this geometry. If analytic solutions are not possible, the integrals can be performed with numerical methods.

In the special case of a discrete charge distribution,

$$\rho(\vec{x}) = \sum_i q_i \delta(\vec{x} - \vec{x}_i), \tag{114}$$

where $\vec{x}_i$ lie inside the free volume, we obtain

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{y} - \vec{x}_i|} + \frac{(-q_i)}{|\vec{y} - \vec{x}_i^*|} \tag{115}$$

This result is familiar! It is the solution of a different electrostatic problem than the one that we have considered here. It is the electrostatic potential due to the charges q_i at positions $\vec{x}_i$ and $-q_i$ charges at reflected (mirrored) positions $\vec{x}_i^*$. The problem of finding the potential for charges q_i at $\vec{x}$ in the presence of an infinite plane conductor has exactly the same solution in the free half-volume not occupied by the conductor as the problem of finding the potential for the charges q_i at $\vec{x}_i$ and the "mirror charges" $-q_i$ at $\vec{x}_i^*$.

3.3.2 Example 2: Reverse engineering and the method of images (mirror charges)

It is often possible to figure out a clever configuration of charges which has the same effect within some volume V as boundary conditions on the surface

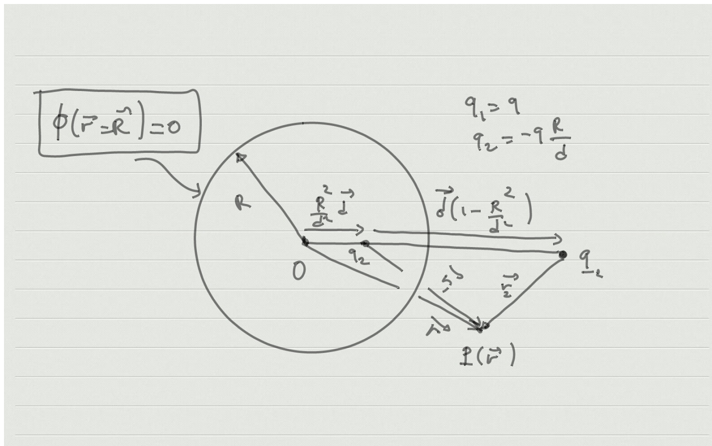


Figure 5: The potential of a positive and a negative charge. There is a sphere on which the potential is zero.

$S(V)$ of the volume. As an example, we place a charge $+q$ at a position $\vec{d}$ and a charge of opposite $-\frac{R}{d}q$ at a position $\frac{R^2}{d^2}\vec{d}$. The contributions of the two charges to the scalar potential at a position $\vec{r}$ are of opposite sign:

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}\right]. \tag{116}$$

There is a surface on which the two contributions cancel against each other. Indeed, Eq. 116 gives zero for all points $\vec{r}=\vec{R}$ on the surface of a sphere with radius $R=\left|\vec{R}\right|$:

$$\begin{aligned}
\Phi(\vec{R}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{R}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{R}-\frac{R^2}{d^2}\vec{d}\right|}\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{R}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left[R^2+\frac{R^4}{d^2}-2\frac{R^2}{d^2}\vec{R}\cdot\vec{d}\right]^{\frac{1}{2}}}\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{R}-\vec{d}\right|} - \frac{1}{\left[d^2+R^2-2\vec{R}\cdot\vec{d}\right]^{\frac{1}{2}}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{R}-\vec{d}\right|} - \frac{1}{\left|\vec{R}-\vec{d}\right|}\right] \\
&= 0.
\end{aligned} \tag{117}
$$

Now, consider a different electrostatic problem of a conducting sphere of a radius R at zero potential and a charge $+q$ at a distance d from the center of the sphere. The potential inside the sphere conductor is everywhere zero. What is the potential outside the sphere? The theorem of Eq. 97 tells us that once we know the potential at the boundary of a volume the solution of Poisson's equation for the potential is unique within that volume. In our case, the volume we are interested in is bounded by the surface of the sphere of the conductor and the surface of all space at infinity. In both (infinite and sphere) surfaces the potential is zero. However, recall that we have identified identical zero potential surfaces with the exact same geometry in our first problem of the two charges in free space. Therefore, the solution of the two problems should be identical in the volume enclosed by the surfaces. Therefore, the solution for the problem with the charge $+q$ at a distance d from the center of a conductor of radius R is,

$$
\Phi(\vec{r}) = \begin{cases} \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}\right], & \forall\vec{r}:\, r \geq R \\ 0, \quad \forall\vec{r}:\, r < R \end{cases} \tag{118}
$$

Let's compare this solution with the general solution of Eq. 116. For a charge density:

$$
\begin{aligned}
\Phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\, G_D(\vec{x},\vec{r}) q\delta(\vec{x}-\vec{d}) \\
&\quad -\frac{1}{4\pi}\int_{S(\vec{x}=\vec{R})} d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G_D(\vec{x},\vec{r}),
\end{aligned} \tag{119}
$$

or, equivalently,

$$
\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}G_D(\vec{d},\vec{r}). \tag{120}
$$

Therefore, the simple solution of the two-charges problem gives us the Green's function for all boundary condition problems with a Green's function. We have,

$$
G_D(\vec{d},\vec{r}) = \frac{4\pi\epsilon_0}{q}\Phi(\vec{r}) = \frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\frac{R^2}{d^2}\vec{d}\right|}. \tag{121}
$$

With the Green's function and Eq. 97 at hand we can solve any other problem with Dirichlet boundary conditions on the same geometry. For, example, one could find the solution for the potential when the conductor is kept at a potential $V \neq 0$ and a single charge $+q$ outside the conductor, or the potential in the presence of a spherical conductor and an arbitrary charge distribution inside the free volume (**exercise:** try it for a spherical segment distribution with the same center as the conductor).

3.4 Green's functions from Laplacian eigenfunctions

In this section, we will present a systematic way to obtain Green's functions for Dirichlet boundary conditions on a surface $S(V)$ surrounding a volume V. We remind that a Dirichlet Green's function vanishes on $S(V)$.

Consider the eigenfunctions $\psi_n(\vec{x})$ of the Laplace operator:

$$\nabla^2 \psi_n = \lambda_n \psi_n, \tag{122}$$

and impose on them that they vanish on the boundary:

$$\psi_n(\vec{x}) = 0, \quad \forall \vec{x} \in S(V). \tag{123}$$

Often, the eigenvalues λ_n turn out to be discrete. Let us assume that this is the case for the rest of the analysis. In the continuous cases, our formulae will need to be modified trivially, changing summations into integrations. In addition, we will assume that the eigenvalues are not degenerate (all λ_n are different from each other).

We now make use of the theorem in Eq. 90. We have

$$\int_V d^3\vec{x} \left[\psi_m^* \nabla^2 \psi_n - \psi_n \nabla^2 \psi_m^*\right] = \int_{S(V)} d\vec{S} \cdot \left[\psi_m^* \vec{\nabla} \psi_n - \psi_n \vec{\nabla} \psi_m^*\right] \tag{124}$$

The rhs of Eq. 124 vanishes since the eigenfunctions ψ_n vanish on the boundary $S(V)$. Thus, we obtain:

$$(\lambda_m^* - \lambda_n) \int_V d^3\vec{x}\, \psi_m^*(\vec{x}) \psi_n(\vec{x}) = 0. \tag{125}$$

For $m = n$, we have that

$$(\lambda_n^* - \lambda_n) \int_V d^3\vec{x}\, |\psi_n(\vec{x})|^2 = 0. \tag{126}$$

Since the above integral is positive definite, we conclude that

$$\lambda_n^* = \lambda_n, \tag{127}$$

stating that the eigenvalues are real. For $m \neq n$, we conclude that

$$\int_V d^3\vec{x}\, \psi_m^*(\vec{x})\psi_n(\vec{x}) = 0. \tag{128}$$

Choosing appropriately the normalization of the eigenfunctions, we have the orthogonality condition:

$$\int_V d^3\vec{x}\, \psi_m^*(\vec{x})\psi_n(\vec{x}) = \delta_{nm} \tag{129}$$

It is often the case that the eigenfunctions form a complete basis, meaning that any other function $f(\vec{x})$ which vanishes on the boundary $S(V)$ can be written as a linear superposition of the Laplace eigenfunctions:

$$f(\vec{x}) = \sum_n c_n \psi_n(\vec{x}) \tag{130}$$

Multiplying with $\psi_m^*(\vec{x})$ and integrating over the volume V we obtain:

$$\int_V d^3\vec{x}\, \psi_m^*(\vec{x}) f(\vec{x}) = \sum_n c_n \int_V d^3\vec{x}\, \psi_m^*(\vec{x})\psi_n(\vec{x}) = \sum_n c_n \delta_{nm} = c_m. \tag{131}$$

Substituting into Eq. 130 we obtain that

$$f(\vec{x}) = \int d^3\vec{y}\, f(\vec{y}) \sum_n \psi_n^*(\vec{y})\psi_n(\vec{x}) \tag{132}$$

which leads to the completeness condition:

$$\sum_n \psi_n^*(\vec{y})\psi_n(\vec{x}) = \delta(\vec{x} - \vec{y}). \tag{133}$$

We are now ready to apply the above to a Dirichlet Green's function $G_D(\vec{x}, \vec{y})$, which we can write as a linear superposition of Laplace eigenfunctions:

$$G_D(\vec{x}, \vec{y}) = \sum_n c_n(\vec{y})\psi_n(\vec{x}). \tag{134}$$

Applying the Laplace operator on both sides we obtain

$$\begin{aligned}
\rightsquigarrow \nabla^2 G_D(\vec{x},\vec{y}) &= \sum_n c_n(\vec{y})\nabla^2\psi_n(\vec{x}) \\
\rightsquigarrow -4\pi\delta(\vec{x}-\vec{y}) &= \sum_n c_n(\vec{y})\lambda_n\psi_n(\vec{x}) \\
\rightsquigarrow -4\pi\sum_n \psi_n^*(\vec{y})\psi_n(\vec{x}) &= \sum_n c_n(\vec{y})\lambda_n\psi_n(\vec{x}) \\
\rightsquigarrow c_n(\vec{y}) &= -\frac{4\pi}{\lambda_n}\psi_n^*(\vec{y}) \qquad (135)
\end{aligned}$$

Knowing the eigenvalues and eigenfunctions of the Laplace operator we can construct the Green's function via:

$$G_D(\vec{x},\vec{y}) = -4\pi\sum_n \frac{\psi_n^*(\vec{y})\psi_n(\vec{x})}{\lambda_n}. \qquad (136)$$

3.4.1 Example: all (infinite) space

The Green's function with boundaries at infinity is the inverse of the distance of the two vectors in the arguments of the function:

$$G_D(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}. \qquad (137)$$

We will derive this result from the eigenfunctions of the Laplacian:

$$\nabla^2\psi(\vec{x}) = \lambda\psi(\vec{x}). \qquad (138)$$

The solutions of this equation (eigenfunctions) are:

$$\psi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\vec{k}\cdot\vec{x}}, \qquad (139)$$

with eigenvalues:

$$\lambda_{\vec{k}} = -\left|\vec{k}\right|^2 \qquad (140)$$

Exercise: *Prove that*

$$\int_{-\infty}^{\infty} dx e^{ixa} = 2\pi\delta(a). \qquad (141)$$

Hint: Think what is the Fourier transform of a delta function and the inverse Fourier transform of 1. These solutions are orthonormal. Indeed,

$$\int d^3\vec{x}\,\psi_{\vec{k}}(\vec{x})^*\psi_{\vec{k}'}(\vec{x}) = \int \frac{d^3\vec{x}}{(2\pi)^3}\,e^{i\vec{x}\cdot(\vec{k}'-\vec{k})} = \delta(\vec{k}'-\vec{k}). \qquad (142)$$

They are also complete:

$$\int d^3\vec{k}\,\psi_{\vec{k}}(\vec{x})^*\psi_{\vec{k}}(\vec{y}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\,e^{i\vec{k}\cdot(\vec{y}-\vec{x})} = \delta(\vec{x}-\vec{y}). \tag{143}$$

We have established all conditions required for applying Eq. 136 in order to calculate the Green's function from the eigenfunctions of the Laplace operator. We have

$$G(\vec{x},\vec{y}) = -4\pi \int \frac{d^3\vec{k}}{-\left|\vec{k}\right|^2} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{144}$$

We can perform the integrations, by writing

$$\vec{k}\cdot(\vec{x}-\vec{y}) = \left|\vec{k}\right| |\vec{x}-\vec{y}| \cos\theta, \tag{145}$$

using spherical coordinates:

$$d^3\vec{k} = d\left|\vec{k}\right| \left|\vec{k}\right|^2 d\cos\theta\, d\phi \tag{146}$$

and Eq. 141. We find:

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}, \tag{147}$$

which is the anticipated result.

3.4.2 Example: Inside an orthogonal parallelepiped

We will now calculate the Green's function for Dirichlet boundary conditions on the sides of an orthogonal parallelepiped:

$$V:\ x \in [0,a],\ y \in [0,b],\ z \in [0,c]. \tag{148}$$

The eigenfunctions of the Laplace operator which vanish on the boundary of the orthogonal parallelepiped are:

$$\psi_{lmn} = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi x}{c}\right) \tag{149}$$

where

$$n, l, m = 1, 2, \ldots$$

and eigenvalues:

$$\lambda_{lmn} = -\pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) \tag{150}$$

You can check that they are orthonormal and satisfy the completeness condition (bf exercise). The Green's function is then an infinite series:

$$G(\vec{x}, \vec{y}) = -4\pi \sum_{l,m,n=1}^{\infty} \frac{\psi_{lmn}(\vec{x})\psi_{nlm}(\vec{y})}{\lambda_{lmn}}. \tag{151}$$

END OF WEEK 2

3.5 Laplace operator and spherical symmetry

Let us now consider boundary condition problems with spherical symmetry. To find the eigenfunctions of the Laplace operator,

$$\nabla^2\psi = \lambda\psi, \tag{152}$$

it is best to work in spherical coordinates (r, θ, ϕ). The Laplace operator takes the form

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\hat{A}}{r^2}, \tag{153}$$

where $\hat{A}$ a differential operator acting only on the angles θ, ϕ:

$$\hat{A} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}. \tag{154}$$

Setting $x = \cos\theta$, the angular differential operator takes the form:

$$\hat{A} = \frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + \frac{1}{1-x^2}\frac{\partial^2}{\partial^2\phi}. \tag{155}$$

We can solve this differential equation by means of the method os separation of variables. We seek factorizable solutions:

$$\psi(r, \theta, \phi) = \frac{R(r)}{r}Y(\theta, \phi), \tag{156}$$

where the radial and angular parts are factorized. Substituting in Eq. 152, we obtain:

$$r^2\left[\frac{1}{R}\frac{\partial^2 R}{\partial r^2} - \lambda\right] = -\frac{1}{Y}\hat{A}Y. \tag{157}$$

Since the rhs depends only on angular coordinates and the lhs depends only on the radial coordinate, for them to be equal they must be both the same constant $l(l+1)$ (written in such a way for future convenience). We thus have two separate differential equations:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} - \frac{l(l+1)}{r^2} - \lambda = 0 \tag{158}$$

and

$$\hat{A}Y = -l(l+1)Y. \tag{159}$$

In the following we will solve the above differential equations separately.

3.5.1 Radial differential equation

We will defer the general solution of the radial differential equation for later (exercise classes). Here, we consider the special case of a zero eigenvalue: $\lambda = 0$. Notice that this corresponds to solving $\nabla^2 \Psi = 0$, which is to calculate the potential in empty space. The radial equation becomes:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} = \frac{l(l+1)}{r^2} \tag{160}$$

For

$$R = r^a,$$

we obtain:

$$a(a-1) = l(l+1), \tag{161}$$

with solutions:

$$a = -l, 1 + l. \tag{162}$$

Thus, the general solution of the Laplace equation in spherical coordinates is:

$$\Psi(r, \theta, \phi) = \sum_l \frac{1}{r}\left(A_l r^{-l} + B_l r^{l+1}\right) Y_l(\theta, \phi). \tag{163}$$

where the sum could represent an integral if l turns out to be a continuous constant (it will not!).

3.5.2 Angular differential equation

Eq. 159 can also be solved with the method of separation of variables. We write:

$$Y(\theta, \phi) = \Theta(\theta)\Phi(\phi). \tag{164}$$

Then, it becomes:

$$\frac{1-x^2}{\Theta}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]\Theta = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}. \tag{165}$$

As before, the two sides must be equal to a constant, which we call m^2. This leads to the two differential equations:

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m^2 \Phi \tag{166}$$

and

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}+l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0. \tag{167}$$

The differential equation on the azimuthal variable can be solved easily, yielding:

$$\Phi(\phi)=e^{im\phi}. \tag{168}$$

Demanding that this function is single-valued,

$$\Phi(\phi)=\Phi(\phi+2\pi), \tag{169}$$

we obtain that m must be an integer,

$$m=0,\pm1,\pm2,\dots \tag{170}$$

We will find solutions for the differential equation (Eq. 167) on the polar angle θ for $m=0$ first. Int this special case it becomes:

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}+l(l+1)\right]\Theta=0. \tag{171}$$

Consider a polynomial of degree l:

$$p_l(x)=(x^2-1)^l \tag{172}$$

Let's introduce an abbreviation for the differential operator:

$$D\equiv\frac{d}{dx},\quad D^m\equiv\frac{d^m}{dx^m}.$$

The product rule for multiple derivatives is:

$$D^m(fg)=\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}D^kfD^{m-k}g. \tag{173}$$

We start with the identity:

$$(x^2-1)Dp_l(x)=2lxp_l(x) \tag{174}$$

Applying the differential operator D^{l+1} on the above and using the product rule of Eq. 173, we arrive at the identity

$$D(1-x^2)D\left(D^lp_l(x)\right)+l(l+1)\left(D^lp_l(x)\right)=0. \tag{175}$$

Therefore, the polynomials $p_l(x)$ are solutions of the differential equation 171. These polynomials, normalized as:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \tag{176}$$

are known as the Legendre polynomials. The first few of them are:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= 3/2\,x^2 - 1/2 \\
P_3(x) &= 5/2\,x^3 - 3/2\,x \\
P_4(x) &= \frac{35}{8}\,x^4 - \frac{15}{4}\,x^2 + 3/8 \\
&\dots
\end{aligned} \tag{177}$$

The Legendre polynomials can be cast as the sum:

$$P_l(x) = \frac{1}{2^l}\sum_{k=0}^{l}\left(\begin{array}{c} l \\ k \end{array}\right)^2 (x-1)^{l-k}(x+1)^k \tag{178}$$

Notice that they are normalized so that

$$P_l(1) = 1. \tag{179}$$

Legendre polynomials vanish when integrated with any other polynomial of a lesser degree in the range $[-1,1]$:

$$\int_{-1}^{1} dx x^k P_l(x) = 0, \quad \forall\, k = 0,1,\dots(l-1). \tag{180}$$

Indeed,

$$\begin{aligned}
&\int_{-1}^{1} dx x^k D^l(x^2-1)^l = \int_{-1}^{1} dx x^k D D^{l-1}(x^2-1)^l \\
&= x^k D^{l-1}(x^2-1)^l\Big|_{-1}^{1} - k\int_{-1}^{1} dx x^{k-1} D^{l-1}(x^2-1)^l
\end{aligned} \tag{181}$$

The first term vanishes at the boundary, since

$$D^{l-1}(x^2-1)^l \propto (x^2-1)$$

We can perform integration by parts $k < l$ times where we always end up with vanishing boundary terms proportional to $(x^2 - 1)$. It is then obvious that two different Legendre polynomials are necessarily orthogonal:

$$\int_{-1}^{1} dx P_l(x) P_m(x) = 0, \quad \forall\, l \neq m. \tag{182}$$

With the above procedure, it is also easy to prove that:

$$\int_{-1}^{1} dx x^l P_l(x) = \frac{l!^2 2^{l+1}}{(1+2l)!} \tag{183}$$

To calculate the normalization of a Legendre polynomial, we first note that the highest order term is

$$P_l(x) = \frac{(2l)!}{2^l l!^2} x^l + \mathcal{O}(x^{l-1}) \tag{184}$$

and therefore

$$\int_{-1}^{1} dx P_l(x)^2 = \frac{(2l)!}{2^l l!^2} \int_{-1}^{1} dx x^l P_l(x) = \frac{2}{1+2l}. \tag{185}$$

We then write the orthogonality relation

$$\int_{-1}^{1} dx P_l(x) P_m(x) = \frac{2}{1+2l} \delta_{lm}. \tag{186}$$

Legendre polynomials form a basis for all continuous functions $f(x)$ in $[-1, 1]$.

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{187}$$

where the coefficients are found as:

$$\begin{aligned} \int_{-1}^{1} dx P_m(x) f(x) &= \sum_{n=0}^{\infty} c_n \int_{-1}^{1} dx P_n(x) P_m(x) \\ \rightsquigarrow \int_{-1}^{1} dx P_m(x) f(x) &= c_m \frac{2}{1+2m}, \end{aligned} \tag{188}$$

and finally,

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} dx P_n(x) f(x). \tag{189}$$

Inserting this coefficient into Eq. 187, we find:

$$f(x) = \int_{-1}^{1} dy f(y) \sum_{n=0}^{\infty} P_n(x) P_n(y) \frac{2n+1}{2} \tag{190}$$

which gives the completeness relation:

$$\sum_{n=0}^{\infty} P_n(x) P_n(y) \frac{2n+1}{2} = \delta(x-y). \tag{191}$$

We now turn to the more complicated differential equation 167 for arbitrary values of m. Finite solutions in the range: $x \in [-1, 1]$ are:

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{\partial^m}{\partial x^m} P_l(x) \tag{192}$$

or,

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \frac{\partial^{l+m}}{\partial x^{l+m}} (x^2-1)^l. \tag{193}$$

where the constant m must be an integer:

$$m = -l, -l+1, \ldots 0, \ldots l-1, l. \tag{194}$$

The polynomials $P_l^m(x)$ are the associated Legendre polynomials. Polynomials with negative values of the integer m are related to polynomials with positive values of the integer m via:

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \tag{195}$$

They satisfy the orthogonality relation

$$\int_{-1}^{1} dx P_k^m(x) P_l^m(x) = \frac{2(m+l)!}{(2l+1)(l-m)!} \delta_{kl} \tag{196}$$

exercise: Prove that

$$\int_{-1}^{1} dx \frac{P_l^m(x) P_l^n(x)}{1-x^2} = \frac{(l+m)!}{m(l-m)!} \delta_{m,n}, \quad \text{if } m \neq 0. \tag{197}$$

Putting together the polar and azimuthal solutions, we find that the functions:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi} P_l^m(\cos\theta) \tag{198}$$

are partial solutions of Eq 159. These functions are called **spherical harmonics** and appear in almost every physics problem with spherical symmetry.

Spherical harmonics are orthogonal. The orthogonality condition is that:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta Y^*_{l'm'}(\theta,\phi) Y^*_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm} \tag{199}$$

and it follows directly from the separate orthogonality conditions of the associated Legendre polynomials (Eq. 196) and the azimuthal solutions $e^{im\phi}$:

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im\phi} e^{im'\phi} = \delta_{m'm}. \tag{200}$$

Every function of the polar and azimuthal angles can be written as a linear superposition of spherical harmonics:

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta,\phi) \tag{201}$$

where the coefficients are:

$$c_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta Y^*_{lm}(\theta,\phi) f(\theta,\phi). \tag{202}$$

Inserting this expression back to Eq. 201 we obtain the completeness identity for spherical harmonics:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) = \delta(\phi'-\phi)\delta(\cos\theta' - \cos\theta). \tag{203}$$

We note that for $m = 0$, the spherical harmonics collapse to the Legendre polynomials:

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{1+2l}{4\pi}} P_l(\cos\theta). \tag{204}$$

and are independent of the angle ϕ. In addition, spherical harmonics with $m \neq 0$ vanish at $\theta = 0$. To be convinced about it, recall the factor $(1 - x^2)^{m/2} \sim \sin^m\theta$ in the definition of the associated Legendre polynomials $P_l^m(\cos\theta)$.

Finally, we note that spherical harmonics with same "polar" integer l and opposite "azimuthal" integer m are related by:

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi). \tag{205}$$

We are now in a position to write down the general solution of the Laplace differential equation:

$$\nabla^2\Psi(r,\theta,\phi) = 0. \tag{206}$$

This is:

$$\Psi(r,\theta,\phi) = \sum_l \sum_{m=-l}^{l} \left(A_{lm} r^{-l-1} + B_{lm} r^l\right) Y_{lm}(\theta,\phi) \tag{207}$$

exercise: Find the eigenfunctions of the Laplace operator in terms of spherical harmonics

$$\nabla^2\Psi_\lambda(r,\theta,\phi) = \lambda\Psi_\lambda(r,\theta,\phi) \tag{208}$$

with the boundary conditions $\Psi_\lambda(0,\theta,\phi) = \Psi_\lambda(a,\theta,\phi) = 0$.

3.5.3 Expansion of inverse distance in Legendre polynomials

Consider two vectors:

$$\vec{r}_L, \vec{r}_S, \quad r_L > r_S.$$

Their distance is:

$$|\vec{r}_L - \vec{r}_S| = \left[r_L^2 + r_s^2 - 2r_L r_s \cos\theta\right]^{\frac{1}{2}} \tag{209}$$

where θ is the angle of the two vectors. Any function of $\cos\theta$ can be expanded as a series in Legendre polynomials. In particular, for the inverse of a distance we have

$$\frac{1}{|\vec{r}_L - \vec{r}_S|} = \sum_{l=0}^{\infty} c_l P_l(\cos\theta). \tag{210}$$

To compute the coefficients, we consider the special case of the two vectors being parallel $\theta = 0$. Recall that Legendre polynomials are normalized to be one for a unit argument:

$$P_l(\cos 0) = P_l(1) = 1.$$

Then

$$\sum_{l=0}^{\infty} c_l = \frac{1}{r_L - r_S} = \sum_{l=0}^{\infty} \frac{r_S^l}{r_L^{l+1}} \rightsquigarrow c_l = \frac{r_S^l}{r_L^{l+1}}. \tag{211}$$

We then have the result

$$\frac{1}{|\vec{r}_L - \vec{r}_S|} = \sum_{l=0}^{\infty} \frac{r_S^l}{r_L^{l+1}} P_l(\cos\theta). \tag{212}$$

Similarly, we can derive the more general result:

$$\frac{1}{|\vec{r}_L - \vec{r}_S|^a} = \sum_{l=0}^{\infty} \frac{(a,l)}{l!} \frac{r_S^l}{r_L^{l+a}} P_l(\cos\theta), \tag{213}$$

where the Pochhammer symbol is defined as:

$$(a,l) = \frac{\Gamma(a+l)}{\Gamma(a)}. \tag{214}$$

3.6 Multipole expansion

Consider a charge distribution $\rho(\vec{x})$ which occupies a small volume V'. We are interested in the potential that this distribution creates at a distance $\vec{r}$ outside the region of the charge distribution. Intuitively, we expect that if we are sufficiently far from the charge distribution the potential will resemble a Coulomb potential. This is correct to a first approximation, but not exact; there should be corrections which become more important the closer we approach to the charges of the distribution.

The potential is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3\vec{x} \frac{\rho(\vec{x})}{|\vec{x} - \vec{r}|}. \tag{215}$$

Since $r > x$, we can expand the inverse distance in the integrand in $\frac{x}{r}$, obtaining:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_{V'} d^3\vec{x} \rho(\vec{x}) x^l P_l(\cos\gamma), \tag{216}$$

where γ is the angle formed by the two vector $\vec{x}$ and $\vec{r}$. We write the two vectors as:

$$\vec{x} \equiv (x, \theta_x, \phi_x), \quad \vec{r} \equiv (r, \theta, \phi). \tag{217}$$

It is an easy geometry exercise to prove that:

$$\cos\gamma = \frac{\vec{x} \cdot \vec{r}}{xr} = \cos\theta\cos\theta_x + \sin\theta\sin\theta_x\cos(\phi - \phi_x). \tag{218}$$

We would like to perform the $d^3\vec{x} = x^2 dx d\Omega_x$ integration in spherical coordinates. However, the expression for $\cos\gamma$ in terms of θ_x, ϕ_x seems overly complicated. To our rescue comes the following identity, called the "addition theorem":

$$P_l(\cos\gamma) = \frac{4\pi}{1+2l} \sum_{m=-l}^{l} Y^*_{lm}(\theta_x, \phi_x) Y_{lm}(\theta, \phi). \tag{219}$$

It is a consequence of the fact that the lhs is a function of angular co-ordinates and it can therefore be expressed as a linear superposition of spherical harmonics. It is also manifest that the expression is symmetric under the exchange: $(\theta, \phi) \leftrightarrow (\theta_x, \phi_x)$. We will not prove this theorem here, since it can be proven easier in a future course of Quantum Mechanics. Substituting into Eq. 216, we obtain:

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{1+2l} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} \left[\int_{V'} x^2 dx d\Omega_x Y^*_{lm}(\theta_x, \phi_x) \rho(\vec{x}) x^l \right] \frac{Y_{lm}(\theta, \phi)}{r^{1+l}}, \tag{220}$$

or, in a more compact form:

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{1+2l} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} \frac{q_{lm} Y_{lm}(\theta, \phi)}{r^{1+l}}, \tag{221}$$

where we have defined the quantities:

$$q_{lm} = \int_{V'} d^3\vec{x}\, Y^*_{lm}(\theta_x, \phi_x) \rho(\vec{x}) x^l \tag{222}$$

which are called multipole moments. The moments characterise the geometry of the charge distribution. For example,

$$q_{00} = \int d^3\vec{x} \rho(\vec{x}) Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \int d^3\vec{x} \rho(\vec{x}) = \frac{Q}{\sqrt{4\pi}} \tag{223}$$

and it is proportional to the total charge in the distribution. For the higher moments, we find:

$$q_{11} = -\sqrt{\frac{3}{8\pi}} (p_1 - i p_2), \tag{224}$$

$$q_{00} = \sqrt{\frac{3}{4\pi}} (p_3), \tag{225}$$

with

$$\vec{p} = (p_1, p_2, p_3) = \int d^3\vec{x}\, \vec{x} \rho(\vec{x}) \tag{226}$$

the dipole moment. For the next moments:

$$
\begin{aligned}
q_{22} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2iQ_{12} - Q_{22}), && (227)\\
q_{21} &= -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}), && (228)\\
q_{21} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{33} && (229)
\end{aligned}
$$

where

$$Q_{ij} = \int d^3\vec{x}\,(x_i x_j - x^2\delta_{ij})\rho(\vec{x}) \tag{230}$$

the quadrupole tensor, etc. The most important moments are the first ones, since the higher moments are suppressed by powers of $1/r$ in their contribution to the potential. A measurement of the angular distribution of the potential due to some unknown charge distribution can be used for the extraction of the multipole moments of the distribution. This is our best way to learn something about the geometrical characteristics of the distribution.

END OF WEEK 3

4 Magnetic field

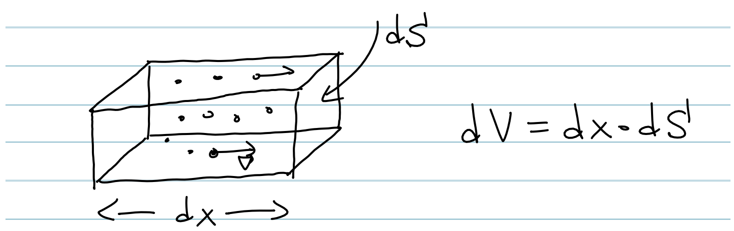
The force acting on a charge q which moves with velocity $\vec{v}$ inside an electromagnetic field $(\vec{E}, \vec{B})$ is:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{231}$$

The direction of the electric component of the force is along or against the direction of the electric field $\vec{E}$. The magnetic component of the force is at right angles with both the magnetic field and the direction of motion of the charge.

4.1 Currents

Macroscopically, it is easy to observe the magnetic force acting on a large number of charges which move together inside a magnetic field $\vec{B}$. Currents can be materialized inside conductors which allow electrons to move freely within their body.



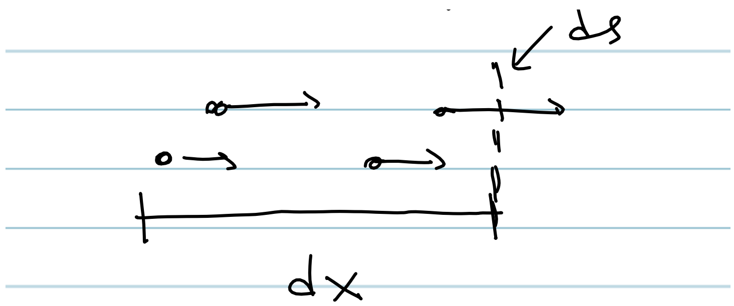
Let us consider a volume element $dV = dxdS$ with a charge density

$$\rho = \frac{\Delta Q_0}{dV}$$

at an initial time t_0. ΔQ_0 is the total charge contained inside dV at t_0. Assume that all elementary charges are identical (q). Then

$$\Delta Q_0 = N_0 q, \tag{232}$$

where N_0 is their number inside dV at t_0. Let's assume that all charges are moving with a velocity $\vec{v}$ (which we have aligned with the $x-$axis).



In a small time dt a number of charges will move outside the volume dV. This number is:

$$qN_{\text{escaping}} = \rho(\vec{v}dt) \cdot d\vec{S}. \tag{233}$$

We define the current density $\vec{j}$ as the charge transversing a surface $d\vec{S}$ per unit time and surface. Specifically:

$$qN_{\text{escaping}} = \vec{J} \cdot d\vec{S}dt \tag{234}$$

By comparing the last two equations we conclude that:

$$\vec{J} = \rho\vec{v}. \tag{235}$$

After time dt, the remaining charge inside the volume dV is

$$\begin{aligned} \Delta Q_{t_0+dt} &= \Delta Q_0 - qN_{\text{escaping}} \\ \rightsquigarrow \frac{\Delta Q_{t_0+dt} - \Delta Q_0}{dt} &= -\vec{J} \cdot d\vec{S} \\ \rightsquigarrow dV\frac{\partial\rho}{\partial t} &= -\vec{J} \cdot d\vec{S}. \end{aligned} \tag{236}$$

Eq. 236 states that the change in the charge density during dt is equal with the flux of the current density through the surface $d\vec{S}$. Integrating over the surface $S(V)$ of an arbitrary volume V, we have:

$$\begin{aligned} \int_V dV\frac{\partial\rho}{\partial t} &= -\int_{S(V)} \vec{J} \cdot d\vec{S} \\ \rightsquigarrow \int_V d^3\vec{x}\,\frac{\partial\rho}{\partial t} &= -\int_V \vec{\nabla} \cdot \vec{J} \\ \rightsquigarrow \qquad & \int_V d^3\vec{x}\left[\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot \vec{J}\right] = 0. \end{aligned} \tag{237}$$

Eq. 237 must hold for **any** volume V. In order for this to happen, the integrand must vanish. Therefore, we arrive to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{238}$$

The above equation is a consequence of charge conservation.

4.2 Magnetic field of steady currents

We will now study a very common situation where the charge density is constant:

$$\frac{\partial \rho}{\partial t} = 0.$$

This can be materialized in simple circuits, where charge flows in and out (at the same rate) of any volume element. The divergence of the electric current density is therefore:

$$\vec{\nabla} \cdot \vec{J} = 0. \tag{239}$$

In such situations, the electric and magnetic fields are constant in time. The magnetic field is determined via two of Maxwell equations, which become:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{240}$$

and

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0}. \tag{241}$$

We can derive an integrated form for the above. From Eq. 240, we obtain that:

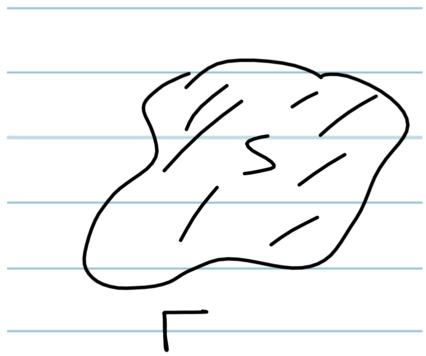
$$\int_V d^3\vec{x}\, \vec{\nabla} \cdot \vec{B} = 0 \rightsquigarrow \int_{S(V)} \vec{B} \cdot d\vec{S} = 0. \tag{242}$$

Therefore, the flux of the magnetic field out of any volume is zero. Recall than in electrostatics,

$$\int_{S(V)} \vec{E} \cdot d\vec{S} = \frac{Q_{\text{inside } V}}{\epsilon_0}.$$

The analogous Eq. 242 states that there is no magnetic charge to be found anywhere.

Integrating Eq. 241 over an open surface $S(\Gamma)$ bounded by a closed loop Γ,



we obtain:

$$c^2 \int_{S(\Gamma)} d\vec{S} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \frac{\int_{S(\Gamma)} \vec{J} \cdot d\vec{S}}{\epsilon_0}$$

$$\leadsto \quad c^2 \oint_{\Gamma} \vec{B} \cdot d\vec{l} = \frac{\int_{S(\Gamma)} \vec{J} \cdot d\vec{S}}{\epsilon_0}. \tag{243}$$

The rhs is proportional to the current intensity passing through the loop Γ:

$$I_{\text{through }\Gamma} \equiv \int_{S(\Gamma)} \vec{J} \cdot d\vec{S}. \tag{244}$$

and denotes the total charge passing through the closed loop Γ per unit time. We then arrive at:

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \frac{I_{\text{through }\Gamma}}{\epsilon_0 c^2}, \tag{245}$$

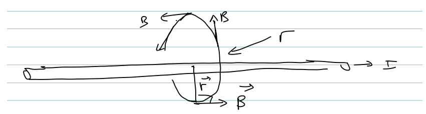
which is known as Ampere's law.

4.3 Applications of Ampere's law

For problems with a manifest symmetry, Ampere's law can be sufficient to determine the magnetic field. Let's review here two very well known applications.

4.3.1 Long straight wire

Consider a long straight wire with a current I flowing through it.

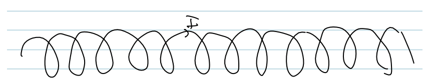


Close to the wire, in a distance r which is much smaller than the length of the wire, the magnetic field is by symmetry the same at all positions of the same distance r. Applying Ampere's law for a loop which is a circle of radius r around the wire we obtain:

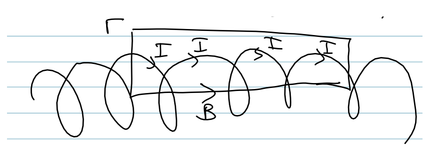
$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \frac{I}{\epsilon_0 c^2} \\ \rightsquigarrow B 2\pi r &= \frac{I}{\epsilon_0 c^2} \\ \rightsquigarrow B &= \frac{1}{4\pi\epsilon_0 c^2} \frac{2I}{r}. \end{aligned} \tag{246}$$

4.3.2 Solenoid

Consider a long solenoid with a current I through it.



Very long solenoids have a homogeneous magnetic field inside them and a zero magnetic field outside them, to a good approximation. These two assumptions allow us to estimate the magnetic field inside the solenoid.



Let's apply Ampere's law for the loop Γ in the picture above. The total current through Γ is the number $N_{\text{in }\Gamma}$ of spires in it times the current I through one spire:

$$I_{\text{through }\Gamma} = I N_{\text{in }\Gamma} \tag{247}$$

Then, from Ampere's law:

$$BL = \frac{I N_{\text{in }\Gamma}}{\epsilon_0 c^2} \tag{248}$$

which gives the magnetic field:

$$B = \frac{nI}{\epsilon_0 c^2}. \tag{249}$$

n is the number of spires per unit length in the solenoid.

4.4 Vector potential

Let's examine further the basic equations of magnetostatics (Eqs 240-241). Eq. 240 is automatically satisfied if we cast the magnetic field as the curl of a vector $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}. \tag{250}$$

We introduce the abbreviation:

$$\partial_i \equiv \frac{\partial}{\partial x_i} \tag{251}$$

and Einstein's summation convention:

$$A_i B_i \equiv \sum_{i=1}^{3} A_i B_i = \vec{A} \cdot \vec{B}. \tag{252}$$

Then

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \partial_i (\vec{\nabla} \times \vec{A})_i = \partial_i \left(\epsilon_{ijk} \partial_j A_k \right) = \epsilon_{ijk} \partial_i \partial_j A_k = 0. \tag{253}$$

The tensor ϵ_{ijk} is fully antisymmetric, while the second derivative is symmetric and therefore their contraction vanishes.

From Eq. 241 we obtain:

$$\begin{aligned} &(\vec{\nabla} \times \vec{B})_i = \frac{J_i}{\epsilon_0 c^2} \\ &\rightsquigarrow \epsilon_{ijk} \partial_j B_k = \frac{J_i}{\epsilon_0 c^2} \\ &\rightsquigarrow \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l A_m = \frac{J_i}{\epsilon_0 c^2} \\ &\rightsquigarrow \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m = \frac{J_i}{\epsilon_0 c^2}. \end{aligned} \tag{254}$$

The contraction of two epsilon antisymmetric tensors yields:

$$\epsilon_{ijk}\epsilon_{klm} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{255}$$

Thus, we have:

$$\begin{aligned} \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\partial_l A_m &= \frac{J_i}{\epsilon_0 c^2} \\ \partial_i\partial_j A_j - \partial_j^2 A_i &= \frac{J_i}{\epsilon_0 c^2}. \end{aligned} \tag{256}$$

In full vector form:

$$\nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = -\frac{\vec{J}}{c^2\epsilon_0}. \tag{257}$$

This is a second order differential equation for the components of the vector potential. Had it not been for the second term in the lhs, this equation would become our familiar Poisson differential equation from electrostatics. In fact, we can choose to eliminate this term by exploiting a property of the vector potential, known as gauge invariance. Consider two vector potential functions $\vec{A}$ and $\vec{A}'$ which differ from each other by the gradient of a scalar function f:

$$\vec{A}' = \vec{A} + \vec{\nabla} f \tag{258}$$

These two vector potentials are physically equivalent and give rise to the same magnetic field

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}. \tag{259}$$

Indeed:

$$(\vec{\nabla}\times\vec{\nabla} f)_i = \epsilon_{ijk}\partial_j\partial_k f = 0. \tag{260}$$

It is easy to verify that $\vec{A}'$ satisfies the same differential equation 257. Indeed, under a "gauge transformation":

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f, \tag{261}$$

we have

$$\begin{aligned} &\nabla^2\vec{A}' - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}') \\ &= \nabla^2(\vec{A} + \vec{\nabla} f) - \vec{\nabla}[\vec{\nabla}\cdot(\vec{A} + \vec{\nabla} f)] \\ &= \nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \\ &= -\frac{\vec{J}}{c^2\epsilon_0}. \end{aligned} \tag{262}$$

We therefore have the liberty to add any gradient to the vector potential without changing the physics. This freedom that we have is called a gauge choice. We can exercise this choice to simplify the differential equation for the potential. For example, assume a potential $\vec{A}'$ with

$$\vec{\nabla} \cdot \vec{A}' \neq 0. \tag{263}$$

Now, let us find a solution of the Poisson equation:

$$\nabla^2 f = -\vec{\nabla} \cdot \vec{A}'. \tag{264}$$

The solution is our familiar:

$$f(\vec{r}) = \frac{1}{4\pi} \int d^3\vec{x} \, \frac{(\vec{\nabla} \cdot \vec{A}')(\vec{x})}{|\vec{r} - \vec{x}|} \tag{265}$$

Then for

$$\vec{A} = \vec{A}' + \vec{\nabla} f = \vec{A}' + \frac{1}{4\pi} \vec{\nabla} \int d^3\vec{x} \, \frac{(\vec{\nabla} \cdot \vec{A}')(\vec{x})}{|\vec{r} - \vec{x}|}$$

we obtain

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{266}$$

The "gauge" for which the above happens is called the "Coulomb" gauge. In that gauge, the vector potential satisfies a Poisson equation:

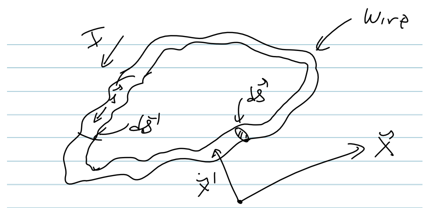
$$\nabla^2 \vec{A} = -\frac{\vec{J}}{c^2 \epsilon_0}. \tag{267}$$

which can be solved with the same techniques as we have developed in electrostatics. If we know all currents in all space, we can write the solution as:

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \, \frac{\vec{J}(\vec{y})}{|\vec{x} - \vec{y}|}. \tag{268}$$

4.5 Wires of a small thickness

A common situation is the calculation of the vector potential $\vec{A}$ and the corresponding magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ due to electric currents circulating in thin wires.



The integral of Eq. 268

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{x}' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{269}$$

receives zero contributions from points outside the wires, where the current density is zero. We therefore have

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int_{\text{in wires}} d^3\vec{x}' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{270}$$

Let's assume that the thickness of the wires is small and that the density $\vec{J}(\vec{x}')$ of the current as well as the distance $|\vec{x} - \vec{x}'|$ are practically constant on a cross-section $\vec{S}$ of the wire. We can then integrate over the transverse directions $d\vec{S}$, obtaining:

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \oint_{wire} dl(\vec{x}') \, \frac{\vec{I}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{271}$$

In the above, $l(\vec{x}')$ parameterises the curve of the wire. $\vec{I}$ is a vector with a magnitude equal to the current intensity

$$I(\vec{x}) = \int d\vec{S} \cdot \vec{J}(\vec{x}) \tag{272}$$

and direction the one of the current. We can now define a vector $d\vec{l}$ which is tangential to the loop. Usually, we take $d\vec{l}(\vec{x})$ to circulate anti-clockwise. The current intensity vector $\vec{I}(\vec{x})$ at a point $\vec{x}$ is either parallel or anti-parallel to $d\vec{l}(\vec{x})$:

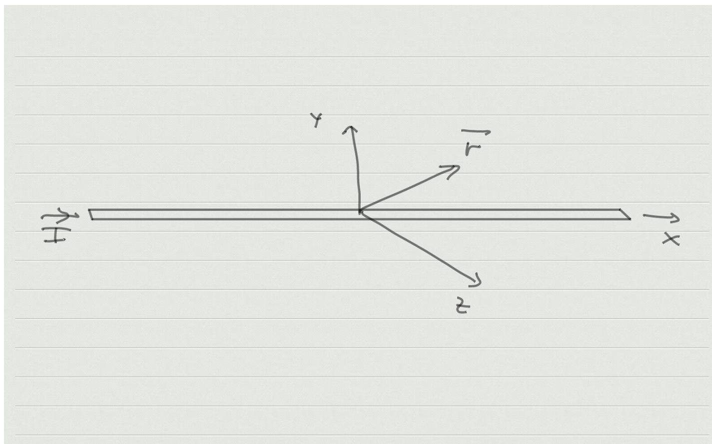
$$d\hat{l}(\vec{x}) = \pm \hat{I}(\vec{x}). \tag{273}$$

Assuming an anti-clockwise steady current we have:

$$\vec{A}(\vec{r}) = \frac{I}{4\pi\epsilon_0 c^2} \oint_{wire} d\vec{l}(\vec{x}) \, \frac{1}{|\vec{r} - \vec{x}|} \tag{274}$$

For a clock-wise current one should remember to include a minus sign or to change the direction convention for $d\vec{l}$.

Let us now see an application of the above result. Consider a very long wire along the $x-$axis with current intensity I.



Since the current flows only on the $x-$axis the vector potential in the y, z axes is zero:

$$A_y = A_z = 0. \tag{275}$$

The $x-$component of the vector potential is:

$$\begin{aligned} A_x &= \frac{I}{4\pi\epsilon_0 c^2}\int_{-\Delta}^{\Delta}\frac{dx}{\sqrt{r^2+x^2}} \\ &= \frac{I}{4\pi\epsilon_0 c^2}\log\left[\frac{\sqrt{1+\frac{r^2}{\Delta^2}}+1}{\sqrt{1+\frac{r^2}{\Delta^2}}-1}\right] \approx \frac{I}{4\pi\epsilon_0 c^2}\left[-\log r^2+\log(4\Delta^2)\right] \end{aligned} \tag{276}$$

where we have taken the limit $\Delta \to \infty$. Notice that the expression has a logarithmic singularity in this limit. However, this is not of any physical consequence. The magnetic field can be computed from the curl of the vector potential, where this infinite constant drops out upon differentiation. We have:

$$B_i = \epsilon_{ijk}\partial_j A_k, \tag{277}$$

which gives for the components:

$$B_x = 0 \tag{278}$$

$$B_y = -\frac{I}{2\pi\epsilon_0 c^2}\frac{z}{r^2} \quad (279)$$

$$B_z = +\frac{I}{2\pi\epsilon_0 c^2}\frac{y}{r^2} \quad (280)$$

$$(281)$$

The magnitude of the magnetic field is

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \frac{I}{2\pi\epsilon_0 c^2}\frac{1}{r}, \quad (282)$$

which is in agreement with the result obtained from Ampere's law.

4.6 Magnetic dipole

Consider a steady current $\vec{J}$ circulating in a small region of space. We are interested in computing the vector potential and the magnetic field in a position $\vec{r}$ which is far from the current.

The vector potential is given by Eq. 268:

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{x}\frac{\vec{J}(\vec{x})}{|\vec{r}-\vec{x}|} \quad (283)$$

We could expand the inverse distance in the ratio $\frac{x}{r}$ by means of spherical harmonics. Since, here, we are interested only in the leading terms of the expansion we do it with a simpler method by means of a Taylor series:

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{\left[r^2 + x^2 - 2\vec{x}\cdot\vec{r}\right]^{\frac{1}{2}}} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{r}}{r^3} + \mathcal{O}\left(\frac{x^2}{r^3}\right) \quad (284)$$

The first term of the rhs in Eq. 284 gives a zero contribution to the vector potential of Eq. 283. Indeed, total derivatives of a localized current density integrated over all infinite volume should vanish (using the divergence theorem) since the current density vanishes at infinity. We therefore have:

$$\begin{aligned} 0 &= \int d^3\vec{x}\,\partial_i\left(x_k J_i\right) = \int d^3\vec{x}\,\left[\delta_{ik}J_i + x_k\partial_i J_i\right] = \int d^3\vec{x}\,J_k \\ &\rightsquigarrow \int d^3\vec{x}\,\vec{J} = 0. \end{aligned} \quad (285)$$

In the above, we have used that

$$\vec{\nabla}\cdot\vec{J} = \partial_i J_i = -\frac{\partial\rho}{\partial t} = 0.$$

Similarly, we can prove that:

$$\begin{aligned} 0 &= \int d^3\vec{x}\, \partial_i \left(x_k J_i x_l \right) \\ &\rightsquigarrow \int d^3\vec{x} \left[J_k x_l + J_l x_k \right] = 0. \end{aligned} \tag{286}$$

Now we are ready to calculate the contribution of the second term in the inverse distance expansion to the vector potential:

$$\begin{aligned} A_i &\approx \frac{1}{4\pi\epsilon_0 c^2\, r^3} \int d^3\vec{x}\; J_i x_j r_j \\ &= \frac{r_j}{4\pi\epsilon_0 c^2\, r^3} \int d^3\vec{x}\, \left[\frac{J_i x_j + J_j x_i}{2} + \frac{J_i x_j - J_j x_i}{2} \right] \\ &= \frac{1}{8\pi\epsilon_0 c^2\, r^3} \int d^3\vec{x}\; \left[J_i (x_j r_j) - (J_j r_j) x_i \right] \end{aligned} \tag{287}$$

In vector notation, we have

$$\vec{A} \approx \frac{1}{8\pi\epsilon_0 c^2\, r^3} \int d^3\vec{x}\; \left[\vec{J}(\vec{x}\cdot\vec{r}) - (\vec{J}\cdot\vec{r})\vec{x} \right] \tag{288}$$

The bracket in the integral is a double cross-product:

$$\vec{r} \times \left(\vec{x} \times \vec{J} \right) = \vec{x} \left(\vec{r} \cdot \vec{J} \right) - \vec{J} \left(\vec{r} \cdot \vec{x} \right). \tag{289}$$

Therefore,

$$\vec{A} = \frac{1}{4\pi\epsilon_0 c^2} \frac{\vec{m} \times \vec{r}}{r^3}, \tag{290}$$

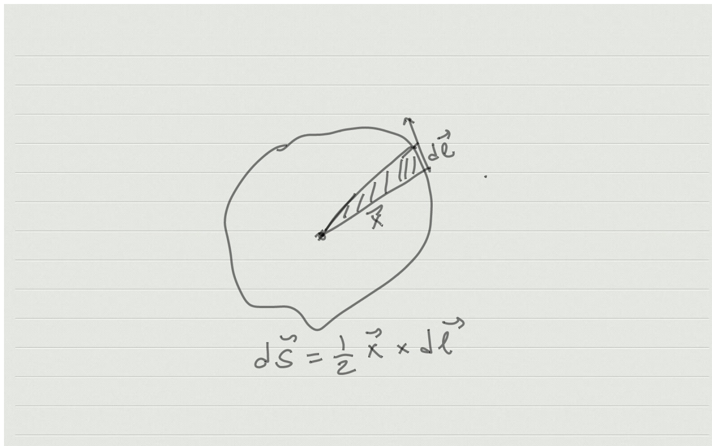
where the vector $\vec{m}$ depends on the geometric characteristics of the distribution and it is known as the magnetic moment:

$$\vec{\mu} = \frac{1}{2} \int d^3\vec{x}\, (\vec{x} \times \vec{J}). \tag{291}$$

As an example, consider the special case of a steady current circulating anti-clockwise in a wire which lays on a plane. The magnetic moment becomes:

$$\vec{\mu} = \oint \frac{I}{2} \vec{x} \times d\vec{l}, \tag{292}$$

where I is the intensity of the current. Notice that integrand is a surface element



$$d\vec{S} = \frac{1}{2}\vec{x} \times d\vec{l}. \tag{293}$$

Therefore, the magnetic moment is simply:

$$\vec{m} = I\vec{S}, \tag{294}$$

where

$$\vec{S} = \int d\vec{S},$$

the total area of the loop.

As a second example, consider the current created by a single charge q and mass M moving along a closed loop. The current density is:

$$\vec{J} = \rho\vec{v} = q\delta(\vec{x} - \vec{r}(t))\vec{v}. \tag{295}$$

where

$$\vec{v} = \frac{d\vec{r}}{dt}$$

the velocity of the charge. The magnetic moment is

$$\vec{m} = \frac{1}{2}\int d^3\vec{x}\,\vec{x} \times \vec{J} = \frac{q}{2}\vec{r} \times \vec{v} = \frac{q}{2M}\vec{L}, \tag{296}$$

where

$$\vec{L} = \vec{r} \times (M\vec{v}) \tag{297}$$

the angular momentum of the charge.

Having computed the vector potential of a "magnetic dipole", we can compute the magnetic field as:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{\vec{m} \times \vec{r}}{r^3} = -\vec{\nabla} \times \left[\vec{m} \times \vec{\nabla}\frac{1}{r}\right]. \tag{298}$$

Performing the differentiations and using the identity

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r}) = 0 \quad \forall\, r > 0, \tag{299}$$

we find:

$$\vec{B} = \left(\vec{m} \cdot \vec{\nabla}\right) \vec{\nabla}\frac{1}{r} = -\frac{\vec{m} - 3\hat{r}\,(\hat{r} \cdot \vec{m})}{r^3}. \tag{300}$$

4.6.1 Force on a "magnetic dipole" inside a magnetic field

Let's place a "magnetic dipole" inside a homogeneous magnetic field $\vec{B}$. The force acting on it is

$$\vec{F} = \int d^3\,\vec{x}\rho(\vec{x})\vec{v} \times \vec{B}(\vec{x}) = \int d^3\vec{x}\,\vec{J}(\vec{x}) \times \vec{B}(\vec{x}) \tag{301}$$

If the current is contained in a small region of space, we can assume that the magnetic field varies slowly within this region and approximate it with the first few terms of a Taylor expansion:

$$\vec{B}(\vec{x}) \approx \vec{B}(0) + \left(\vec{x} \cdot \vec{\nabla}\right) \vec{B}(0) + \dots \tag{302}$$

The first term vanishes upon integration. The second term yields (**exercise**)

$$\vec{F} \approx \nabla(\vec{m} \cdot \vec{B}). \tag{303}$$

The potential energy associated with this force is

$$\vec{F} = -\vec{\nabla}U, \tag{304}$$

where

$$U = -\vec{m} \cdot \vec{B}. \tag{305}$$

exercise: Prove that the torque is

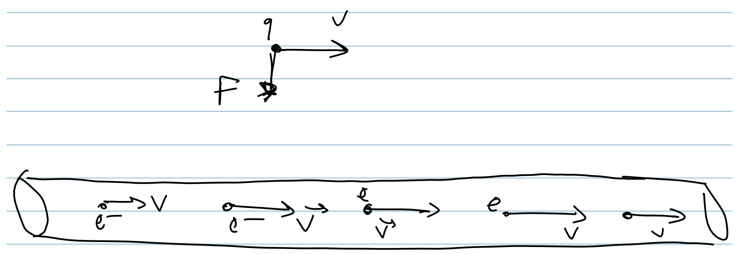
$$\tau = \int \vec{x} \times d\vec{F} = \vec{m} \times \vec{B}(0). \tag{306}$$

4.7 Relativity of the electric and magnetic field

Consider a wire on the $x-$axis producing a magnetic field

$$B = \frac{1}{2\pi\epsilon_0 c^2}\frac{I}{r} \tag{307}$$

at a distance r due to a current with an intensity I. Let's assume that the induction electrons inside the wire have a velocity $\vec{v}$. We now imagine another electron outside the wire to move parallel to the wire at a distance r from it with the same velocity $\vec{v}$.



The force acting on the electron outside the wire is

$$F = qvB = \frac{1}{2\pi\epsilon_0}\frac{qvI}{c^2 r}. \tag{308}$$

The intensity I of the current is:

$$I = S\rho_- v, \tag{309}$$

where S is the cross-section of the wire and ρ_- is the electron charge density inside the wire. The force is then:

$$F = \frac{qS}{2\pi\epsilon_0}\frac{\rho_-}{r}\frac{v^2}{c^2}. \tag{310}$$

We now change reference frame and choose one which moves with a relative velocity $\vec{v}$ along the wire. In the new frame the electron outside is static and has zero velocity:

$$v' = 0. \tag{311}$$

In this frame, the magnetic component of the force must vanish! This is a paradox. It appears that the charge is accelerated in one frame and it is free in another. The paradox is resolved due to Einstein's special relativity and the development of an equivalent electric force in the new frame.

In the original frame, the positive charge density inside the wire cancels the negative charge density of the free electrons:

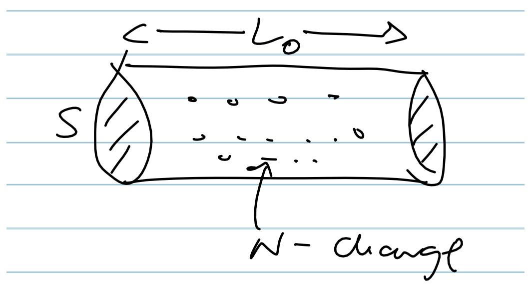
$$\rho = \rho_+ + \rho_- = 0. \tag{312}$$

The total charge density in the moving frame is

$$\rho' = \rho'_+ + \rho'_-. \tag{313}$$

It turns out that it is not zero!

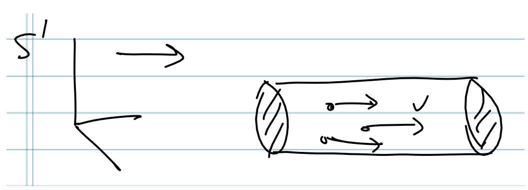
Consider N charges inside a volume $V = SL$ which are at rest in a certain reference frame.



The total charge is

$$qN = \rho_{rest} V = \rho_{rest} LS. \tag{314}$$

In a reference frame where the charges are



moving with velocity $\vec{v}$ the same number of charges is included in the same piece of material, which, however will appear contracted due to the laws of special relativity:

$$Nq = \rho_{moving} SL_{moving} = \rho_{moving} SL\sqrt{1 - \frac{v^2}{c^2}}. \tag{315}$$

We can therefore obtain a relation for charge densities in two different frames, where in one the charges appear at rest and a frame where the charges appear moving:

$$\rho_{moving} = \frac{\rho_{rest}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{316}$$

For the electron and positive charge densities inside the wire we obtain:

$$\rho'_+ = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} \tag{317}$$

and

$$\rho_- = \frac{\rho'_-}{\sqrt{1-\frac{v^2}{c^2}}} \rightsquigarrow \rho'_- = \rho_-\sqrt{1-\frac{v^2}{c^2}}. \tag{318}$$

Given that $\rho_+ = -\rho_-$ we have:

$$\rho' = -\rho_- \frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{319}$$

This charge distribution spreads over an infinitely long wire and creates an electric field:

$$E' = \frac{1}{2\pi\epsilon_0} S \frac{\rho'}{r}. \tag{320}$$

The force is:

$$F' = qE' = -\frac{qS}{2\pi\epsilon_0}\frac{\rho_-}{r}\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}} \tag{321}$$

Notice that these two forces cause the same physical effect in their respective reference frames. Both are vertical to the wire and point towards it. The momentum change in the vertical direction for the charge q is identical:

$$\Delta p_T = F\Delta t = F'\sqrt{1-\frac{v^2}{c^2}}\frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = F'\Delta t' = \Delta p'_T. \tag{322}$$

From this discussion, we conclude that the magnetic and electric fields can transform to each other by changing reference frame. In view of this observation it is desired to find a unified description of the two field as components of a unique electromagnetic field.

END of WEEK 4

5 Time varying electromagnetic fields

It is now time to consider the full Maxwell equations.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{323}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{324}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{325}$$

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t} \tag{326}$$

5.1 Charge conservation

These equations imply charge conservation. Taking the divergence of Eq. 326 we obtain:

$$c^2 \epsilon_0 \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial_t} \left(\epsilon_0 \vec{\nabla} \cdot \vec{E}\right) \rightsquigarrow 0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}. \tag{327}$$

which is the continuity equation derived from charge conservation. Integrating over all space,

$$0 = \int d^3\vec{x} \left[\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right] = \frac{\partial Q_{\text{universe}}}{\partial t} \tag{328}$$

we obtain that the total charge in the universe is a constant.

5.2 Vector and scalar potential

It is now time to find the properties of the scalar and vector potentials in the general case of time-dependent electromagnetic fields. The equations

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \vec{A} = -\frac{\vec{J}}{c^2 \epsilon_0} \tag{329}$$

are not general and they are correct only for static fields.

The equation $\vec{\nabla} \cdot \vec{B} = 0$ is generally valid and it can be satisfied by introducing a vector potential as before:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{330}$$

Maxwell Eq. 325 becomes

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} B \rightsquigarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0. \tag{331}$$

This is satisfied if we introduce a scalar potential

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi. \tag{332}$$

Indeed,

$$\vec{\nabla} \times \vec{\nabla} \phi = \epsilon_{ijk} \partial_j \partial_k \phi = 0. \tag{333}$$

Substituting Eq. 332 and Eq. 330 into the Maxwell equation 326, we obtain:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \vec{A} + \vec{\nabla} \left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right] = \frac{\vec{J}}{c^2 \epsilon_0}. \tag{334}$$

In this derivation, we have used the identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}. \tag{335}$$

With the same substitutions, Maxwell equation 323 becomes

$$\nabla^2 \phi - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \frac{\rho}{\epsilon_0}. \tag{336}$$

Equivalently,

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \phi - \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \frac{\rho}{\epsilon_0}. \tag{337}$$

The differential operator on the left of Eqs 334-337 is known as the D' Alembert or "box" operator:

$$\Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{338}$$

5.3 Gauge invariance

The electric and magnetic field remain invariant under a simultaneous tranformation, known as gauge transformations, of the scalar and vector potential:

$$\phi \to \phi' = \phi - \frac{\partial}{\partial t} f, \tag{339}$$

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f, \tag{340}$$

where

$$f = f(\vec{x}, t)$$

an arbitrary function of space-time. Indeed:

$$\begin{aligned} \vec{E}' &= -\partial_t \vec{A}' - \vec{\nabla}\phi' = -\partial_t \vec{A} - \partial_t \vec{\nabla} f - \vec{\nabla}\phi + \vec{\nabla}\partial_t f \\ &= -\partial_t \vec{A} - \vec{\nabla}\phi = \vec{E}. \end{aligned} \tag{341}$$

For the magnetic field,

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{A} = \vec{B}. \tag{342}$$

Equations 334-337 also remain invariant under gauge transformations. This can be verified easily observing that the combination:

$$\begin{aligned} & \vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\partial_t \phi' = \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla} \cdot f\right) + \frac{1}{c^2}\partial_t \left(\phi - \partial_t f\right) \\ = & \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t \phi - \Box f. \end{aligned} \tag{343}$$

Given our freedom to modify our vector and scalar potential by choosing the function $f(\vec{x}, t)$, we have an opportunity to make a choice which simplifies their differential equations. For example, we can choose a gauge f so that the potentials $\vec{A}_L, \phi_L$ in this gauge satisfy

$$\vec{\nabla} \cdot \vec{A}_L + \frac{1}{c^2}\partial_t \phi_L = 0. \tag{344}$$

This is called the "Lorentz gauge". The subscript $_L$ is to remind us that this equation is valid only for this specific gauge. Maxwell equations give for the potentials in the Lorentz gauge the differential equations:

$$\Box \phi_L = \frac{\rho}{\epsilon_0} \tag{345}$$

and

$$\Box \vec{A}_L = \frac{\vec{J}}{c^2 \epsilon_0}. \tag{346}$$

exercise: Show that the D'Alembert operator is not invariant under Galileian transformations of Newtonian mechanics but it is invariant under the Lorentz transformations of special relativity.

5.4 Electromagnetic waves in empty space

Maxwell equations admit non-zero solutions for the electric and magnetic field in places where there exist no currents and charges. This is of course not a surprise since already in magnetostatics and electrostatics we have found that the scalar and vector potentials extend over infinite distances from their sources. However, now we are set to find more exciting solutions of Maxwell equations which correspond to ripples of electromagnetic energy propagating through empty space with a constant speed, the speed of light.

In empty space and in the Lorentz gauge, Maxwell equations take the form:

$$\Box \vec{A} = \Box \phi = 0. \tag{347}$$

The electric and magnetic fields are given by:

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{348}$$

Acting on the above equations with the D'Alembert operator, we obtain:

$$\Box \vec{E} = \Box \vec{B} = 0. \tag{349}$$

In empty space, all fields satisfy the same equation:

$$\Box f = 0, \quad f \in \{\phi, \vec{A}, \vec{E}, \vec{B}\}. \tag{350}$$

A solution of this equation is:

$$f(\vec{x}, t) = f(\hat{\eta} \cdot \vec{x} - ct) \tag{351}$$

with

$$\hat{\eta}^2 = 1. \tag{352}$$

In this solution, f depends on a single combination

$$u = \hat{\eta} \cdot \vec{x} - ct. \tag{353}$$

We can verify easily that $f(u)$ is indeed a solution. We have

$$\partial_t f(u) = (\partial_t u)\partial_u f(u) = -c\partial_u f. \tag{354}$$

For the second time derivative, we find:

$$\partial_t^2 = c^2 \partial_u^2 f. \tag{355}$$

Similarly,

$$\vec{\nabla} f(u) = (\vec{\nabla} u)\partial_u f = \hat{\eta}\partial_u f, \tag{356}$$

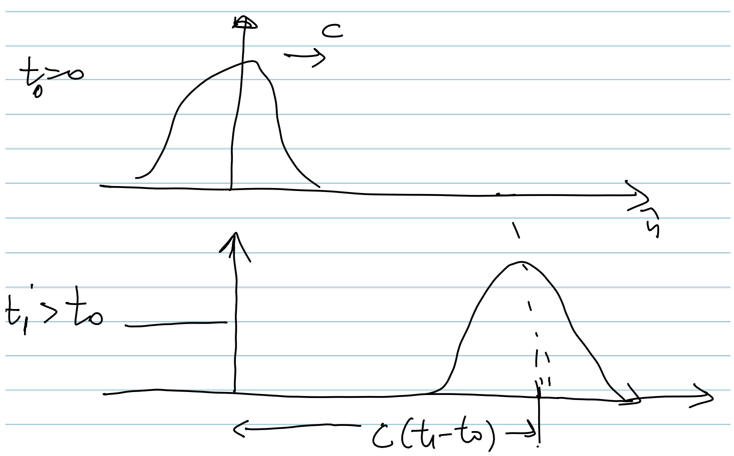
and

$$\nabla^2 f(u) = \hat{\eta}^2 \partial_u^2 f = \partial_u^2 f. \tag{357}$$

Therefore

$$\left[\frac{1}{c^2}\partial_t^2 - \nabla^2\right] f = \partial_u^2 f - \partial_u^2 f = 0.$$

The solution $f(\hat{\eta} \cdot \vec{x} - ct)$ is a travelling wave with a speed equal to the speed of light c, moving along the direction of the unit vector $\hat{\eta}$.



The greatest success of Maxwell's theory of electrodynamics has been to realize that electromagnetic energy (light) can propagate as electromagnetic waves.

For the electric and magnetic field, we write:

$$\vec{E} = \hat{e} E(\hat{\eta} \cdot \vec{x} - ct), \tag{358}$$

$$\vec{B} = \hat{b} B(\hat{\eta} \cdot \vec{x} - ct), \tag{359}$$

Then

$$0 = \vec{\nabla} \cdot \vec{E} = \partial_i \hat{e}_i E(\hat{\eta}_k x_k - ct) = \hat{e}_i \hat{\eta}_i \partial_u E = 0 \tag{360}$$

which leads to

$$\hat{e} \cdot \hat{\eta} = 0. \tag{361}$$

Similarly, from

$$\vec{\nabla} \cdot \vec{B} = 0$$

we conclude that also

$$\hat{b} \cdot \hat{\eta} = 0. \tag{362}$$

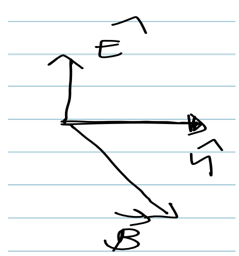
Therefore, the electric and magnetic field in an electromagnetic wave propagating in empty space are vectors which are transverse to the direction $\vec{\eta}$ of the propagation of the wave:

$$\vec{E}, \vec{B} \perp \hat{\eta}$$

We now turn to the remaining Maxwell equations:

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \rightsquigarrow (\hat{\eta} \times \vec{e})\, \partial_u E = c \partial_u B \hat{b}. \tag{363}$$

The magnetic field is therefore perpendicular to both the electric field and the direction of the wave propagation.



For the magnitudes of the electric and magnetic fields we have the equation:

$$\partial_u E = c \partial_u B \tag{364}$$

For dynamical fields (time-varying with no constant component) the electric and magnetic fields are proportional to each other

$$E = cB. \tag{365}$$

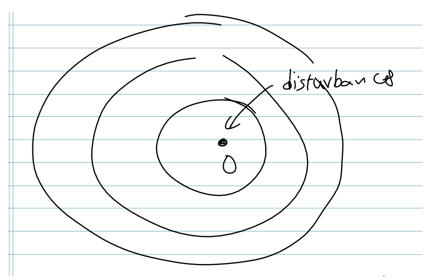
The last of Maxwell equations

$$c^2 \vec{\nabla} \times \vec{B} = \partial_t \vec{E} \tag{366}$$

yields an equivalent identity to Eq. 363.

5.4.1 Spherical waves

In the last section we found plane wave solutions for Maxwell equations in empty space. These are not the only wave solutions that there exist. One can find a diversity of solutions depending on the sources and boundary conditions that generate them, For example, if we are able to change the charge and current density at one point in the entire space only, we will generate an electromagnetic wave with spherical symmetry, i.e. no preferred direction.



Maxwell equations give

$$\Box f = 0, \quad \forall f \in \{\vec{A}, \vec{B}, \vec{E}\}. \tag{367}$$

A wave solution with spherical symmetry is

$$f(\vec{r}, t) = f(r, t) \tag{368}$$

and is is independent of the azimuthal and polar angles ϕ, θ. In spherical coordinates, we have

$$\Box f(r, t) = \frac{1}{c^2} \partial_t^2 f - \frac{1}{r} \partial_r^2 (rf) = 0, \tag{369}$$

which gives:

$$\left[\frac{1}{c^2}\partial_t^2 - \partial_r^2\right](rf) = 0. \tag{370}$$

A solution of this equation is:

$$rf = A(r - ct) + B(r + ct) \tag{371}$$

and, equivalently,

$$f(r,t) = \frac{A(r - ct)}{r} + \frac{B(r + ct)}{r}. \tag{372}$$

The first term corresponds to a spherical wave which propagates outwards. The second term is a spherical wave that propagates inwards. It is very hard to realize boundary conditions where a wave can surge at a periphery simultaneously at a time t_0 and allow this disturbance to propagate towards a center. Typically, outwards propagation is realistic, where a disturbance takes place at t_0 at one and only point.

6 General solutions of Maxwell equations with sources

We will now develop tools for finding solutions of Maxwell equations in the presence of sources $(\vec{J}, \rho \neq 0)$. We have seen that in Lorentz gauge Maxwell equations for the scalar and vector potentials become:

$$\Box\phi = \frac{\rho}{\epsilon_0}, \quad \Box\vec{A} = \frac{\vec{J}}{c^2\epsilon_0}. \tag{373}$$

6.1 Green's functions

To solve the above differential equations, we will use the technique of Green's functions. We seek functions

$$G\left(\vec{r}, t; \vec{r}', t'\right) \tag{374}$$

which satisfy:

$$\Box_{\vec{r},t} G(\vec{r}, t; \vec{r}', t') = \delta(\vec{r} - \vec{r}')\delta(t - t'). \tag{375}$$

Then a solution of Maxwell's equation for the scalar potential is:

$$\phi(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{r}' dt' G(\vec{r}, t; \vec{r}', t') \frac{\rho(\vec{r}', t')}{\epsilon_0} \tag{376}$$

This is easy to verify by acting with the D'Alembert operator on both sides of Eq. 376. To the above expression, we can add solutions $\phi_{\text{free}}(\vec{r}, t)$ in free space, as the ones we have found in the last chapter for electromagnetic waves, which satisfy

$$\Box \phi_{\text{free}}(\vec{r}, t) = 0 \tag{377}$$

Our general solution is then:

$$\phi(\vec{r}, t) = \phi_{\text{free}}(\vec{r}, t) + \int_{-\infty}^{\infty} d^3\vec{r}' dt' G(\vec{r}, t; \vec{r}', t') \frac{\rho(\vec{r}', t')}{\epsilon_0}. \tag{378}$$

The solution is fixed by boundary conditions.

Equivalently, the general solution for the vector potential is

$$\vec{A}(\vec{r}, t) = \vec{A}_{\text{free}}(\vec{r}, t) + \int_{-\infty}^{\infty} d^3\vec{r}' dt' G(\vec{r}, t; \vec{r}', t') \frac{\vec{J}(\vec{r}', t')}{c^2 \epsilon_0}. \tag{379}$$

END WEEK 5

6.2 Fourier transformations

In this section, we would like to recall the salient properties of Fourier transformations. As an exercise, please prove that

$$\int_{-\infty}^{\infty} dw e^{-iwt} = 2\pi\delta(t). \tag{380}$$

(Hint: Use that

$$\partial_t \Theta(t) = \delta(t)$$

and that the integral

$$-\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dw \frac{e^{-iwt}}{w + i\epsilon}$$

is a representation of the heavyside Θ function.) For a general smooth function $f(t)$ we have that

$$f(t) = \int_{-\infty}^{\infty} dt' \delta(t - t') f(t') = \int_{-\infty}^{\infty} dt' \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik(t-t')} f(t') \tag{381}$$

which leads to

$$f(t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{-ikt} \cdot \int_{-\infty}^{\infty} \frac{dt'}{\sqrt{2\pi}} e^{ikt'} f(t') \tag{382}$$

We define the function:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dt'}{\sqrt{2\pi}} e^{ikt'} f(t') \tag{383}$$

Then, we have

$$\tilde{f}(t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{-ikt} \tilde{f}(k) \tag{384}$$

Eqs 383-384 define the Fourier transform of the function f and its inverse. It is often beneficial to work with $\tilde{f}(k)$ rather that $f(t)$ in solving differential equations. We will make a good use of Fourier transforms in finding Green's functions.

6.3 Fourier transformations and Green's functions

We seek Green's functions which are

$$G(\vec{x}, \vec{x}'; t, t') = G(\vec{x} - \vec{x}', t - t'), \tag{385}$$

anticipating that they depend on space-time differences rather than values of absolute positions and time. Writing $\vec{x} - \vec{x}' = \Delta\vec{x}$ and $t - t' = \Delta t$, the Green's function must satisfy:

$$\Box G(\Delta\vec{x}, \Delta t) = \delta(\Delta\vec{x})\delta(\Delta t). \tag{386}$$

We substitute in the above the Fourier transform:

$$G(\Delta\vec{x}, \Delta t) = \int \frac{d^3\vec{k} dE}{(2\pi)^4} e^{-i\left(Ec\Delta t - \vec{k}\cdot\Delta\vec{x}\right)} \tilde{G}(\vec{k}, E). \tag{387}$$

We then find:

$$\int \frac{d^3\vec{k} dE}{(2\pi)^4} e^{-i\left(Ect - \vec{k}\cdot\vec{x}\right)} \left[-E^2 + \vec{k}^2\right] \tilde{G}(\vec{k}, E) = c \int \frac{d^3\vec{k} dE}{(2\pi)^4} e^{-i\left(Ect - \vec{k}\cdot\vec{x}\right)} \tag{388}$$

We then find for the Fourier transform of the Green's function:

$$\tilde{G}(\vec{k}, E) = -\frac{c}{E^2 - \vec{k}^2}. \tag{389}$$

This solution is a bit naive and it does not impose any physical boundary condition to the Green's function. In physical situation, we require that the effect to the vector and scalar potentials $\phi(\vec{x}, t), \vec{A}(\vec{x}, t)$ from the sources $\rho(\vec{x}', t'), \vec{j}(\vec{x}', t')$ takes place for times $t > t'$. In other words, if we switch on a source at a given prime t' we expect physically that this source will cause an effect to the potentials only at a later time and not earlier. Physical Green's function must then satisfy the boundary condition

$$G(\Delta\vec{x}, \Delta t) = 0, \quad \text{for } \Delta t < 0. \tag{390}$$

It is possible to implement the above by modifying the Fourier transformation of the Green's function. We write

$$G(\Delta\vec{x}, \Delta t) = \int \frac{d^3\vec{k} dE}{(2\pi)^4} e^{-i\left((E+i\delta)c\Delta t - \vec{k}\cdot\Delta\vec{x}\right)} \tilde{G}(\vec{k}, E) \tag{391}$$

where δ is a infinitesimally small positive variable. This Fourier transformation differs from the one of Eq. 387 in that the "energy" E integration variable has a small positive imaginary part in the exponent. For $\Delta t \to -\infty$ the Green's function vanishes:

$$G(\Delta\vec{x}, \Delta t) \propto e^{\Delta t \delta} \to e^{-\infty\delta} \to 0, \tag{392}$$

which is in accordance with the physical requirement for the Green's function to vanish before the sources are turned on. Up to corrections of $\mathcal{O}(\delta)$, we now find for the Fourier transform of the Green's function:

$$\tilde{G}(\vec{k}, E) = \frac{-c}{(E+i\delta)^2 - \vec{k}^2} = \frac{c}{2k}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right] \tag{393}$$

where $k \equiv \left|\vec{k}\right|$. We can now compute the Green's function as

$$G(\Delta\vec{x}, \Delta t) = \frac{c}{2(2\pi)^4}\int \frac{d^3\vec{k}dE}{k} e^{-i\left((E+i\delta)c\Delta t - \vec{k}\cdot\Delta\vec{x}\right)}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right] \tag{394}$$

To perform the integrations we use spherical coordinates:

$$\begin{aligned} d^3\vec{k} &= k^2 dk d\phi d\cos\theta \\ \vec{k}\cdot\Delta\vec{x} &= k\Delta x\cos\theta. \end{aligned} \tag{395}$$

After the angular ϕ, θ integrations we arrive at:

$$\begin{aligned} G(\Delta\vec{x}, \Delta t) &= \frac{ic}{2(2\pi)^3\,\Delta x}\int dk\,\left[e^{-ik\Delta x} - e^{+ik\Delta x}\right] \\ &\quad \int dE\; e^{-i(E+i\delta)c\Delta t}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right]. \end{aligned} \tag{396}$$

We will do the "energy" dE integration using Cauchy's theorem. First, we promote E to a complex variable:

$$E = E_r + iE_i.$$

The integration takes place over the horizontal axis of the (E_r, E_i) plane. The time dependent phase factor behaves as:

$$e^{-ic\Delta t(E_r + iE_i)} = e^{-ic\Delta t E_r} e^{E_i c\Delta t} \to 0 \quad \text{for} \begin{cases} \Delta t < 0, E_i \to +\infty, \\ \quad\text{or} \\ \Delta t > 0, E_i \to -\infty. \end{cases} \tag{397}$$

Therefore, we can close the contour of integration on the upper (E_r, E_i) half-plane for $\Delta t < 0$ and on the lower half-plane for $\Delta t > 0$. The upper half-plane contains no poles and therefore the integral is zero for negative Δt. We then have

$$G(\Delta\vec{x}, \Delta t) = 0, \quad \text{if } \Delta t < 0. \tag{398}$$

This is in agreement with what we have required for our Green's function, i.e. to vanish for times earlier than the times for which the sources are turned on. The integral for $\Delta t > 0$ does not vanish, since the lower half-plane contains two poles at $E = \pm k - i\delta$. With residues theorem, we obtain:

$$\begin{aligned} G(\Delta\vec{x}, \Delta t > 0) &= \frac{c}{2(2\pi)^2 \Delta x} \int_0^\infty dk \left\{ e^{-ik(c\Delta t - \Delta x)} + e^{+ik(c\Delta t - \Delta x)} \right. \\ &\quad \left. - e^{-ik(c\Delta t + \Delta x)} - e^{+ik(c\Delta t + \Delta x)} \right\} \\ &= \frac{c}{2(2\pi)^2 \Delta x} \int_{-\infty}^\infty dk \left\{ e^{-ik(c\Delta t - \Delta x)} - e^{+ik(c\Delta t + \Delta x)} \right\} \end{aligned} \tag{399}$$

Finally, the last integral yields $\delta-$functions, according to Eq. 380. We then find

$$G(\Delta\vec{x}, \Delta t > 0) = \frac{c}{4\pi \left|\vec{x} - \vec{x}'\right|} \left[\delta(c\Delta t - \Delta x)) - \delta(c\Delta t + \Delta x)) \right] \tag{400}$$

Since Δt and Δx are both positive, the argument of the second delta function never vanishes,

$$\delta(c\Delta t + \Delta x)) = 0, \quad \text{for } \Delta x, \Delta t > 0. \tag{401}$$

Thus the Green's function becomes

$$G(\vec{x} - \vec{x}', t - t') = \frac{1}{4\pi \left|\vec{x} - \vec{x}'\right|} \delta\left(t - t' - \frac{\left|\vec{x} - \vec{x}'\right|}{c} \right) \Theta(t > t'), \tag{402}$$

where the Θ-function is $+1$ for $t > t'$ and 0 otherwise.

Substituting into Eq. 378 and assuming no prior electromagnetic fields $\phi_{\text{free}}, \vec{A}_{\text{free}}$ in the absence of sources, we find

$$\phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' dt' \rho(\vec{x}', t') \frac{\delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c} \right)}{\left|\vec{x} - \vec{x}'\right|} \Theta(t > t') \tag{403}$$

Performing the t' integration yields:

$$\phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{\rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} \tag{404}$$

We observe that for time-independent charge densities:

$$\rho(\vec{x}', \vec{t'}) = \rho(\vec{x}')$$

Eq. 404 reproduces our known result from electrostatics. In the limit of an infinite speed of light $c \to \infty$, the potential becomes once again the known potential from electrostatics:

$$\lim_{c\to\infty} \phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{\rho\left(\vec{x}', t\right)}{|\vec{x} - \vec{x}'|}. \tag{405}$$

If c were infinite, we would need simply Coulomb's law in order to compute the potential $\phi(\vec{x}, t)$ at a time t, integrating over the contributions of the charge density $\rho(\vec{x}, t)$ as it appears *at the same time* t. For a finite value of c, the potential $\Phi(\vec{x}, t)$ is due to the charge density $\rho(\vec{x}', t')$ as it appears at an earlier time $t - \frac{|\vec{x}-\vec{x}'|}{c}$. We interpret the time delay (retardation)

$$\delta t = \frac{|\vec{x} - \vec{x}'|}{c}$$

as the time required for the action of the source to propagate with the speed of light, in the form of an electromagnetic wave, up to the point $\vec{x}$ where we measure the potential.

Similarly, we can calculate the vector potential $\vec{A}(\vec{x}, t)$. We find:

$$\vec{A}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{x}' \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} \tag{406}$$

The Green's function of Eq. 402 is known as the *retarded Green's function* due to its property to account for the retardation due to the propagation of the electromagnetic signals from one point to another with a finite speed. It is useful to derive an alternative form for the Green's function. We first prove that for real a, b

$$\begin{aligned} & \delta(a^2 - |b|^2)\Theta(a > 0) = \delta((a - |b|)(a + |b|))\Theta(a > 0) \\ = & \frac{1}{2|b|} \left[\delta(a - |b|) + \delta(a + |b|)\right] \Theta(a > 0) \\ = & \frac{1}{2|b|} \delta(a - |b|)\Theta(a > 0) \end{aligned} \tag{407}$$

Applying the above for $a = t - t'$ and $b = \frac{|\vec{x}-\vec{x}'|}{c}$ we can cast the retarded Green's function as:

$$G(\vec{x}-\vec{x}', t-t') = \frac{1}{2\pi}\delta\left((t-t')^2 - \frac{|\vec{x}-\vec{x}'|^2}{c^2}\right)\Theta(t>t'). \qquad (408)$$

6.4 Potential of a moving charge with a constant velocity

As a first application of Eqs 404-406 we will compute the scalar and vector potential of a point-like charge q, moving with a velocity $\vec{v}$. The charge density corresponding to the moving charge is

$$\rho(\vec{x}', t') = q\delta(\vec{x}' - \vec{v}t'). \qquad (409)$$

Using the above charge density and the retarded Green's function in the form of Eq. 408 we obtain for the scalar potential:

$$\begin{aligned}\Phi(\vec{x}, t) &= \frac{q}{2\pi\epsilon_0}\int d^3\vec{x}' dt' \delta(\vec{x}' - \vec{v}t')\delta\left((t-t')^2 - \frac{|\vec{x}-\vec{x}'|^2}{c^2}\right)\Theta(t>t') \\ &= \frac{q}{2\pi\epsilon_0}\int dt' \delta\left((t-t')^2 - \frac{|\vec{x}-\vec{v}t'|^2}{c^2}\right)\Theta(t>t'). \qquad (410)\end{aligned}$$

We now need to find the zeros of the argument of the delta function. We can write:

$$0 = (t-t')^2 - \frac{|\vec{x}-\vec{v}t'|^2}{c^2} = t'^2 - 2t't + t^2 - \frac{(x_\parallel - vt')^2 + x_\perp^2}{c^2} \qquad (411)$$

where we have decomposed

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp \qquad (412)$$

into its parallel and perpendicular components to the velocity $\vec{v}$. We now define the "boosted" variables:

$$x_b = \gamma\left(x_\parallel - vt\right), \quad t_b = \gamma\left(t - \frac{x_\parallel v}{c^2}\right), \qquad (413)$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (414)$$

and we introduce the quantity

$$\tau^2 = c^2t^2 - (x_\parallel^2 + x_\perp^2) = c^2t_b^2 - (x_b^2 + x_\perp^2). \tag{415}$$

Eq. 411 can then be cast in the form:

$$\frac{t'^2}{\gamma^2} - 2\frac{t'}{\gamma} + \frac{\tau^2}{c^2} = 0. \tag{416}$$

The discriminant is

$$\Delta = \frac{4(t_b^2 - \tau^2/c^2)}{\gamma^2} = \frac{4r_b^2}{\gamma^2 c^2}, \tag{417}$$

where we have defined the "boosted" distance

$$r_b^2 = x_b^2 + x_\perp^2. \tag{418}$$

The solutions are:

$$t_\pm = \gamma\left(t_b \pm \frac{r_b}{c}\right). \tag{419}$$

The delta function becomes:

$$\delta\left((t-t')^2 - \frac{|\vec{x} - \vec{v}t'|^2}{c^2}\right)\Theta(t > t') = \frac{c\gamma}{2r_b}\Theta(t > t')\delta(t' - t_-). \tag{420}$$

It is now trivial to perform the t' integration in Eq. 410, obtaining:

$$\Phi(\vec{x}, t) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{r_b}. \tag{421}$$

Explicitly,

$$\Phi(\vec{x}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{1}{\left[\left(\frac{x_\parallel - vt}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + x_\perp^2\right]^{\frac{1}{2}}}. \tag{422}$$

Following the same steps, we find that the vector potential of a moving charge is (**exercise**):

$$\vec{A}(\vec{x}, t) = \frac{\vec{v}}{c^2}\Phi(\vec{x}, t). \tag{423}$$

In this derivation of the scalar and vector potential for a moving charge, we see the "rise" of special relativity. In a reference frame where the charge is at rest, the scalar and vector potentials are:

$$\Phi(\vec{x}, t)\big|_{\text{rest}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left[x_\parallel^2 + x_\perp^2\right]^{\frac{1}{2}}}, \quad \vec{A}(\vec{x}, t)\Big|_{\text{rest}} = 0. \tag{424}$$

Comparing with the corresponding expressions in a frame where the charge is moving, we obtain an excellent hint that space co-ordinates transform under boosts according to Lorentz transformations,

$$x_\parallel \to \gamma\left(x_\parallel - vt\right), \quad x_\perp \to x_\perp, \tag{425}$$

and not Galileian transformations.

How about the emergence of the factor γ and of a non-vanishing vector potential in the frame where the charge is moving? These are hints that the vector and a scalar potential are components of a single relativistic "four-vector":

$$\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} \equiv A^\mu.$$

7 Special Relativity

Special relativity is based on the assumption that the laws of nature are the same for inertial observers where their co-ordinates are related via Lorentz transformations:

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + \rho^\mu. \tag{426}$$

where

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{427}$$

a "four-vector" comprising space-time coordinates with $x^0 = ct$, ρ^μ is a constant four-vector and Λ^μ_ν satisfies:

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}. \tag{428}$$

The 4×4 matrix $g_{\mu\nu}$ is the so-called *metric*, defined as:

$$g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3 \\ 0, & \mu \neq \nu \end{cases} \tag{429}$$

In the above we have used Einstein's summation convention. For example, one would write explicitly

$$\Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_0 x^0 + \Lambda^\mu{}_1 x^1 + \Lambda^\mu{}_2 x^2 + \Lambda^\mu{}_3 x^3. \tag{430}$$

This is a convention that we will use extensively from now on.

7.1 Proper time

Lorentz transformations leave invariant "proper-time" intervals. These are defined as:

$$d\tau^2 \equiv c^2 dt^2 - d\vec{x}^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{431}$$

Indeed, in a different reference frame we have from Eq. 426:

$$dx^{\mu\prime} = \Lambda^\mu_{\ \nu} dx^\nu. \tag{432}$$

A proper-time interval in the new frame is

$$\begin{aligned} d\tau'^2 &= g_{\mu\nu} dx'^\mu dx'^\nu \\ &= g_{\mu\nu} \left(\Lambda^\mu_{\ \rho} dx^\rho\right) \left(\Lambda^\nu_{\ \sigma} dx^\sigma\right) \\ &= \left(g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma}\right) dx^\rho dx^\sigma \\ &= g_{\rho\sigma} dx^\rho dx^\sigma = d\tau^2. \end{aligned} \tag{433}$$

As a consequence of the invariance of proper-time intervals, *the speed of light is the same in all inertial frames.* Indeed, for light we have:

$$\left|\frac{d\vec{x}}{dt}\right| = c \rightsquigarrow d\tau^2 = c^2 dt^2 - d\vec{x}^2 = 0 \tag{434}$$

In a new frame,

$$d\tau^{2\prime} = d\tau^2 = 0 \rightsquigarrow \left|\frac{d\vec{x}'}{dt'}\right| = c \tag{435}$$

Lorentz transformations are the only non-singular transformations which preserve proper-time intervals:

$$\begin{aligned} d\tau^2 &= d\tau'^2 \\ \rightsquigarrow g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} dx'^\mu dx'^\nu \\ \rightsquigarrow g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma} dx^\rho dx^\sigma, \end{aligned} \tag{436}$$

concluding that:

$$g_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma}. \tag{437}$$

Differentiating with dx^ϵ, we obtain:

$$0 = g_{\mu\nu} \left[\frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} + \frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho} \right]. \tag{438}$$

To this, we add the same equation with $\epsilon \leftrightarrow \rho$ and subtract the same equation with $\epsilon \leftrightarrow \sigma$. We obtain:

$$0 = 2g_{\mu\nu} \frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \tag{439}$$

Assuming that the transformation $x^\mu \to x'^\mu$ is a well behaved differentiable function and that the inverse of the transformation also exists,

$$\frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\nu} = \delta_{\mu\nu}, \tag{440}$$

we obtain that

$$\frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} = 0. \tag{441}$$

Therefore, the transformation $x^\mu \to x'^\mu$ ought to be linear:

$$x'^\mu = \Lambda^\mu \ \nu + \rho^\mu. \tag{442}$$

END OF WEEK 6

7.2 Subgroups of Lorentz transformations

The set of all Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + \rho^{\mu}. \quad g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma} \tag{443}$$

form a group (**exercise:** prove it), which is known as the group of *inhomogeneous Lorentz group* or the Poincare' group. The subset of transformations with $\rho^{\mu} = 0$ is known as the *homogeneous Lorentz group*.

From

$$g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have:

$$(\Lambda^{0}_{\ 0})^2 - \sum_{i=1}^{3}(\Lambda^{i}_{\ 0})^2 = 1 \rightsquigarrow (\Lambda^{0}_{\ 0})^2 \geq 1. \tag{444}$$

Also, in matrix form the definition of the Lorentz transformation becomes:

$$g = \Lambda^T g \Lambda \rightsquigarrow \det g = \det(\Lambda^T g \Lambda) \rightsquigarrow (\det\Lambda)^2 = 1. \tag{445}$$

The subgroup of transformations with

$$\det\Lambda = 1, \ \Lambda^{0}_{\ 0} \geq 1,$$

which contains the unity $\mathbf{1} = \delta^{\mu}_{\nu}$, is known as the proper group of Lorentz transformations. All other transformations are known as *improper Lorentz transformations*. It is impossible with a continuous change of parameters to change

$$\det\Lambda = 1 \to \det\Lambda = -1 \text{ or } \Lambda^{0}_{\ 0} \geq 1 \to \Lambda^{0}_{\ 0} \leq -1.$$

Improper Lorentz transformations involve either space-reflection ($\det\Lambda = -1, \Lambda^{0}_{\ 0} \geq 1$) or time-inversion ($\det\Lambda = 1, \Lambda^{0}_{\ 0} \leq -1$) or both ($\det\Lambda = -1, \Lambda^{0}_{\ 0} \leq -1$).

Proper homogeneous or inhomogeneous Lorentz transformations have a further subgroup: the group of rotations,

$$\Lambda^{0}_{\ 0} = 1, \ \Lambda^{i}_{\ 0} = \Lambda^{0}_{\ i} = 0, \ \Lambda^{i}_{\ j} = R_{ij}, \tag{446}$$

with

$$\det R = 1, \ R^T R = 1. \tag{447}$$

Thus, for rotations and translations ($x^\mu \to x'^\mu = x^\mu + \rho^\mu$) Lorentz transformations are no different than Galilei transformations.

A difference with Galilei transformations arises in *boosts*. Assume a reference frame O in which a certain particle appears at rest, and O' a reference frame where the particle appears to move with a velocity $\vec{v}$. Space-time intervals in the two frames are related via

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu = \Lambda^\mu{}_0 cdt, \tag{448}$$

given that $d\vec{x} = 0$ in the frame O. For $\mu = 0$, this equation gives

$$dt' = \Lambda^0{}_0 dt. \tag{449}$$

For $\mu = i = 1, 2, 3$ we have:

$$dx'^i = \Lambda^i{}_0 cdt \tag{450}$$

Dividing the two, we have

$$v^i \equiv \frac{dx'^i}{dt'} = c\frac{\Lambda^i{}_0}{\Lambda^0{}_0} \rightsquigarrow \Lambda^i{}_0 = \frac{v^i}{c}\Lambda^0{}_0. \tag{451}$$

From

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have:

$$\begin{aligned} & (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1 \\ \rightsquigarrow \quad & \Lambda^0{}_0 = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}. \end{aligned} \tag{452}$$

and thus

$$\Lambda^i{}_0 = \gamma\frac{v^i}{c}. \tag{453}$$

The remaining components are not determined uniquely by knowing the velocity $\vec{v}$ of the particle. Indeed, two Lorentz transformations

$$\Lambda^\mu{}_\nu \quad \text{and} \quad \Lambda^\mu{}_\rho R^\rho{}_\nu$$

where R is a rotation, boost a particle to the same velocity. For coordinate systems O and O' with parallel axes we find that (**exercise**)

$$\Lambda^i{}_j = \delta^i_j + \frac{v^i v^j}{v^2}(\gamma - 1) \tag{454}$$

and

$$\Lambda^0{}_j = \gamma\frac{v^j}{c}. \tag{455}$$

7.3 Time dilation

Consider an inertial observer O which looks as a clock at rest. Two ticks of the clock correspond to a space-time interval

$$d\vec{x} = 0, \quad dt = \Delta t. \tag{456}$$

The proper time interval is

$$d\tau = (c^2 dt^2 - d\vec{x}^2)^{\frac{1}{2}} = c\Delta t. \tag{457}$$

A second observer sees the clock with velocity $\vec{v}$. Two ticks of the clock define a space-time interval

$$dt' = \Delta t', \quad d\vec{x}' = \vec{v} dt'. \tag{458}$$

The proper-time interval in the new frame is:

$$d\tau' = (c^2 dt'^2 - d\vec{x}'^2)^{\frac{1}{2}} = c\Delta t' \sqrt{1 - \left| \frac{d\vec{x}'}{cdt'} \right|^2} = c\Delta t' \sqrt{1 - \frac{\vec{v}^2}{c^2}}. \tag{459}$$

The proper-time is invariant under the change of inertial reference frames. Thus we conclude that

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \gamma \Delta t \tag{460}$$

7.4 Doppler effect

Take our clock to be a source of light with a frequency

$$\omega = \frac{2\pi}{\Delta t}.$$

For an observer where the light-source is moving with velocity $\vec{v}$ this time interval is measured to be

$$dt' = \gamma \Delta t.$$

In the same period, the distance of the observer from the light source increases by

$$v_r dt'$$

where v_r is the component of the velocity of the light-source along the direction of sight of the observer. The time elapsing between the reception of two successive light wave-fronts from the observer is

$$cdt_0 = cdt' + v_r dt'. \tag{461}$$

The frequency measured by the observer is

$$\omega' = \frac{2\pi}{dt_0} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v_r}{c}}\omega. \tag{462}$$

If the light-source is moving along the line of sight, $v_r = v$, we have

$$\omega' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\omega. \tag{463}$$

If the light-source moves away from the observer, $v_r > 0$, the frequency decreases and the light appears to be more red (red shift). If the source moves towards the observer, the frequency increases (violet shift).
Exercise: Calculate the angle of the direction of motion of the light-source with respect to the line of sight of the observer for which there is no shift in the frequency.

For an application of the Doppler effect in cosmology, read about Hubble's law.

7.5 Particle dynamics

How can we compute the force of a particle which moves with a relativistic velocity $\vec{v}$? We should expect that our classical formulae from Newtonian mechanics need to be modified. Nevertheless, Newtonian expressions for the force should be valid if a particle is at rest. We can always change reference frame with Lorentz transformations to bring a particle at rest and calculate the change in its velocity for a small time interval using Newtonian mechanics. However, we will need to perform these changes of reference frame at every small increase of the velocity of the particle during its acceleration due to the force.

In a more elegant solution to the problem, we define a relativistic force acting on a particle as

$$f^\mu = mc^2 \frac{d^2 x^\mu}{d\tau^2}, \tag{464}$$

where m is the mass of a particle [2]. If the particle is at rest, the proper-time interval $d\tau$ coincides with the common time-interval dt

$$d\tau = cdt.$$

Therefore, in the rest frame of the particle, the"space"-components of the force four-vector become

$$f^i_{\text{rest}} = m\frac{d^2x^i}{dt^2} = F^i_{\text{Newton}}, \quad \text{for } i = 1, 23, \tag{465}$$

where $\vec{F}_{\text{Newton}}$ is the force-vector as we know it from Newtonian mechanics. The "time" component of the force four-vector vanishes:

$$f^0_{rest} = mc\frac{d^2t}{dt^2} = 0. \tag{466}$$

Under a Lorentz transformation, f^μ transforms as

$$f^\mu = mc^2\frac{d^2x^\mu}{d\tau^2} \to mc^2\frac{d^2x'^\mu}{d\tau^2} = mc^2\frac{d^2\left(\Lambda^\mu_{\ \nu}x^\nu + \rho^\mu\right)}{d\tau^2} = \Lambda^\mu_{\ \nu}\, mc^2\frac{d^2x^\nu}{d\tau^2} \tag{467}$$

Therefore,

$$f'^\mu = \Lambda^\mu_{\ \nu}f^\nu. \tag{468}$$

The components of f^μ transform under Lorentz transformations in exactly the same way as the components of space-time coordinates. It is therefore a four-vector as well.

For a specific transformation from the rest frame of a particle to a frame where the particle moves with a velocity $\vec{v}$, we have

$$f^\mu = \Lambda^\mu_{\ \nu}(\vec{v})f^\nu_{\text{rest}}. \tag{469}$$

where, we have found that,

$$\Lambda^0_{\ 0}(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda^i_{\ 0}(\vec{v}) = \Lambda^0_{\ i}(\vec{v}) = \gamma\frac{v^i}{c}.$$
$$\Lambda^i_{\ j}(\vec{v}) = \delta^i_j + \frac{v^iv^j}{v^2}(\gamma - 1) \tag{470}$$

[2] With mass, we mean the mass of a particle as it is measured in its rest-frame. We will refrain from using the "relativistic", velocity dependent, mass.

Therefore, the force on a moving particle is:

$$\vec{f} = \vec{F}_{\text{Newton}} + (\gamma - 1)\frac{\vec{v}\left(\vec{F}_{\text{Newton}} \cdot \vec{v}\right)}{v^2}, \tag{471}$$

and

$$f^0 = \gamma \frac{\vec{v} \cdot \vec{F}_{\text{Newton}}}{c} = \frac{\vec{v}}{c} \cdot \vec{f}. \tag{472}$$

In Newtonian mechanics, if the force $\vec{F}$ is given, we can compute the trajectory $\vec{x}(t)$ by solving the second order differential equation:

$$\frac{d^2\vec{x}}{dt^2} = \frac{\vec{F}(\vec{x}, t)}{m}. \tag{473}$$

Similarly, in special relativity, when the relativistic force f^μ is known, the differential equation 464 can, in principle, be solved to give the space-time coordinates as a function of the proper time τ:

$$x^\mu = x^\mu(\tau). \tag{474}$$

To calculate the trajectory, we then need to calculate the proper-time in terms of the time coordinate by inverting

$$x^0 = x^0(\tau) \rightsquigarrow \tau = \tau(x^0), \tag{475}$$

which we can use to cast the space components directly as functions of the time-coordinate.

We should not forget a second constrain that must be satisfied for our solutions $x^\mu(\tau)$, namely

$$\Omega \equiv g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1. \tag{476}$$

We have for the derivative of Ω with respect to proper-time:

$$\frac{d\Omega}{d\tau} = 2 g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \frac{dx^\nu}{d\tau} = \frac{2}{mc^2} g_{\mu\nu} f^\mu \frac{dx^\nu}{d\tau}. \tag{477}$$

The rhs is a Lorentz invariant quantity. In a new frame,

$$g_{\mu\nu} f'^\mu \frac{dx'^\nu}{d\tau} = g_{\mu\nu} \left(\Lambda^\mu{}_\rho f^\rho\right) \frac{\left(\Lambda^\nu{}_\sigma dx^\sigma\right)}{d\tau} = \left(g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma\right) f^\rho \frac{dx^\sigma}{d\tau} = g_{\rho\sigma} f^\rho \frac{dx^\sigma}{d\tau}.$$

We are therefore allowed to compute $\frac{d\Omega}{d\tau}$ in any reference frame we wish. Let us choose the rest frame of the particle, where

$$x^\mu = (ct, \vec{0}), \quad f^\mu = (0, \vec{F}_{\text{Newton}}).$$

We obtain:

$$\frac{d\Omega}{d\tau} = \frac{2}{mc^2}\left(f^0 \frac{dx^0}{d\tau} - \vec{f} \cdot \vec{x}\right) = 0. \tag{478}$$

Therefore, the quantity Ω is always a constant:

$$\Omega(\tau) = \text{constant}. \tag{479}$$

If for some initial value τ_0 we choose the constant to be one, we will have

$$\Omega(\tau) = \Omega(\tau_0) = 1, \qquad \forall\, \tau. \tag{480}$$

Exercise: Calculate the trajectory of a particle on which the four-vector force exerted is $f^\mu = 0$.

END OF WEEK 7
(before Easter: only two hours)

7.6 Energy and momentum

We can define a relativistic four-vector analogue of momentum as

$$p^\mu = mc\frac{dx^\mu}{d\tau} \tag{481}$$

We have that

$$d\tau = \left(c^2dt^2 - d\vec{x}^2\right)^{\frac{1}{2}} = cdt\left[1 - \left(\frac{d\vec{x}}{cdt}\right)^2\right]^{\frac{1}{2}} = cdt\left[1 - \frac{\vec{v}^2}{c^2}\right]^{\frac{1}{2}} = \frac{cdt}{\gamma}. \tag{482}$$

Thus, for the time-component ($\mu = 0$) of the four-momentum we have

$$p^0 = mc\frac{dx^0}{d\tau} = m\gamma c. \tag{483}$$

For the space-components ($\mu = i = 1, 2, 3$) we have

$$p^i = mc\frac{dx^i}{d\tau} = m\gamma\frac{dx^i}{dt} = m\gamma v^i. \tag{484}$$

For small velocities, we can expand the factor γ as

$$\gamma = \left[1 - \frac{v^2}{c^2}\right]^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right). \tag{485}$$

Therefore, for small velocities the space-components of the four-momentum become the classical momentum,

$$p^i \approx mv^i + \dots, \tag{486}$$

while the time-component becomes

$$p^0 \approx mc + \frac{1}{2c}mv^2 + \dots \tag{487}$$

In the second term of the above expansion we recognize the kinetic energy $\frac{1}{2}mv^2$ of the particle. We then identify the relativistic energy of a particle with

$$E = cp^0 = m\gamma c^2. \tag{488}$$

Eliminating the velocity $\vec{v}$ from Eqs 484-488, we obtain the relation:

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4} \tag{489}$$

7.7 The inverse of a Lorentz transformation

Recall the metric matrix

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{490}$$

We define an inverse

$$g^{\mu\nu}: \quad g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\ \nu}, \tag{491}$$

where $\delta^{\mu}_{\ \nu}$ is the Kronecker delta. It is easy to verify that the inverse of the metric is the metric itself:

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{492}$$

Now consider a Lorentz transformation $\Lambda^{\mu}_{\ \nu}$, which satisfies the identity:

$$\Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} g_{\mu\nu} = g_{\rho\sigma}. \tag{493}$$

We can prove that the matrix

$$\Lambda_{\mu}^{\ \nu} \equiv g_{\mu\rho} g^{\nu\sigma} \Lambda^{\rho}_{\ \sigma} \tag{494}$$

is the inverse of $\Lambda^{\mu}_{\ \nu}$. Indeed

$$\Lambda^{\mu}_{\ \lambda} \Lambda_{\mu}^{\ \nu} = g_{\mu\rho} g^{\nu\sigma} \Lambda^{\rho}_{\ \sigma} \Lambda^{\mu}_{\ \lambda} = g_{\sigma\lambda} g^{\nu\sigma} = \delta^{\lambda}_{\ \nu}. \tag{495}$$

If $\Lambda^{\mu}_{\ \nu}$ is a velocity $\vec{v}$ boost transformation of Eq. 470, then

$$\Lambda_0^{\ 0}(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda_i^{\ 0}(\vec{v}) = \Lambda_0^{\ i}(\vec{v}) = -\gamma \frac{v^i}{c}.$$
$$\Lambda_i^{\ j}(\vec{v}) = \delta_i^j + \frac{v^i v^j}{v^2}(\gamma - 1) \tag{496}$$

We, therefore have that the inverse of a boost is

$$\Lambda_{\mu}^{\ \nu}(\vec{v}) = \Lambda^{\mu}_{\ \nu}(-\vec{v}), \tag{497}$$

as we also expect physically.

7.8 Vectors and Tensors

It is now time to give officially a definition for vectors in special relativity. We call any set of four components which transform according to the rule:

$$V^\mu \to V'^\mu = \Lambda^\mu_{\ \nu} V^\nu \tag{498}$$

a *contravariant* vector. Contravariant vectors transform in the same way as space-time coordinates x^μ do under homogeneous Lorentz transformations.

Not all vectors transform as contravariant vectors. Consider the derivative $\frac{\partial}{\partial x^\mu}$. Under a Lorentz transformation, it transforms as:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho}. \tag{499}$$

We have that

$$\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \delta^\rho_{\ \nu} \rightsquigarrow \left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\Lambda^\mu_{\ \nu} = \delta^\rho_{\ \nu}. \tag{500}$$

Therefore, $\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$ is the inverse of a Lorentz transformation $\Lambda^\mu_{\ \nu}$:

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu^{\ \nu}. \tag{501}$$

Substituting into Eq. 499, we find:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \Lambda_\mu^{\ \rho}\frac{\partial}{\partial x^\rho}. \tag{502}$$

We found that the derivative does not transform according to the Lorentz transformation but according to its inverse. All vectors which transform with the inverse Lorentz transformation:

$$U_\mu = \Lambda_\mu^{\ \nu} U_\nu, \tag{503}$$

are called covariant vectors.

For every contravariant vector U^μ there is a dual vector

$$U_\mu = g_{\mu\nu} U^\nu. \tag{504}$$

We can invert the above equation multiplying with $g^{\rho\mu}$,

$$g^{\rho\mu} U_\mu = g^{\rho\mu} g_{\mu\nu} U^\nu = \delta^\rho_{\ \nu} U^\nu = U^\rho. \tag{505}$$

The dual vector U_μ is a covariant vector. Indeed, under a Lorentz transformation we have

$$U_\mu \to U'_\mu = g_{\mu\nu} U'^\nu = g_{\mu\nu} \Lambda^\nu{}_\rho U^\rho = g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} U_\sigma = \Lambda_\mu{}^\sigma U_\sigma. \tag{506}$$

The scalar product of a contravariant and a covariant vector

$$A \cdot B \equiv A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu \tag{507}$$

is invariant under Lorentz transformations. Indeed,

$$A \cdot B \to A' \cdot B' = A'^\mu B'_\mu = \Lambda^\mu{}_\rho A^\rho \Lambda_\mu{}^\sigma B_\sigma = \delta^\sigma_\rho A^\rho B_\sigma = A_\rho B_\rho = A \cdot B. \tag{508}$$

Let us define for short:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{509}$$

and the dual contravariant vector:

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu. \tag{510}$$

The D' Alembert operator is the scalar product:

$$\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{511}$$

Due to it being a scalar product, the D'Alembert operator is invariant under Lorentz transformations.

Finally, we define a tensor with multiple "up" and/or "down" indices to be an object

$$T^{\mu_1 \mu_2 \ldots}_{\nu_1 \nu_2 \ldots} \tag{512}$$

which transfroms as:

$$T^{\mu_1 \mu_2 \ldots}_{\nu_1 \nu_2 \ldots} \to \Lambda^{\mu_1}_{\rho_1} \Lambda^{\mu_2}_{\rho_2} \ldots \Lambda^{\sigma_1}_{\nu_1} \Lambda^{\sigma_2}_{\nu_2} \ldots T^{\rho_1 \rho_2 \ldots}_{\sigma_1 \sigma_2 \ldots} \tag{513}$$

7.9 Currents and densities

Consider a set of particles $\{n\}$ with charged q_n at positions $\vec{r}_n(t)$. The charge and current density are

$$\rho(\vec{x}, t) = \sum_n q_n \delta(\vec{x} - \vec{r}_n(t)), \tag{514}$$

$$\vec{j}(\vec{x},t) = \sum_n q_n \frac{d\vec{r}_n(t)}{dt}\delta(\vec{x}-\vec{r}_n(t)) = \sum_n q_n \frac{d\vec{x}}{dt}\delta(\vec{x}-\vec{r}_n(t)), \tag{515}$$

Recall now that for the vector $x^\mu = (ct, \vec{x})$, we have

$$\frac{dx^\mu}{dt} = \frac{d}{dt}(ct, \vec{x}) = \left(c, \frac{d\vec{x}}{dt}\right). \tag{516}$$

We can then combine the charge and current densities into one object:

$$j^\mu \equiv \left(c\rho, \vec{j}\right) \tag{517}$$

with

$$j^\mu(\vec{x},t) = \sum_n q_n \frac{dx^\mu}{dt}\delta(\vec{x}-\vec{r}_n(t)), \tag{518}$$

We can now cast j^μ in a form which manifestly shows that it is a contravariant four-vector. First, we define a delta function in four dimensions as

$$\delta\left(x^\mu - y^\mu\right) = \delta(x^0 - y^0)\delta(\vec{x}-\vec{y}) = \frac{1}{c}\delta(t_x - t_y)\delta(\vec{x}-\vec{y}). \tag{519}$$

Notice that the $\delta-$function of a four-vector is a scalar. Under a Lorentz transformation,

$$\delta\left(U^\mu\right) \to \delta\left(U'^\mu\right) = \delta\left(\Lambda^\mu_\nu U^\nu\right) = \frac{\delta(U^\nu)}{|\det\Lambda|} = \delta(U^\nu). \tag{520}$$

With the use of the delta-function, we can write the current-density four-vector of Eq. 518 as an integral

$$j^\mu(\vec{x},t) = \sum_n q_n \int dt' \frac{dx^\mu}{dt'}\delta(\vec{x}-\vec{r}_n(t))\delta(t'-t) = c\sum_n q_n \int dt' \frac{dx^\mu}{dt'}\delta(x^\mu - r^\mu_n(t)), \tag{521}$$

where

$$x^\mu \equiv (ct', \vec{x}), \quad r^\mu_n(t) \equiv (ct, \vec{r}_n(t)).$$

Now, we change integration variables from $t' \to \tau$. Recall that

$$dt' = \frac{d\tau}{c}\gamma. \tag{522}$$

Then, we have

$$j^\mu(\vec{x},t) = c\sum_n q_n \int d\tau \frac{dx^\mu}{d\tau}\delta(x^\mu - r^\mu_n(\tau)), \tag{523}$$

which shows manifestly that j^μ transforms as $\frac{dx^\mu}{d\tau}$ and it is therefore a contravariant four-vector.

We have already shown that charge-conservation implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{524}$$

In relativistic notation, the continuity equation takes the more elegant form:

$$\partial_\mu j^\mu = 0 \tag{525}$$

7.10 Energy-Momentum tensor

Consider a collection of particles $\{n\}$ at positions $\vec{r}_n(t)$. The energy density is:

$$\text{energy density} = \sum_n E_n(t)\delta(\vec{x} - \vec{r}_n(t)) \tag{526}$$

Changes in the energy density result to a "energy-current density":

$$\text{energy current density} = \sum_n E_n(t)\frac{d\vec{r}_n}{dt}\delta(\vec{x} - \vec{r}_n(t)). \tag{527}$$

As with the charge density and its current-density of the last section, we can combine the two together into a single object:

$$\sum_n E_n(t)\frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)). \tag{528}$$

Similarly to the energy, the density/current-density for the components of the three-dimensional momentum are:

$$\sum_n p_n^i(t)\frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)), \quad i = 1, 2, 3. \tag{529}$$

Collectively, we can form the so called "energy-momentum tensor" which encompasses the density and current-density for all components of the four-momentum:

$$T^{\mu\nu} \equiv \sum_n p_n^\mu \frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)) \tag{530}$$

or, in the equivalent form:

$$T^{\mu\nu} = \sum_n \int d\tau p_n^\mu \frac{dx_n^\nu}{d\tau}\delta(x^\rho - r_n^\rho(\tau)). \tag{531}$$

From the last equation, we can see manifestly that this object is justifiably called a tensor since it transforms as the product of two four-vectors under Lorentz transformations:

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}T^{\rho\sigma} \tag{532}$$

The energy momentum tensor is symmetric:

$$T^{\mu\nu} = T^{\nu\mu}. \tag{533}$$

To verify this, we recall that

$$p_n^{\nu} = m_n \frac{dr_n^{\nu}}{d\tau} = m_n \gamma \frac{dr_n^{\nu}}{dt} = E_n \frac{dr_n^{\nu}}{dt}. \tag{534}$$

The energy momentum tensor takes then the explicitly symmetric form:

$$T^{\mu\nu} \equiv \sum_n \frac{p_n^{\mu} p_n^{\nu}}{E_n} \delta(\vec{x} - \vec{r}_n(t)). \tag{535}$$

For the charge-density four-vector, we have found a continuity identity

$$\partial_\mu j^\mu = 0.$$

This was a consequence of the conservation of charge. If the total energy and momentum of the system of particles is conserved, we anticipate a similar continuity identity for the energy-momentum tensor:

$$\partial_\nu T^{\mu\nu} = 0.$$

We have:

$$\begin{aligned}
\partial_i T^{\mu i} &= \sum_n p_n^{\mu} \frac{dr_n^i}{dt} \frac{\partial}{\partial x^i} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\sum_n p_n^{\mu} \frac{dr_n^i}{dt} \frac{\partial}{\partial r_n^i} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\sum_n p_n^{\mu} \frac{\partial}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\frac{\partial}{\partial t} \sum_n p_n^{\mu} \delta\left(\vec{x} - \vec{r}_n\right) + \sum_n \frac{\partial p_n^{\mu}}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\frac{\partial}{\partial t} T^{00} + \sum_n \frac{\partial p_n^{\mu}}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right).
\end{aligned} \tag{536}$$

In the above we have used that

$$\begin{aligned}\partial_x \delta(x-y) &= 2\pi \partial_x \int_{-\infty}^{\infty} dw e^{-iw(x-y)} = -2i\pi \int_{-\infty}^{\infty} dw\, w e^{-iw(x-y)} \\ &= -2\pi \partial_y \int_{-\infty}^{\infty} dw e^{-iw(x-y)} = -\partial_x \delta(x-y). \end{aligned} \tag{537}$$

We have therefore arrived to the equation

$$\partial_\nu T^{\mu\nu} = G^\mu, \tag{538}$$

where

$$G^\mu = \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) = \sum_n \frac{\partial \tau}{\partial t} f_n^\mu(t) \delta\left(\vec{x} - \vec{r}_n\right) \tag{539}$$

is the "density of force".

For free particles, where the energy and momentum of all particles separately is conserved $p_n^\mu = constant$, the energy-momentum tensor satisfies the continuity equation:

$$\partial_\nu T^{\mu\nu} = 0. \tag{540}$$

The energy momentum tensor is also conserved if the particles interact only at the points where they collide with each other. In that case, the force density is

$$\begin{aligned} G^\mu &= \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\ &= \sum_{coll.} \delta(\vec{x} - \vec{x}_{coll}(t)) \frac{d}{dt} \sum_{n \in coll.} p_n^\mu(t). \end{aligned} \tag{541}$$

We have grouped the sum over all particles contributing to the force density according to the collision points that they meet at. In each collision point, the sum of the momenta of the colliding particles is conserved

$$\frac{d}{dt} \sum_{n \in coll.} p_n^\mu(t) = 0 \rightsquigarrow \partial_\nu T^{\mu\nu} = 0. \tag{542}$$

If the continuity equation is satisfied, then

$$\begin{aligned} 0 &= \partial_\nu T^{\mu\nu} \\ \rightsquigarrow 0 &= \partial_0 T^{\mu 0} + \partial_i T^{\mu i} \\ \rightsquigarrow 0 &= \partial_0 \int d^3\vec{x} T^{\mu 0} + \int d^3\vec{x} \partial_i T^{\mu i} \\ 0 &= \partial_0 \int d^3\vec{x} T^{\mu 0} \end{aligned} \tag{543}$$

which means that the vector

$$P^\mu \equiv \int d^3\vec{x} T^{\mu 0} = \text{constant} \tag{544}$$

is conserved. We find that the conserved vector is the sum of all the total four-momentum of the particles in the distribution:

$$P^\mu = \sum_n \int d^3\vec{x} p_n^\mu \delta(\vec{x} - \vec{r}_n(t)) = \sum_n p_n^\mu. \tag{545}$$

END of WEEK 8

8 Relativistic formulation of Electrodynamics

From now on we will set $\epsilon_0 = c = 1$. It will be easy to restore the full dependence on these constants with dimensional analysis when necessary. Maxwell equations are

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{546}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{547}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{548}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{549}$$

$$\tag{550}$$

We construct an "electromagnetic field tensor" $F^{\mu\nu}$ from the components of the electric $\vec{E} \equiv (E^1, E^2, E^3)$ and magnetic field $\vec{B} \equiv (B^1, B^2, B^3)$ as:

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon_{ijk} B^k, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{551}$$

(we use $\epsilon_{123} = +1$)
Explicitly,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{552}$$

Conversely, If the tensor $F^{\mu\nu}$ is given, we can obtain the magnetic field via (**exercise**):

$$B^i = -\frac{1}{2} \epsilon_{ijk} F^{jk}. \tag{553}$$

Exercise: What is the covariant tensor $F_{\mu\nu}$ in matrix form?
Exercise: Write the matrix:

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{554}$$

Exercise: Prove that:

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{\vec{E}^2 - \vec{B}^2}{2}, \quad \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 8 \vec{E} \cdot \vec{B}. \tag{555}$$

As we have already noted, the charge and current densities form a four-vector:

$$j^\mu = (\rho, \vec{j}).$$

We can observe that two of Maxwell equations can be combined into one:

$$\partial_\nu F^{\nu\mu} = j^\mu. \tag{556}$$

Indeed, for $\mu = 0$, we have

$$\partial_0 F^{00} + \partial_i F^{i0} = j^0 \rightsquigarrow \vec{\nabla} \cdot \vec{E} = \rho.$$

For $\mu = i = 1, 2, 3$ we have

$$\partial_0 F^{0i} + \partial_j F^{ji} = j^i \rightsquigarrow -\partial_0 E^i + \epsilon_{ijk} \partial_j B^k = j^i \rightsquigarrow \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}.$$

The remaining two Maxwell equations tell us, as we have found earlier, that we can derive the electric and magnetic fields by means of the scalar and vector potential $\phi, \vec{A}$, via

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{557}$$

We can combine the scalar and vector potential into a single four-vector:

$$A^\mu \equiv (\phi, \vec{A}) = (\phi, A^1, A^2, A^3). \tag{558}$$

Then, the above equations take the elegant form:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{559}$$

Indeed, for $\mu = 0, \nu = i = 1, 2, 3$ we have

$$F^{0i} = \partial^0 A^i - \partial^i A^0 \rightsquigarrow -\vec{E} = \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi$$

For $\mu = i, \nu = j,\ i, j = 1, 2, 3$ we have

$$F^{ij} = \partial^i A^j - \partial^j A^i \rightsquigarrow -\frac{1}{2}\epsilon_{ijk} F^{ij} = \epsilon_{ijk}\partial_i A^j \rightsquigarrow \vec{B} = \vec{\nabla} \times \vec{A}.$$

It is now straightforward to prove the following identities (**exercise**):

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \tag{560}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{561}$$

Substituting Eq. 559 into Eq. 556 we find

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu. \tag{562}$$

Explicitly, for $\nu = 0$ and $\nu = i = 1, 2, 3$ we recover the known differential equations for the scalar and vector potentials respectively:

$$\Box\phi - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right) = \rho, \tag{563}$$

$$\Box\vec{A} + \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right] = \vec{J}. \tag{564}$$

The property of gauge invariance becomes more elegant as well in relativistic notation. The gauge transformations of the vector and scalar potentials which leave Maxwell equations invariant are written as:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi, \tag{565}$$

where χ is a scalar function. Indeed, under a gauge transformation the electromagnetic field tensor remains invariant:

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu \chi) - \partial_\nu(A_\mu + \partial_\mu \chi) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{566}$$

The Lorentz gauge-fixing condition

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0 \tag{567}$$

can be written simply as

$$\partial_\mu A^\mu = 0. \tag{568}$$

In the Lorentz gauge, Maxwell equations for the four-vector potential become:

$$\partial^2 A^\mu = j^\mu. \tag{569}$$

Earlier, we have found that the vector and scalar potential can be computed via

$$\phi(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{r}' dt' G\left(\vec{r} - \vec{r}'; t - t'\right) \rho(\vec{r}', t') \tag{570}$$

and

$$\vec{A}(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{r}' dt' G\left(\vec{r} - \vec{r}'; t - t'\right) \vec{j}(\vec{r}', t') \tag{571}$$

where the Green's function can be written in the form:

$$G(\Delta\vec{r}, \Delta t) = \frac{1}{2\pi}\delta\left(\Delta t^2 - \Delta\vec{r}^2\right)\Theta(\Delta t > 0). \tag{572}$$

In relativistic notation, the solutions for the four-vector potential take the form:

$$A^\nu(x^\mu) = \frac{1}{2\pi}\int d^4x' j^\nu(x'^\mu)\delta\left((x'^\mu - x^\mu)^2\right)\Theta\left(x^0 > x'^0\right). \tag{573}$$

The electromagnetic force acting on a particle with a charge q is:

$$f^\mu = qF^{\mu\nu}\frac{dx_\nu}{d\tau}. \tag{574}$$

Indeed, in the rest frame of the particle $d\tau = dt, d\vec{x} = 0$, we get

$$f^\mu_{\text{rest}} = qF^{\mu 0} \rightsquigarrow f^0_{\text{rest}} = 0,\ \vec{f}_{\text{rest}} = q\vec{E}. \tag{575}$$

Since our expression for the electromagnetic force is correct in one reference frame, it should hold in every inertial reference frame due to it being written as an equation of four-vectors. In a frame where the charge is moving with a velocity $\vec{v}$, the three-dimensional force is:

$$\begin{aligned} f^i &= qF^{i0}\frac{dx^0}{d\tau} - qF^{ij}\frac{dx^j}{d\tau} = q\gamma\left(E^i + \epsilon_{ijk}B^k v^j\right) \\ \rightsquigarrow \vec{f} &= q\gamma\left(\vec{E} + \vec{v}\times\vec{B}\right). \end{aligned} \tag{576}$$

Recall that

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt}. \tag{577}$$

We therefore have recovered our familiar expression for Lorentz' force:

$$\frac{d\vec{p}}{dt} = \left(\vec{E} + \vec{v}\times\vec{B}\right). \tag{578}$$

8.1 Energy-Momentum Tensor in the presence of an electromagnetic field

Consider a number of charges q_n which interact via the electromagnetic field. The energy-momentum tensor is not conserved:

$$\partial_\nu T^{\mu\nu} = G^\mu \tag{579}$$

where the force density is given by

$$\begin{aligned} G^\mu &= \sum_n \frac{\partial \tau}{\partial t} f_n^\mu(t) \delta\left(\vec{x} - \vec{r}_n\right) \\ &= \sum_n \frac{\partial \tau}{\partial t} q_n F^\mu{}_\nu \frac{dr_n^\nu}{d\tau} \delta\left(\vec{x} - \vec{r}_n\right) \\ &= F^\mu{}_\nu \sum_n q_n \frac{dr_n^\nu}{dt} \delta\left(\vec{x} - \vec{r}_n\right) \\ &= F^{\mu\nu} j_\nu. \end{aligned} \tag{580}$$

Consider the tensor

$$T_{em}^{\mu\nu} \equiv F^\mu_\rho F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{581}$$

This tensor is explicitly

- symmetric
- gauge-invariant.

The components of the tensor are (**exercise**):

$$T_{em}^{00} = \frac{\vec{E}^2 + \vec{B}^2}{2}, \quad T_{em}^{0i} = T_{em}^{i0} = (\vec{E} \times \vec{B})_i \tag{582}$$

Exercise: Find the remaining components.

We find

$$\begin{aligned} \partial_\nu T_{em}^{\mu\nu} &= \partial_\nu \left\{ F^{\mu\rho} F_\rho{}^\nu + \frac{g^{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma} \right\} \\ &= F^{\mu\rho} \partial_\nu F_\rho{}^\nu + \left(\partial^\nu F^{\mu\rho}\right) F_{\rho\nu} + \frac{1}{2} F_{\rho\sigma} \partial^\mu F^{\rho\sigma} \\ &= -F^{\mu\rho} j_\rho + \frac{1}{2}\left(\partial^\nu F^{\mu\rho} - \partial^\rho F^{\mu\nu}\right) F_{\rho\nu} + \frac{1}{2} F_{\rho\sigma} \partial^\mu F^{\rho\sigma} \\ &= -F^{\mu\rho} j_\rho + \frac{1}{2}\left(\partial^\sigma F^{\mu\rho} + \partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\mu}\right) F_{\rho\sigma} \end{aligned} \tag{583}$$

which, due to Eq. 561, yields:

$$\partial_\nu T^{\mu\nu}_{em} = -F^{\mu\nu} j_\nu. \tag{584}$$

$T^{\mu\nu}_{em}$ is purely electromagnetic. While neither $T^{\mu\nu}$ nor $T^{\mu\nu}_{em}$ satisfy a continuity equation, but their sum

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + T^{\mu\nu}_{em} = \sum_n \frac{p^\mu_n p^\nu_n}{E_n} \delta(\vec{x} - \vec{r}_n(t)) + F^{\mu\rho} F_\rho^{\ \nu} + \frac{g^{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma}. \tag{585}$$

does:

$$\partial_\nu \Theta^{\mu\nu} = 0. \tag{586}$$

From the continuity equation, we obtain that there is a conserved four-vector:

$$\partial_t P^\mu \equiv \partial_t \int d^3\vec{x}\, \Theta^{\mu 0} = -\int d^3\vec{x}\, \partial_i \Theta^{0i} = -\left. \Theta^{0i} \right|_\infty = 0. \tag{587}$$

The conserved vector is:

$$P^\mu = \int d^3\vec{x}\, \Theta^{\mu 0} = \sum_n p^\mu_n + \int d^3\vec{x}\, T^{\mu 0}_{em}. \tag{588}$$

The four-momentum of the charges

$$P^\mu_{\text{charges}} = \sum_n p^\mu_n \tag{589}$$

is not conserved on its own. Some momentum

$$P^\mu_{em} = \int d^3\vec{x}\, T^{\mu 0} \tag{590}$$

is also carried by the electromagnetic field itself. This is not conserved either. Momentum can be exchanged between the charges and the field, however this is done in such a way so that the total momentum is always the same:

$$P^\mu = P^\mu_{\text{charges}} + P^\mu_{em} = \text{constant}. \tag{591}$$

The time-component of the four-vector is the total energy. The energy stored in the electromagnetic field is:

$$E_{em} = \int d^3\vec{x}\, T^{00}_{em}. \tag{592}$$

Therefore, the energy density w of the electromagnetic field is:

$$w = T^{00}_{em} = \frac{\vec{E}^2 + \vec{B}^2}{2}, \quad \text{EM field energy density.} \tag{593}$$

Similarly, we find that the three-momentum density $\vec{S}$ of the electromagnetic field is:

$$\vec{S} = T^{0i}_{em} = T^{i0}_{em} = (\vec{E} \times \vec{B})_i \text{ EM field momentum density.} \tag{594}$$

The vector

$$\vec{S} \equiv \vec{E} \times \vec{B}, \tag{595}$$

is known as the *Poynting vector*.

Setting $\mu = 0$ in Eq. 584 we have

$$\partial_0 T^{00} + \partial_i T^{i0} = -F^{0i} j_i, \tag{596}$$

and equivalently,

$$\frac{\partial w}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{E} \cdot \vec{j}. \tag{597}$$

9 Radiation from moving charges

In this section we will study the electromagnetic field which is created by moving charges.

9.1 The vector potential from a moving charge

A moving charge q has a current density

$$j^\mu(x) = \left(q\delta(\vec{x} - \vec{r}(t)), q\frac{d\vec{r}}{dt}\delta(\vec{x} - \vec{r}(t))\right) \tag{598}$$

In an explicitly covariant form, we have found:

$$j^\mu = q \int d\tau v^\mu \delta\left((x - r(\tau))^2\right), \tag{599}$$

where τ is the proper time and

$$v^\mu \equiv \frac{dr^\mu}{d\tau}, \tag{600}$$

the four-velocity of the charge.

The solution of Maxwell equations for the vector potential at a position x^μ is:

$$A^\mu(x) = \frac{q}{2\pi} \int d^4x' G_{\text{ret}}(x - x') j^\mu(x'), \tag{601}$$

with the "retarded" Green's function given by

$$G_{\text{ret}}(x - x') = \frac{1}{2\pi}\delta((x - x')^2)\Theta(x^0 > x'^0). \tag{602}$$

Substituting the expression of the Green's function into the potential integral solution and performing the d^4x' integration, we obtain:

$$A^\mu(x) = \frac{q}{2\pi} \int d\tau v^\mu(\tau)\delta((x - r(\tau))^2)\Theta(x^0 > r^0(\tau)) \tag{603}$$

To perform the τ integration, we need to solve for the contraint imposed by the delta function:

$$f(\tau) = (x - r(\tau))^2 = 0 \tag{604}$$

with the additional constraint

$$x^0 > r^0(\tau). \tag{605}$$

We have

$$\begin{aligned} 0 &= (x - r(\tau))^2 = (x^0 - r^0(\tau))^2 - |\vec{x} - \vec{r}(\tau)|^2 \\ \rightsquigarrow x^0 - r^0(\tau) &= \pm |\vec{x} - \vec{r}(\tau)| = \pm R(\tau), \end{aligned} \tag{606}$$

where $R(\tau)$ is the distance of the charge from the observation point and it is a positive quantity. Due to the constraint $x^0 > r^0(\tau)$, we keep only the solution originating from:

$$x^0 = r^0(\tau) + |\vec{x} - \vec{r}(\tau)| \tag{607}$$

Once we know the roots τ_i of Eq. 604, $f(\tau_i) = 0$, which satisfy Eq. 605, we can use that

$$\delta\left[f(\tau)\right] = \sum_i \frac{\delta(\tau - \tau_i)}{\left|\frac{df}{d\tau}\right|_{\tau=\tau_i}}. \tag{608}$$

For relativistic particles with velocity $v < c = 1$ we find only one such solution τ_0. We will demonstrate how this is done geometrically for a charge moving with a constant velocity later. Let us now proceed to perform the τ integration assuming that this solution τ_0 has been found.

The derivative is

$$\begin{aligned} \frac{df(\tau)}{d\tau} &= \frac{d}{d\tau}(x - r(\tau))^2 = \frac{d}{d\tau}\left[g_{\mu\nu}(x - r(\tau))^\mu (x - r(\tau))^\nu\right] \\ &= 2g_{\mu\nu}(x - r(\tau))^\mu \frac{d}{d\tau}(x - r(\tau))^\nu = -2g_{\mu\nu}(x - r(\tau))^\mu \frac{dr^\nu(\tau)}{d\tau} \\ &= -2g_{\mu\nu}(x - r(\tau))^\mu v^\nu(\tau) = -2v \cdot (x - r(\tau)). \end{aligned} \tag{609}$$

The vector potential becomes:

$$A^\mu(x) = \frac{q}{4\pi} \int d\tau \frac{\delta(\tau - \tau_0) v^\mu(\tau)}{|(x - r(\tau)) \cdot v(\tau)|} \tag{610}$$

As it can be seen in the rest frame of the charge, the quantity $(x - r(\tau)) \cdot v(\tau)$ is positive definite and we can therefore drop the absolute value from the denominator. Performing the τ integration is trivial, yielding

$$A^\mu(x) = \frac{q}{4\pi} \left.\frac{v^\mu}{v \cdot (x - r(\tau))}\right|_{\text{retarded}}. \tag{611}$$

The subscript denotes that all quantities in this expression (distance and velocity of the charge) must be computed at a retarded proper time $\tau = \tau_0$

and not at the current time of the measurement. The expression of Eq. 611 is known as the Lienard-Wiechert potential. Notice that for a charge in its rest frame, $v^\mu = (1, \vec{0})$ it reproduces our known Coulomb potential:

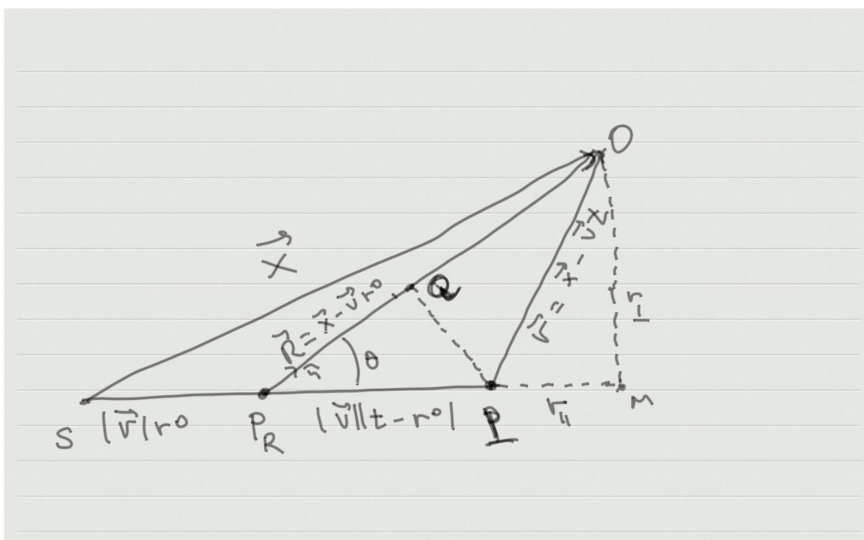
$$A^\mu(x) = \frac{q}{4\pi}\frac{(1,\vec{0})}{x^0 - r^0} = \frac{q}{4\pi}\frac{(1,\vec{0})}{|\vec{x} - \vec{r}|} \tag{612}$$

9.1.1 Potential from a moving charge with a constant velocity

Let us now see how we can use Eq. 611 in order to recover the results of Section 6.4 for the scalar and vector potential due to a charged particle moving with a constant velocity $\vec{v}$. At a time t, we assume the particle to be at a position P which is at a distance

$$(PS) = |\vec{v}|t,$$

from the origin S. We are interested in knowing the potential $A^\mu(\vec{x}, t)$ at a position $O(\vec{x})$ at the time t.



The electromagnetic wave which is measured at $(\vec{x}, t)$ has been emitted at an earlier time r^0 when the charge was at a retarded position $P_R(\vec{v}r^0)$. The time needed for the signal to travel is

$$t - r^0 \tag{613}$$

and (in units of $c = 1$) the distance covered is

$$(P_r O) = R = \left| \vec{x} - \vec{r}^0 \right| = t - r^0. \tag{614}$$

In the same time, the charge advanced by a distance:

$$(P_R P) = |\vec{v}|(t - t^0) = |\vec{v}| R. \tag{615}$$

Let Q be the projection of the position P of the charge on the trajectory of the light signal. Then,

$$(P_R Q) = |\vec{v}| R \cos\theta = R \vec{v} \cdot \hat{n}, \tag{616}$$

where $\hat{n}$ points to the direction of travel of the light signal. Then,

$$(QO) = R(1 - \vec{v} \cdot \hat{n}). \tag{617}$$

Now consider the projection M of the observation point O on the trajectory line of the charge. From Pythagoras' theorem we have:

$$(PQ)^2 + (QM)^2 = (PM)^2 + (MO)^2 = r^2. \tag{618}$$

The vertical distance of the charge and the observation point is:

$$(MO) \equiv r_\perp, \tag{619}$$

and the parallel distance is

$$(MP) \equiv r_\parallel = x_\parallel - vt. \tag{620}$$

From Eq. 616 and a bit of trigonometry, we obtain (**exercise**)

$$R(1 - \vec{n} \cdot \vec{v}) = \sqrt{1 - v^2} \left[\left(\frac{x_\parallel - vt}{\sqrt{1 - v^2}} \right)^2 + r_\perp^2 \right]^{\frac{1}{2}} \tag{621}$$

In the lhs, we recognise the scalar product:

$$(x - r) \cdot v\big|_{\text{retarded}} = R(1, \hat{n}) \cdot \gamma(1, \vec{v}) = \gamma R(1 - \vec{v} \cdot \hat{n}). \tag{622}$$

Thus, the potential is:

$$A^\mu = \frac{q}{4\pi} \frac{v^\mu}{(x - r) \cdot v} = \frac{q}{4\pi} \frac{\gamma(1, \vec{v})}{\gamma R(1 - \vec{v} \cdot \hat{n})} = \frac{q}{4\pi} \frac{\gamma(1, \vec{v})}{\left[\left(\frac{x_\parallel - vt}{\sqrt{1 - v^2}} \right)^2 + r_\perp^2 \right]^{\frac{1}{2}}} \tag{623}$$

This is in agreement with the result found in Section 6.4.

9.2 The electromagnetic field tensor from a moving charge

The electromagnetic field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{624}$$

for a moving charge can be derived by differentiating the expression for the Lienard-Wiechert potential. Due to the requirement of calculating the terms of Eq. 611 at retarded positions, it is easier to work with the integral of Eq. 603 and perform the integration after we have carried out the differentiations. We define the four-vector:

$$R^\mu \equiv x^\mu - r^\mu(\tau). \tag{625}$$

The potential is then:

$$A^\nu(x) = \frac{q}{2\pi}\int d\tau v^\nu(\tau)\delta(R^2)\Theta(R^0 > 0) \tag{626}$$

Its derivative is

$$\begin{aligned} \partial^\mu A^\nu &= \frac{q}{2\pi}\int d\tau v^\nu(\tau)\partial^\mu\left[\delta(R^2)\Theta(R^0 > 0)\right] \\ &= \frac{q}{2\pi}\int d\tau v^\nu(\tau)\left[\Theta(R^0 > 0)\partial^\mu\delta(R^2) + \delta(R^2)\partial^\mu\Theta(R^0 > 0)\right] \\ &= \frac{q}{2\pi}\int d\tau v^\nu(\tau)\left[\Theta(R^0 > 0)\partial^\mu\delta(R^2) + \delta(R^2)\delta(R^0)g^{\mu 0}\right]. \end{aligned} \tag{627}$$

The second term in the bracket vanishes,

$$\delta(R^2)\delta(R^0) = \delta((R^0)^2 - \vec{R}^2)\delta(R^0) = \delta(-\vec{R}^2)\delta(R^0) = 0, \tag{628}$$

unless we are interested in calculating the electromagnetic field exactly at a space-time point occupied by the electric charge q. For all other points, where in fact classical electrodynamics should be valid, we have

$$\partial^\mu A^\nu = \frac{q}{2\pi}\int d\tau v^\nu(\tau)\Theta(R^0 > 0)\partial^\mu\delta(R^2). \tag{629}$$

The derivative on the delta-function is:

$$\begin{aligned} \partial^\mu\delta(F) &= \partial^\mu\delta(R^2) = \partial^\mu F\frac{\partial\delta(F)}{\partial F} \\ &= \frac{\partial^\mu F}{\frac{\partial F}{\partial\tau}}\frac{\partial\delta(F)}{\partial\tau} = \frac{\partial^\mu R^2}{\frac{\partial R^2}{\partial\tau}}\frac{\partial\delta(R^2)}{\partial\tau} \\ &= \frac{2R^\mu}{2R\cdot\frac{\partial R}{\partial\tau}}\frac{\partial\delta(R^2)}{\partial\tau} = \frac{-R^\mu}{R\cdot v(\tau)}\frac{\partial\delta(R^2)}{\partial\tau} \end{aligned} \tag{630}$$

Thus, the derivative of the potential becomes:

$$\partial^\mu A^\nu = -\frac{q}{2\pi}\int d\tau \Theta(R^0 > 0)\frac{R^\mu v^\nu}{R\cdot v}\frac{\partial\delta(R^2)}{\partial\tau}. \tag{631}$$

The field tensor is then

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ &= -\frac{q}{2\pi}\int d\tau \Theta(R^0 > 0)\frac{R^\mu v^\nu - R^\nu v^\mu}{R\cdot v}\frac{\partial\delta(R^2)}{\partial\tau} \\ &= -\frac{q}{2\pi}\int d\tau \frac{\partial}{\partial\tau}\left[\Theta(R^0 > 0)\frac{R^\mu v^\nu - R^\nu v^\mu}{R\cdot v}\delta(R^2)\right] \\ &\quad +\frac{q}{2\pi}\int d\tau \delta(R^2)\frac{\partial}{\partial\tau}\left[\Theta(R^0 > 0)\frac{R^\mu v^\nu - R^\nu v^\mu}{R\cdot v}\right] \end{aligned} \tag{632}$$

The first term requires the evaluation of the integrand in the boundaries of integration and it is zero; $\delta(R^2)$ has a solution inside the integration region and not at the edges. Also, the term which is produced by differentiating $\Theta(R^0)$ vanishes for the same reason as we have seen earlier. Thus, the field tensor becomes:

$$F^{\mu\nu} = +\frac{q}{2\pi}\int d\tau \delta(R^2)\Theta(R^0 > 0)\frac{\partial}{\partial\tau}\left[\frac{R^\mu v^\nu - R^\nu v^\mu}{R\cdot v}\right]. \tag{633}$$

We can now perform the τ integration in exactly the same manner as in Section 9.1. We obtain:

$$F^{\mu\nu} = +\frac{q}{4\pi}\left.\frac{\frac{\partial}{\partial\tau}\left[\frac{R^\mu v^\nu - R^\nu v^\mu}{R\cdot v}\right]}{R\cdot v}\right|_{\text{retarded}} \tag{634}$$

Performing the differentiations, and using that:

$$v_\mu v^\mu = \frac{dr_\mu}{d\tau}\frac{dr^\mu}{d\tau} = \frac{d\tau^2}{d\tau^2} = 1, \tag{635}$$

as well the definition of the acceleration four-vector:

$$a^\mu \equiv \frac{dv^\mu}{d\tau} \tag{636}$$

we find that

$$F^{\mu\nu} = \frac{q}{4\pi}\left.\frac{R^\mu Q^\nu - R^\nu Q^\mu}{(R\cdot v)^3}\right|_{\text{retarded}} \tag{637}$$

where

$$Q^\nu = v^\nu + a^\nu R\cdot v - v^\nu R\cdot a. \tag{638}$$

The acceleration four-vector is:

$$\begin{aligned} a^\nu &= \frac{dv^\nu}{d\tau} = \frac{d}{d\tau}\gamma(1,\vec{v}) \\ &= \frac{d\gamma}{d\tau}(1,\vec{v}) + \gamma\left(0,\frac{d\vec{v}}{d\tau}\right) \\ &= \frac{1}{\gamma}\frac{d\gamma}{d\tau}v^\nu + a_T^\mu, \end{aligned} \tag{639}$$

with

$$a_T^\mu = \gamma\left(0,\frac{d\vec{v}}{d\tau}\right) = \gamma^2\left(0,\dot{\vec{v}}\right) \tag{640}$$

and

$$\dot{\vec{v}} \equiv \frac{d\vec{v}}{dt} \tag{641}$$

Question: Is a_T^μ a four-vector?
Substituting into Eq. 638, we find that

$$Q^\nu = v^\nu + a_T^\nu R\cdot v - v^\nu R\cdot a_T. \tag{642}$$

9.2.1 The electric and magnetic field of a moving charge

We can read the components of the electric and magnetic field from the electromagnetic field tensor of Eq. 637. Let's now write the distance vector R^μ as

$$R^\mu = |\vec{R}|(1,\hat{n}). \tag{643}$$

Then, the components of the electric field are

$$\begin{aligned} E^i = F^{i0} &= \left.\frac{q}{4\pi}\frac{R^iQ^0 - R^0Q^i}{(R\cdot v)^3}\right|_{\text{retarded}} \\ \rightsquigarrow \vec{E} &= \left.\frac{q|\vec{R}|}{4\pi}\frac{Q^0\hat{n} - \vec{Q}}{(R\cdot v)^3}\right|_{\text{retarded}} \end{aligned} \tag{644}$$

The components of the magnetic field are

$$B^i = -\frac{1}{2}\epsilon_{ijk}F^{jk} = -\frac{1}{2}\epsilon_{ijk}\left.\frac{q}{4\pi}\frac{R^jQ^k - R^kQ^j}{(R\cdot v)^3}\right|_{\text{retarded}}$$

$$= -\frac{q|\vec{R}|}{4\pi}\frac{\epsilon_{ijk}\hat{n}^i Q^k}{(R\cdot v)^3}\bigg|_{\text{retarded}}$$

$$\rightsquigarrow \vec{B} = -\frac{q|\vec{R}|}{4\pi}\frac{\hat{n}\times\vec{Q}}{(R\cdot v)^3}\bigg|_{\text{retarded}}. \tag{645}$$

We have therefore found explicitly that

$$\vec{B} = \hat{n}\times\vec{E}. \tag{646}$$

To compute the magnitude of the electric field, we need the products:

$$R\cdot v = |\vec{R}|\gamma(1-\hat{n}\cdot\vec{v}), \tag{647}$$

and

$$R\cdot a_T = -|\vec{R}|\gamma^2\,\dot{\vec{v}}\cdot\hat{n}. \tag{648}$$

Finally, we arrive at the result (**exercise**):

$$\vec{E} = \frac{q}{4\pi(1-\hat{n}\cdot\vec{v})^3}\left\{\frac{(1-\vec{v}^2)}{|\vec{R}|^2}(\hat{n}-\vec{v}) + \frac{1}{|\vec{R}|}\hat{n}\times\left[(\hat{n}-\vec{v})\times\dot{\vec{v}}\right]\right\}\bigg|_{\text{retarded}} \tag{649}$$

9.3 Radiation from an accelerated charge in its rest frame: Larmor's formula

For a charge q which is accelerated but it is momentarily at rest, the electric field is:

$$\vec{E} = \frac{q}{4\pi}\left\{\frac{\hat{n}}{|\vec{R}|^2} + \frac{1}{|\vec{R}|}\hat{n}\times\left[\hat{n}\times\dot{\vec{v}}\right]\right\}\bigg|_{\text{retarded}} \tag{650}$$

To study the flow of energy (radiation) we compute the Poynting vector:

$$\vec{S} = \vec{E}\times\vec{B} = \vec{E}\times\left(\hat{n}\times\vec{E}\right) = \left|\vec{E}\right|^2\hat{n} - (\vec{E}\cdot\hat{n})\vec{E} \tag{651}$$

Expanding in $1/|\vec{R}|$, we have:

$$\vec{S} = \hat{n}\frac{q^2}{16\pi^2|\vec{R}|^2}\left|\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right|^2 + \mathcal{O}\left(\frac{1}{|\vec{R}|^3}\right). \tag{652}$$

Equivalently,

$$\vec{S} = \hat{n}\frac{q^2}{16\pi^2|\vec{R}|^2}|\,\dot{\vec{v}}\,|^2\sin^2\Theta + \mathcal{O}\left(\frac{1}{|\vec{R}|^3}\right). \tag{653}$$

where Θ is the angle between $\hat{n}$ and $\dot{\vec{v}}$:

$$\dot{\vec{v}}\cdot\hat{n} = |\,\dot{\vec{v}}\,|\cos\Theta. \tag{654}$$

The power (energy per unit time) dP emitted through a segment $d\vec{A}$ of a closed surface around the retarded position of the charge q is given by

$$dP \equiv \frac{dW}{dt} = d\vec{A}\cdot\vec{S} \tag{655}$$

For a segment of a sphere with radius $|\vec{R}|$ centered around the retarded position of the charge, we have:

$$d\vec{A} = \hat{n}|\vec{R}|^2 d\Omega \tag{656}$$

and therefore

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2}|\,\dot{\vec{v}}\,|^2\sin^2\Theta + \mathcal{O}\left(\frac{1}{|\vec{R}|}\right) \tag{657}$$

The radiation power, i.e. the power which is radiated at infinitely large distances, per solid angle $d\Omega$ is:

$$\frac{dP_{\text{rad.}}}{d\Omega} = \frac{q^2}{16\pi^2}|\,\dot{\vec{v}}\,|^2\sin^2\Theta \tag{658}$$

The total power radiated at all solid angles surrounding the retarded position of the charge is:

$$P_{\text{rad.}} = \int d\Omega\frac{dP_{\text{rad.}}}{d\Omega} \tag{659}$$

Writing

$$d\Omega = \sin\Theta d\Theta d\phi \tag{660}$$

and performing the angular integrations we obtain:

$$P_{\text{rad.}} = \frac{q^2}{4\pi}\frac{2}{3}|\,\dot{\vec{v}}\,|^2. \tag{661}$$

Restoring the ϵ_0, c constants, we have:

$$P_{\text{rad.}} = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3}\frac{|\dot{\vec{v}}|^2}{c}. \tag{662}$$

This is known as Larmor's formula. It gives the total power of radiation of an accelerated charge when it has zero velocity. However, it can also be used as a good approximation for the radiation power emitted by accelerated charges with a small (non-relativistic) velocity $|\vec{v}| \ll c$.

9.4 Radiation from an accelerated charge with a relativistic velocity

The power of radiation

$$\text{Power} = \frac{\text{Energy}}{\text{Time}}$$

is invariant under Lorentz transformations. If a charge $+q$ is accelerated and moving with a velocity $\vec{v}$, we can make a Lorentz transformation to its rest frame and use Larmor's formula. Alternatively, we can write Larmor's formula in a manifestly Lorentz invariant form expressing 3-dimensional vectors in the rest frame with the corresponding four-vectors. Larmor's formula reads:

$$P_{\text{rad.}} = \frac{q^2}{4\pi}\frac{2}{3}|\dot{\vec{v}}|^2 = \frac{q^2}{4\pi}\frac{2}{3m^2}\left(\frac{d\vec{p}}{dt}\right)^2. \tag{663}$$

The relativistic force in the rest-frame is:

$$\frac{dp^\mu}{d\tau} = \left(0, \frac{d\vec{p}}{dt}\right) \tag{664}$$

and

$$\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = -\left(\frac{d\vec{p}}{dt}\right)^2. \tag{665}$$

We can then write:

$$P_{\text{rad.}} = -\frac{q^2}{4\pi}\frac{2}{3m^2}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}. \tag{666}$$

In a reference frame where the particle moves with velocity $\vec{v}$ the above expression gives:

$$P_{\text{rad.}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6\left[|\dot{\vec{v}}|^2 - \left|\vec{v}\times\dot{\vec{v}}\right|^2\right]. \tag{667}$$

9.4.1 Circular motion

Let us examine the case of a charge moving in a circular motion. The acceleration in that case is perpendicular to the velocity of the charge:

$$\vec{v} \perp \dot{\vec{v}} \,. \tag{668}$$

Eq. 667 gives that

$$P_{\text{rad.}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^4 |\, \dot{\vec{v}} \,|^2. \tag{669}$$

A charged particle in a cyclical orbit will radiate constantly. This observation tells us that a planetary-type model for atoms does not work. According to classical electrodynamics, electrons orbiting a nucleus would lose energy from emitting radiation continuously falling to lower and lower orbits and eventually collapsing on the nucleus. This does not happen due to quantum mechanics and the uncertainty principle. The electrons of an atom occupy discrete energy levels. In the ground state, which is the state of minimum allowed energy, they never radiate.

We also notice in Eq. 669 that the radiated energy

$$P_{\text{rad.}} \sim \gamma^4$$

is larger for particles with larger velocity (larger γ).

In a circular collider, such as the Large Hadron Collider, one has to compensate for the radiation loss (synchrotron radiation) due to the acceleration of the charged particles which keeps them in a circular orbit.

9.4.2 Linear accelerators

Let us now assume that a charge is accelerated in a straight line:

$$\vec{v} \parallel \dot{\vec{v}} \,.$$

Eq. 667 gives that the radiated power is:

$$P_{\text{rad.}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6 |\, \dot{\vec{v}} \,|^2. \tag{670}$$

For a relativistic charge the energy loss is

$$P_{\text{rad.}} \sim \gamma^6$$

9.5 Angular distribution of radiation from a linearly accelerated relativistic charge

In this section, we will study the angular dependence of radiation from a charge which is accelerated in a straight line. The electric field of an accelerated charge is:

$$\vec{E} = \frac{q}{4\pi(1-\hat{n}\cdot\vec{v})^3}\frac{1}{|\vec{R}|}\hat{n}\times\left[(\hat{n}-\vec{v})\times\dot{\vec{v}}\right] + \mathcal{O}\left(\frac{1}{\left|\vec{R}\right|^2}\right)\Bigg|_{\text{retarded}} \tag{671}$$

and for

$$\vec{v}\,\|\dot{\vec{v}},$$

it becomes

$$\vec{E} = \frac{q}{4\pi(1-\hat{n}\cdot\vec{v})^3}\frac{1}{|\vec{R}|}\hat{n}\times\left[\hat{n}\times\dot{\vec{v}}\right] + \mathcal{O}\left(\frac{1}{\left|\vec{R}\right|^2}\right)\Bigg|_{\text{retarded}} \tag{672}$$

The Poynting vector is:

$$\vec{S} = \hat{n}\frac{q^2}{16\pi^2|\vec{R}|^2}\frac{\left|\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right|^2}{(1-\hat{n}\cdot\vec{v})^6} + \mathcal{O}\left(\frac{1}{|\vec{R}|^3}\right). \tag{673}$$

The power of radiation through a solid angle $d\Omega$ at a retarded distance $|\vec{R}|$ is:

$$\frac{dP_{\text{rad.}}}{d\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\frac{\sin^2\Theta}{(1-v\cos\Theta)^6} \tag{674}$$

where Θ is the angle between the direction of the emitted radiation and the velocity (or, equivalently, the acceleration) of the particle.

Let us now examine the limit where the velocity of the particle is very close to the speed of light:

$$v \approx c = 1.$$

In that limit, the denominator of Eq. 674 becomes large for a collinear emission of radiation to the direction of motion:

$$\frac{1}{(1-v\cos\Theta)^6} \overset{v\approx 1}{\longrightarrow} \frac{1}{(1-\cos\Theta)^6} \overset{\Theta\to 0}{\longrightarrow} \infty.$$

Therefore, radiation tends to be collinear. To study better this double limit, $\Theta \to 0, v \to 1$, we observe that (**exercise**):

$$1 - v\cos\Theta \approx \frac{1}{2\gamma^2}(1 + \gamma^2\Theta^2) \tag{675}$$

and

$$\frac{dP_{\text{rad.}}}{d\Omega} \approx \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2 \frac{(\gamma\Theta)^2}{(1 + (\gamma\Theta)^2)^6}. \tag{676}$$

The distribution (**exercise:** plot it) vanishes for small and large values of $\gamma\Theta$. The maximum of the distribution is for values of

$$\gamma\Theta = \frac{1}{\sqrt{5}} = 0.4472\ldots$$

Therefore, we conclude that the radiation is emitted within a characteristic angle:

$$\Theta \sim \frac{1}{\gamma}. \tag{677}$$

10 Electrodynamics in a medium

In a macroscopic amount of matter we can identify two types of motions of charged particles:

- fast currents, due to the motions of charges in atoms and molecules,
- slow currents, which extend at distances much larger than the size of atoms due to free electrons or ions.

In our macroscopic measurements of the currents and the electromagnetic field, we are able to identify only the slow variations and we are insensitive to the fast ones taking place at the atomic level. We can write

$$
\begin{aligned}
A^{\mu} &= \langle A^{\mu} \rangle + A^{\mu}_{\text{atomic}} && (678)\\
j^{\mu} &= \langle j^{\mu} \rangle + j^{\mu}_{\text{atomic}} && (679)\\
\vec{E} &= \left\langle \vec{E} \right\rangle + \vec{E}_{\text{atomic}} && (680)\\
\vec{B} &= \left\langle \vec{B} \right\rangle + \vec{B}_{\text{atomic}} && (681)
\end{aligned}
$$

, where $\langle X \rangle$ is the macroscopic average of the quantity X. It is not clear how one should average over the atomic distances. However, we can parametrize our ignorance by means of macroscopic properties of the material which reflect its atomic structure and we can measure experimentally.

We introduce an averaging of a function over some distances via:

$$
\langle F(\vec{x}, t) \rangle = \int d^3\vec{y} f(\vec{y}) F(\vec{x} - \vec{y}, t). \qquad (682)
$$

The weighting factor $f(\vec{y})$ is unknown, but let's assume that it is a well-behaved smooth function. Starting from Maxwell equations, we have

$$
\begin{aligned}
\partial_{\mu} F^{\mu\nu} &= J^{\nu} \\
\rightsquigarrow \langle \partial_{\mu} F^{\mu\nu} \rangle &= \langle J^{\nu} \rangle \\
\rightsquigarrow \int d^3\vec{y} \partial_{\mu} F^{\mu\nu} (\vec{x} - \vec{y}, t) f(\vec{y}) &= \langle J^{\nu} \rangle \\
\partial_{\mu} \rightsquigarrow \int d^3\vec{y} F^{\mu\nu} (\vec{x} - \vec{y}, t) f(\vec{y}) &= \langle J^{\nu} \rangle \\
\partial_{\mu} \langle F^{\mu\nu} \rangle &= \langle J^{\nu} \rangle && (683)
\end{aligned}
$$

where we could pull out the derivative $\partial_{\mu} \equiv (\partial_t, \vec{\nabla}_{\vec{x}})$ from the integration, since it does not act on the integration variable $\vec{y}$.

10.1 Average of the charge density $J^0 = \rho$.

We separate the density of charges into

$$\rho = \rho_{\text{free}} + \rho_{\text{bound}} \tag{684}$$

where ρ_{free} is the density of charges which are capable of moving at large distances. We write

$$\rho_{free} = \sum_{i \in \text{ free}} q_i \delta(\vec{x} - \vec{x}_i). \tag{685}$$

The density ρ_{bound} corresponds to the charge density of molecules/atoms where the electric charges are constrained within regions of a small atomic size. We write

$$\rho_{\text{bound}} = \sum_{n \in \text{ molecules}} \rho_{(n)}, \tag{686}$$

where $\rho_{(n)}$ is the charge density of the $n-$th molecule. Let's assume that the $n-$th molecule consists of j charges q_j:

$$\rho_{(n)} = \sum_{j \in \ (n)} q_j \delta(\vec{x} - \vec{x}_n - \vec{x}_j), \tag{687}$$

where $\vec{x}_n$ is the position of the molecule and $\vec{x}_j$ is the position of the charge with respect to the "centre" of the molecule. The average is:

$$\begin{aligned} \left\langle \rho_{(n)} \right\rangle &= \sum_{j \in (n)} \langle q_j \delta(\vec{x} - \vec{x}_n - \vec{x}_j) \rangle \\ &= \sum_{j \in (n)} q_j \int d^3\vec{y} f(\vec{y}) \delta(\vec{x} - \vec{x}_n - \vec{x}_j - \vec{y}) \\ &= \sum_{j \in (n)} q_j f(\vec{x} - \vec{x}_n - \vec{x}_j). \end{aligned} \tag{688}$$

We can now perform a Taylor expansion in

$$\frac{|\vec{x}_j|}{|\vec{x} - \vec{x}_n|}, \tag{689}$$

the atomic scale over the macroscopic scale of observation. We obtain:

$$\begin{aligned} \left\langle \rho_{(n)} \right\rangle &= \sum_{j \in (n)} \left[q_j f(\vec{x} - \vec{x}_n) - (q_j \vec{x}_j) \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \ldots \right] \\ \rightsquigarrow \left\langle \rho_{(n)} \right\rangle &= \left(\sum_{j \in (n)} q_j \right) f(\vec{x} - \vec{x}_n) - \left(\sum_{j \in (n)} q_j \vec{x}_j \right) \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \ldots \end{aligned} \tag{690}$$

In the rhs, we recognise the total charge

$$q_n = \sum_{j \in (n)} q_j \tag{691}$$

and the dipole moment

$$\vec{p}_n = \sum_{j \in (n)} q_j \vec{x}_j \tag{692}$$

of the $n-$th molecule. Thus:

$$\left\langle \rho_{(n)} \right\rangle = q_n f(\vec{x} - \vec{x}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots \tag{693}$$

The total density, averaged over atomic distances, is:

$$\begin{aligned}
\langle \rho \rangle &= \langle \rho_{\text{free}} \rangle + \sum_n q_n f(\vec{x} - \vec{x}_n) - \sum_n \vec{p}_n \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots \\
&= \langle \rho_{\text{free}} \rangle + \int d^3\vec{y} f(\vec{y}) \sum_n q_n \delta(\vec{x} - \vec{x}_n - \vec{y}) - \vec{\nabla} \cdot \int d^3\vec{y} f(\vec{y}) \sum_n \vec{p}_n \delta(\vec{x} - \vec{x}_n - \vec{y}) + \dots \\
&= \langle \rho_{\text{free}} \rangle + \left\langle \sum_n q_n \delta(\vec{x} - \vec{x}_n) \right\rangle - \vec{\nabla} \cdot \left\langle \sum_n \vec{p}_n \delta(\vec{x} - \vec{x}_n) \right\rangle + \dots
\end{aligned} \tag{694}$$

The macroscopic charge density is now decomposed into

$$\langle \rho \rangle = \langle \rho_{\text{eff}} \rangle + \langle \rho_{\text{polarisation}} \rangle \tag{695}$$

where

$$\langle \rho_{\text{eff}} \rangle = \langle \rho_{\text{free}} \rangle + \left\langle \sum_n q_n \delta(\vec{x} - \vec{x}_n) \right\rangle \tag{696}$$

is an effective charge density and a macroscopic polarisation term:

$$\langle \rho_{\text{polarisation}} \rangle = -\vec{\nabla} \cdot \vec{P} \tag{697}$$

where

$$\vec{P} = \left\langle \sum_n \vec{p}_n \delta(\vec{x} - \vec{x}_n) \right\rangle \tag{698}$$

is the so-called polarisation of the medium.

To summarise, the average charge density can be written as:

$$\langle \rho \rangle = \langle \rho_{\text{eff}} \rangle - \vec{\nabla} \cdot \vec{P}. \tag{699}$$

10.2 Average of current density

The current density is:

$$\vec{J}(\vec{x},t) = \sum_k q_k \vec{v}_k \delta(\vec{x} - \vec{x}_k(t)). \tag{700}$$

We will restrict our analysis to motions which are not relativistic, $v_k \ll c$. We decompose

$$\vec{J} = \vec{J}_{\text{free}} + \vec{J}_{\text{bound}} \tag{701}$$

with

$$\vec{J}_{\text{bound}} = \sum_n \vec{J}_n, \quad n \in \text{ molecules} \tag{702}$$

and

$$\vec{J}_n = \sum_k q_k \frac{d\vec{x}}{dt} \delta(\vec{x} - \vec{x}_n - \vec{x}_k) = \sum_k q_k (\vec{v}_n + \vec{v}_k) \delta(\vec{x} - \vec{x}_n - \vec{x}_k). \tag{703}$$

For the average, we have:

$$\begin{aligned} \vec{J}_n &= \sum_k q_k \int d^3\vec{y} f(\vec{y})(\vec{v}_n + \vec{v}_k)\delta(\vec{x} - \vec{x}_n - \vec{x}_k - \vec{y}) \\ \rightsquigarrow \left\langle \vec{J}_n \right\rangle &= \sum_k q_k (\vec{v}_n + \vec{v}_k) f(\vec{x} - \vec{x}_n - \vec{x}_k) \end{aligned} \tag{704}$$

Once again, we expand

$$f(\vec{x} - \vec{x}_n - \vec{x}_k) \approx f(\vec{x} - \vec{x}_n) - \vec{x}_k \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \dots \tag{705}$$

Using identities such as

$$(\vec{a} \times \vec{b}) \times \vec{\nabla} f = \vec{a}(\vec{b} \cdot \vec{\nabla} f) - \vec{b}(\vec{a} \cdot \vec{\nabla} f) \tag{706}$$

and performing the summation over k and the atoms n we find that:

$$\begin{aligned} \left\langle \vec{J}_{\text{bound}} \right\rangle &= \sum_n q_n \delta(\vec{x} - \vec{x}_n)\vec{v}_n + \frac{d}{dt} \left\langle \sum_n \vec{p}_n \delta(\vec{x} - \vec{x}_n) \vec{v}_n \right\rangle \\ &\quad + \vec{\nabla} \times \left\langle \sum_n \sum_{j \in (n)} \frac{q_j}{2} \left(x_j \times \frac{dx_j}{dt} \right) \right\rangle + \left\langle \sum_n (\vec{p}_n \times \vec{v}_n) \delta(\vec{x} - \vec{x}_n) \vec{v}_n \right\rangle \\ &\quad + \text{higher order terms} \end{aligned} \tag{707}$$

Thus,

$$\begin{aligned}\left\langle \vec{J}_{\text{bound}} \right\rangle &\approx \left\langle \vec{J}_{\text{atoms}} \right\rangle + \vec{J}_{\text{polarisation}} + \vec{J}_{\text{magnetic}} \\ &\quad + \text{a small term for small } \vec{v}_n \end{aligned} \tag{708}$$

where

$$\vec{J}_{\text{polarisations}} = \frac{\partial \vec{P}}{\partial t}, \quad \vec{P} = \left\langle \sum_n \vec{p}_n \delta(\vec{x} - \vec{x}_n) \right\rangle, \tag{709}$$

$$\vec{J}_{\text{magnetic}} = \vec{\nabla} \times \vec{M}, \quad \vec{M} = \left\langle \sum_n \vec{m}_n \delta(\vec{x} - \vec{x}_n) \right\rangle \tag{710}$$

and $\vec{m}_n$ is the magnetic moment of an atom:

$$\vec{m}_n = \sum_j \frac{q_j}{2} (\vec{x}_j \times \vec{v}_j). \tag{711}$$

The total average of the current is then

$$\left\langle \vec{J} \right\rangle = \left\langle \vec{J}_{\text{free}} \right\rangle + \left\langle \vec{J}_{\text{bound}} \right\rangle. \tag{712}$$

Defining an effective current density

$$\vec{J}_{\text{eff}} = \left\langle \vec{J}_{\text{free}} \right\rangle + \left\langle \vec{J}_{\text{atoms}} \right\rangle \tag{713}$$

we have

$$\left\langle \vec{J} \right\rangle \approx \vec{J}_{free} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M}. \tag{714}$$

10.3 Maxwell equations in a medium

Having computed the average charge and current density $\langle J^\mu \rangle$, and substituting their expressions from Eq. 699 and Eq. 714 into

$$\partial_\mu \langle F^{\mu\nu} \rangle = \langle J^\nu \rangle, \tag{715}$$

we obtain:

$$\vec{\nabla} \cdot \left\langle \vec{B} \right\rangle = 0 \tag{716}$$

$$\vec{\nabla} \times \left\langle \vec{E} \right\rangle = -\frac{\partial \left\langle \vec{B} \right\rangle}{\partial t}, \tag{717}$$

$$\vec{\nabla}\cdot\left(\left\langle\vec{E}\right\rangle+\frac{\vec{P}}{\epsilon_0}\right) = \frac{\rho_{\rm eff}}{\epsilon_0} \tag{718}$$

and

$$\vec{\nabla}\times\left(\left\langle\vec{B}\right\rangle-\frac{\vec{M}}{c^2\epsilon_0}\right) = \frac{\vec{J}_{\rm eff}}{c^2\epsilon_0}+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E}+\frac{\vec{P}}{\epsilon_0}\right). \tag{719}$$

10.3.1 The $\vec{D}$ and $\vec{H}$ field

Historically, Maxwell equations in matter have been often presented in terms of the combinations:

$$\vec{D}\equiv\epsilon_0\left\langle\vec{E}\right\rangle+\vec{P} \tag{720}$$

and

$$\vec{H}\equiv\left\langle\vec{B}\right\rangle-\frac{\vec{M}}{c^2\epsilon_0}. \tag{721}$$

Dropping the averaging symbol (averaging is always assumed implicitly), Maxwell equations in the medium take the form:

$$\vec{\nabla}\cdot\vec{D} = \rho \tag{722}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{723}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{724}$$

$$\vec{\nabla}\times\vec{H} = \frac{\vec{J}}{c^2\epsilon_0}+\frac{1}{\epsilon_0 c^2}\frac{\partial\vec{D}}{\partial t}. \tag{725}$$

10.4 Maxwell equations inside a dielectric material

Assume a dielectric material, such as air or water, with no magnetisation $\vec{M}=0$ but with the ability to acquire a polarisation $\vec{P}$. Maxwell equations take the form:

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{E}+\frac{\vec{P}}{\epsilon_0}\right) &= \frac{\rho}{\epsilon_0}\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\cdot\vec{B} &= 0\end{aligned} \tag{726}$$

$$\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0 c^2} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\vec{E} + \frac{\vec{P}}{\epsilon_0} \right) \tag{727}$$

To solve these equations, we need information about the polarisability $\vec{P}$ of the medium. $\vec{P}$ and $\vec{E}$ are not truly independent. In fact, the larger the electric field $\vec{E}$ the more it will stretch the atoms and molecules in the medium inducing a larger polarisation $\vec{P}$. Ignoring non-linear effects, we can write a phenomenological proportionality relation of the electric field and the polarisability,

$$\vec{P} = \chi \epsilon_0 \vec{E}. \tag{728}$$

where χ is known as the "electric susceptibility" of the dielectric medium. Maxwell equations then become:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0 (1+\chi)} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} &= 0 && (729) \\
\vec{\nabla} \times \vec{B} &= \frac{\vec{J}}{\epsilon_0 c^2} + \frac{1+\chi}{c^2} \frac{\partial \vec{E}}{\partial t}. && (730)
\end{aligned}$$

Defining

$$c_m = \frac{c}{\sqrt{1+\chi}} \tag{731}$$

and

$$\epsilon = (1+\chi)\epsilon_0, \tag{732}$$

Maxwell equations take the same form as in the vacuum,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} &= 0 && (733) \\
\vec{\nabla} \times \vec{B} &= \frac{\vec{J}}{\epsilon c_m^2} + \frac{1}{c_m^2} \frac{\partial \vec{E}}{\partial t}. && (734)
\end{aligned}$$

with the electric permittivity and the speed of light constants replaced by

$$\epsilon_0 \to \epsilon, \quad c \to c_m. \tag{735}$$

10.5 Waves in a dielectric medium

Obviously, the solutions that we have found for Maxwell equations in the vacuum are also solutions of Maxwell equations in a dielectric, as long as we make the replacement of Eq. 735. Away from macroscopic charges and currents, we have (as in the vacuum) that the electric and magnetic field satisfy differential equations of the type:

$$\left[\frac{1}{c_m^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{E}(\vec{x},t) = 0, \tag{736}$$

which admit solutions

$$\vec{E} = \vec{E}_0 E^{i(\omega t - \vec{k}\cdot\vec{x})}. \tag{737}$$

This represents a wave with frequency

$$\omega = 2\pi f \tag{738}$$

and phase-velocity

$$v_{\text{phase}} = \frac{\omega}{\left|\vec{k}\right|}. \tag{739}$$

Substituting the solution into the differential equation, we obtain

$$k^2 = \frac{\omega^2}{c_m^2} = \frac{\omega^2}{c^2}(1+\chi) \tag{740}$$

For the phase-velocity, we have

$$u_{\text{phase}} = \frac{c}{n} \tag{741}$$

where

$$n = \sqrt{1+\chi} \tag{742}$$

is the so-called refraction index and characterises the dielectric material.

10.6 A model for the dielectric susceptibility χ

To make further progress in understanding the properties of electromagnetic waves inside dielectrics, we need to calculate $\chi = n^2 - 1$ which characterises the material. Equivalently, we need to calculate the polarisability $\vec{P}$ of the material when we subject it in an electric field $\vec{E}$.

In a dielectric, positive and negative charges are bound together inside atoms or molecules. To a good approximation, we can view the dielectric as a collection of dipoles each with a positive and a negative charge $(+q, -q)$. The charges in a dipole are bound together due to their electric attraction but they cannot come to arbitrarily close distances due to the laws of quantum mechanics. Making a full account of the quantum effects which are at play is beyond our scope. However, we can approximate phenomenologically the resulting binding force as a harmonic oscillator. If a charge separates away from the dipole, a force will pull it back and if it moves closer to the other charge a force will push it away. As we have seen, electric charges which accelerate radiate and lose energy. To account for this effect, we can also include a friction or drag force. Let's assume that the electric field is in the $\hat{x}$ direction. The force acting on a charge $+q$ of a dipole is then

$$F = qE = m(\ddot{x} + \gamma \dot{x} + \omega_0^2 x). \tag{743}$$

For a time varying electric field

$$E = E_0 e^{i\omega t}, \tag{744}$$

we find solutions of the form:

$$x = x_0 e^{i\omega t}, \tag{745}$$

with

$$x_0 = \frac{\frac{qE_0}{m}}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{746}$$

Therefore,

$$\vec{x} = \frac{\frac{q}{m}}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}, \tag{747}$$

and the electric dipole moment is

$$\vec{p} = q\vec{x} = \frac{\frac{q^2}{m}}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}. \tag{748}$$

Assuming a constant density of $+q$ charges N, the polarisability of the dielectric is:

$$\vec{P} = N\vec{p} = Nq\vec{x} = \frac{Nq^2}{m}\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}. \tag{749}$$

The dielectric susceptibility is then

$$\chi = n^2 - 1 = \frac{Nq^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{750}$$

Equivalently, the square of the index of refraction is

$$n^2 = 1 + \frac{Nq^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{751}$$

10.7 The complex index of refraction

We have found that the index of refraction is a complex number:

$$n = n_R - i n_I \tag{752}$$

A plane wave propagating in the dielectric is:

$$\vec{E} = \vec{E}_0 e^{i\omega\left[t - n\hat{k}\cdot\vec{x}\right]} = \vec{E}_0 e^{i\omega\left[t - \frac{n_R}{c}\hat{k}\cdot\vec{x}\right]} e^{-\omega\frac{n_I}{c}\hat{k}\cdot\vec{x}}. \tag{753}$$

The amplitude of the plane wave is

$$\left|\vec{E}\right| = \left|\vec{E}_0\right| e^{-\omega\frac{n_I}{c}\hat{k}\cdot\vec{x}} \tag{754}$$

and it decreases (for the physical case $n_I > 0$) as the wave penetrates further inside the material.

10.8 Waves in metals

Metals are conductors permitting electrons to move freely at large distances. We can compute their index of refraction as a special limit of Eq. 751 by setting

$$\omega_0 = 0.$$

This corresponds to a zero force for binding the charges to a fixed position as it was the case for dielectrics. The index of refraction is then:

$$n^2 = 1 + \frac{Nq^2}{m\epsilon_0} \frac{1}{-\omega^2 + i\omega\gamma}. \tag{755}$$

The density N can be obtained from macroscopic properties of the metal. The constant γ is an intrinsic parameter of our model. To make contact with reality and test our theory of wave propagation in metals, we must find a way to relate γ to an experimentally measured quantity.

In our model, the equation of motion for the electrons in the metal can be obtained by solving the differential equation:

$$qE = m(\ddot{x} + \gamma \dot{x}) \tag{756}$$

which is a first order differential equation for the velocity v of the electrons:

$$qE = m(\dot{v} + \gamma v) \tag{757}$$

with a general solution:

$$\vec{v} = \frac{q\vec{E}}{m\gamma} + \vec{v}_0 e^{-\gamma t} \tag{758}$$

The second term is exponentially decreasing with time. Therefore, the electrons after some time drift in the metal with a practically constant velocity:

$$\vec{v}_{\text{drift}} = \frac{q\vec{E}}{m\gamma}. \tag{759}$$

This corresponds to a current density

$$\vec{J} = Nq\vec{v}_{\text{drift}} = \frac{Nq^2}{m\gamma}\vec{E}. \tag{760}$$

Our model has lead us to Ohm's law:

$$\vec{J} = \sigma \vec{E}, \tag{761}$$

where σ is the conductivity of the metal.
Exercise: What is the relation of σ and the resistance R in

$$V = IR?$$

We therefore obtain:

$$\gamma = \frac{Nq^2}{m\sigma} \tag{762}$$

To get a physical picture for the propagation of waves in metals, let us look at the low and high frequency limits.

10.8.1 Low frequency approximation

In the low frequency limit, $\omega \to 0$, we will assume that $\omega\gamma \gg \omega^2$. Eq. 755 gives:

$$n^2 = -i\frac{\sigma}{\epsilon_0 \omega}. \tag{763}$$

The refraction index is then

$$n = \sqrt{\frac{\sigma}{2\epsilon_0 \omega}}(1-i). \tag{764}$$

The index of refraction has a large imaginary part (as large as the real part). The amplitude of a wave reduces exponentially with the distance x as it propagates inside the conductor:

$$\left|\vec{E}\right| = \left|\vec{E}\right| e^{-\frac{x}{\delta}} \tag{765}$$

where the penetration distance is:

$$\delta = \sqrt{\frac{2\epsilon_0 c^2}{\sigma\omega}}. \tag{766}$$

For copper and a microwave frequency $\nu = \frac{\omega}{2\pi} = 10^4$MHz the penetration length is

$$\delta \approx 6.710^{-5} cm \tag{767}$$

(see Feynman Lectures Vol. 2, 32-7.).

10.8.2 High frequency approximation

In the high frequency limit, $\omega^2 \gg \omega\gamma$, the refraction index becomes:

$$n^2 \approx 1 - \frac{Nq^2}{m\epsilon_0}\frac{1}{\omega^2} = 1 - \frac{\omega_P^2}{\omega^2}, \tag{768}$$

where

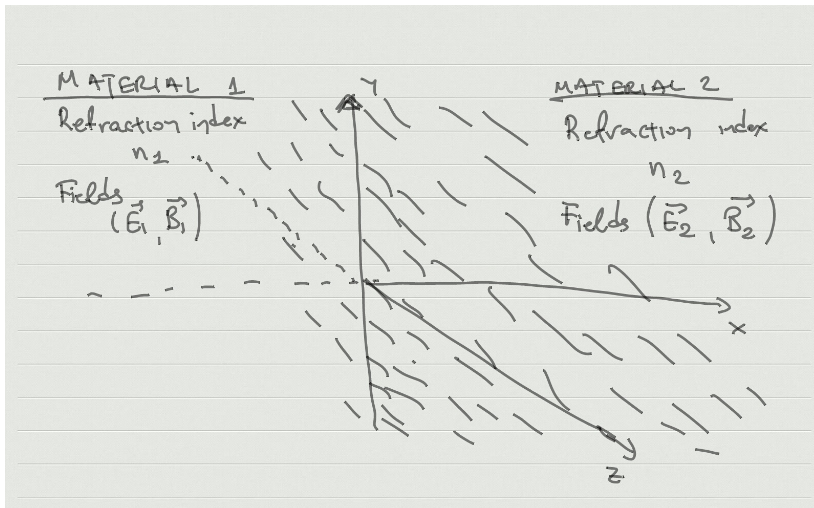
$$\omega_P^2 = \frac{Nq^2}{m\epsilon_0}. \tag{769}$$

For frequencies which are smaller than the "plasma frequency" $\omega < \omega_P$, the refraction index is imaginary and the wave is damped after some distance inside the metal. For frequencies larger than the plasma frequency, $\omega > \omega_P$ the refraction index is real and the metal becomes transparent to the electromagnetic wave.

Exercise: Calculate the wavelengths $\lambda_P = \frac{2\pi c}{\omega_P}$ for Li, Na, K, Rb. (see Feynman Lectures Vol. 2, 32-7.)

10.9 Reflection and refraction

Consider two materials with refraction indices n_1 and n_2 separated by a boundary surface on the $y-z$ plane as in the picture below:



Maxwell equations should govern both regions, as well as the "intermediate" boundary region. Let's assume no macroscopic charges and currents in the two media. The first Maxwell equation is

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \vec{P}. \tag{770}$$

Written explicitly in terms of components, we have:

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = -\frac{1}{\epsilon_0} \left(\partial_x P_x + \partial_y P_y + \partial_z P_z \right) \tag{771}$$

Derivatives with respect to the $x-$variable in the boundary regions compare the fields on the two sides of the boundary. We anticipate them to be larger than derivatives with respect to y or z or t, which compare the fields on the same side of the boundary or at different times. If this is the case, for the boundary region, the above Maxwell equation becomes:

$$\partial_x E_x = -\frac{1}{\epsilon_0} \partial_x P_x \tag{772}$$

or, equivalently,

$$\frac{E_{x2} - E_{x1}}{\Delta x} = -\frac{1}{\epsilon_0}\frac{P_{x2} - P_{x1}}{\Delta x} \tag{773}$$

which yields our first boundary condition:

$$\epsilon_0 E_{x2} + P_{x2} = \epsilon_0 E_{x1} + P_{x1}. \tag{774}$$

Let us now work similarly with another Maxwell equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{775}$$

Decomposing in components, the above encompasses three equations. For example,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \tag{776}$$

contains no large $\frac{\partial}{\partial x}$ derivatives and it yields no boundary condition. However,

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \tag{777}$$

yields

$$\frac{\partial E_z}{\partial x} = 0 \rightsquigarrow E_{z_1} = E_{z_2}. \tag{778}$$

Similarly, we obtain

$$E_{y_1} = E_{y_2} \tag{779}$$

From the remaining Maxwell equations, we obtain (**exercise**) the boundary condition:

$$\vec{B}_2 = \vec{B}_1. \tag{780}$$

In summary, we have obtained the following boundary conditions for the electromagnetic field:

$$\vec{B}_1 = \vec{B}_2 \tag{781}$$

$$\vec{E}_{1,\parallel} = \vec{E}_{2,\parallel} \tag{782}$$

$$(\epsilon_0 \vec{E}_1 + \vec{P}_1)_\perp = (\epsilon_0 \vec{E}_2 + \vec{P}_2)_\perp \tag{783}$$

where $\perp$ denotes the component perpendicular to the boundary plane and $\parallel$ the components parallel to it.

10.9.1 Snell's law

Let us now assume that an electromagnetic plane-wave $(\vec{E}_I, \vec{B}_I)$ approaches from medium n_1 the boundary. Experience tells us that there will be a reflected $(\vec{E}_R, \vec{B}_R)$ and a transmitted electromagnetic field $(\vec{E}_T, \vec{B}_T)$. The electric field and magnetic fields on the two sides of the boundary will be:

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R, \quad \vec{E}_2 = \vec{E}_T, \tag{784}$$

and

$$\vec{B}_1 = \vec{B}_I + \vec{B}_R, \quad \vec{B}_2 = \vec{B}_T, \tag{785}$$

with

$$\vec{E}_I = \hat{e}_I E_I e^{i\left(\omega_I t - \vec{k}_I \cdot \vec{x}\right)}, \tag{786}$$

$$\vec{E}_R = \hat{e}_R E_R e^{i\left(\omega_R t - \vec{k}_R \cdot \vec{x}\right)}, \tag{787}$$

$$\vec{E}_T = \hat{e}_T E_T e^{i\left(\omega_T t - \vec{k}_T \cdot \vec{x}\right)} \tag{788}$$

and

$$\vec{B}_I = \frac{\vec{k}_I \times \vec{E}_I}{\omega_I}, \quad \vec{B}_R = \frac{\vec{k}_R \times \vec{E}_R}{\omega_R}, \quad \vec{B}_T = \frac{\vec{k}_T \times \vec{E}_T}{\omega_T}. \tag{789}$$

The electric and magnetic fields are perpendicular to the direction of propagation:

$$\vec{k}_I \cdot \hat{e}_I = \vec{k}_R \cdot \hat{e}_R = \vec{k}_T \cdot \hat{e}_T = 0. \tag{790}$$

For the magnitudes of the $\vec{k}$ wave-vectors we have:

$$\frac{k_I}{\omega_I} = \frac{k_R}{\omega_R} = \frac{n_1}{c}, \quad \frac{k_T}{\omega_T} = \frac{n_2}{c}. \tag{791}$$

From the boundary condition,

$$\vec{E}_{1,\parallel} = \vec{E}_{2,\parallel},$$

we have that on the boundary

$$\vec{x} = (0, y, z) \tag{792}$$

the electric fields satisfy

$$\hat{e}_{I,\parallel} E_I e^{i\left(\omega_I t - \vec{k}_{I,\parallel} \cdot \vec{x}\right)} + \hat{e}_{R,\parallel} E_R e^{i\left(\omega_R t - \vec{k}_{R,\parallel} \cdot \vec{x}\right)} = \hat{e}_{T,\parallel} E_T e^{i\left(\omega_T t - \vec{k}_{T,\parallel} \cdot \vec{x}\right)}, \tag{793}$$

where $\vec{k}_{\parallel} = (0, k_y, k_z)$. Let us choose to evaluate Eq. 793 at

$$\vec{x} = (0, 0, 0). \tag{794}$$

Then, we obtain:

$$\hat{e}_{I,\parallel} E_I e^{i\omega_I t} + \hat{e}_{R,\parallel} E_R e^{i\omega_R t} = \hat{e}_{T,\parallel} E_T e^{i\omega_T t} \tag{795}$$

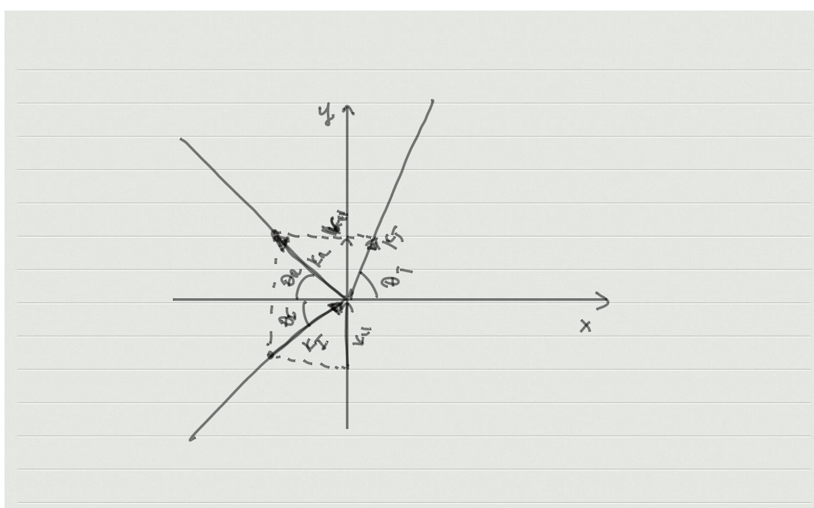
Let's assume that we arrange for the above equation to be satisfied at $t = 0$. If the frequencies $\omega_I, \omega_R, \omega_T$ are different from each other, then the three terms of the equation will change differently at an arbitrary later time $t > 0$ and the equation will not be satisfied any longer. We thus conclude that the frequencies of the transmitted and reflected light are the same as the frequency of the incident light:

$$\omega_I = \omega_R = \omega_T = \omega. \tag{796}$$

Similarly, to satisfy Eq. 793 at $t = 0, \vec{x} = ((), 0, z)$ and $t = 0, \vec{x} = ((), y, 0)$ yields that

$$\vec{k}_{I,\parallel} = \vec{k}_{R,\parallel} = \vec{k}_{T,\parallel} = \vec{k}_{\parallel}. \tag{797}$$

From the above, we conclude that the incident, the reflected and the transmitted waves propagate on the same plane.



Assuming that the refraction indices n_1, n_2 are real, Eq. 797 yields a relation for the incident, reflected and transmitted wave angles:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T. \tag{798}$$

Recalling Eq. 791, we obtain:

$$n_1 \sin\theta_I = n_1 \sin\theta_R = n_2 \sin\theta_T. \tag{799}$$

We conclude that the incident and reflected wave angles are equal,

$$\theta_I = \theta_R \tag{800}$$

and that the transmitted wave angle is:

$$\sin\theta_T = \frac{n_1}{n_2} \sin\theta_I. \tag{801}$$

Eq. 801 is the familiar law of Snell in optics.

10.9.2 Polarisation

Let us align the $y-$axis so that

$$\vec{k}_{\parallel} = k_{parallel}\hat{y}. \tag{802}$$

The electric field and the magnetic fields are perpendicular to the wave-vector $\vec{k}$. We distinguish two possibilities:

- $\vec{E}_I \parallel \vec{z}$ and $\vec{B}_I$ lies on the $x-y$ plane
- $\vec{B}_I \parallel \vec{z}$ and $\vec{E}_I$ lies on the $x-y$ plane

All other polarisations can be calculated as a superposition of these cases.

For, $\vec{e}_I = \hat{y}$, Eq. 793 gives that

$$E_I + E_R = E_T. \tag{803}$$

Also, from

$$\vec{B}_1 = \vec{B}_2 \rightsquigarrow B_{I,x} + B_{R_x} = B_{T,x} \rightsquigarrow k_{I,x}E_I + k_{R,x}E_R = k_{T,x}E_T \tag{804}$$

The two equations yield for the reflected and transmitted waves:

$$E_T = \frac{2k_{I,x}}{k_{I,x} + k_{T,x}} E_I, \tag{805}$$

and

$$E_R = \frac{k_{I,x} - k_{T_x}}{k_{I,x} + k_{T,x}} E_I, \tag{806}$$

where we have used that

$$k_{R,x} = -k_{I,x}.$$

Exercise: Calculate E_T, E_R in terms of θ_I, θ_T.
Exercise: Calculate E_T, E_R for the polarisation $\vec{B} \parallel \hat{z}$.

10.10 Cherenkov radiation

In a dielectric medium, like air of water, Maxwell equations take approximately the same form as in the vacuum where we substitute

$$c \to c_m = \frac{c}{\sqrt{1+\chi}}, \quad \epsilon_0 \to \epsilon = \epsilon_0(1+\chi). \tag{807}$$

Therefore, the solutions of Maxwell equations for the four-vector potential A^μ in the vacuum are also solutions of Maxwell-equations in a dielectric as long as we adjust appropriately the constants c, ϵ_0.

In the vacuum, all particles propagate with a velocity smaller than the speed of light. However, in a medium electrically charged particles can have a velocity which is bigger than the speed of light in that medium:

$$c_m < v < c.$$

Our solutions of Maxwell equations have assumed so far that $v < c_m$. It is interesting to see what is $A^\mu(x)$ in the case that some currents j^μ contain charged particles with

$$v > c_m.$$

Consider a single charge q travelling with a uniform velocity $v > c_m$. In units of $c_m = 1, \epsilon = 1$, the solution of Maxwell equations takes the form:

$$A^\mu(x) = \int d^4x' G(x - x') J^\mu(x') \tag{808}$$

where

$$G(x - x') = \frac{1}{2\pi}\delta\left((x - x')^2\right)\Theta(x^0 - x'^0) \tag{809}$$

and

$$J^\mu(x') = q\frac{dx'^\mu}{dt'}\delta(\vec{x}' - \vec{v}t'). \tag{810}$$

Identifying

$$x^\mu \equiv (t, \vec{x}), \quad x'^\mu \equiv (t', \vec{x}'),$$

we have

$$A^\mu(x) = \frac{(q, q\vec{v})}{2\pi}\int dt'\,\delta\left((t - t')^2 - |\vec{x} - \vec{v}t'|^2\right)\Theta(t > t'). \tag{811}$$

Let us set

$$\vec{R} \equiv \vec{x} - \vec{v}t,$$

and

$$\delta t \equiv t - t'.$$

As usual, we will now find the roots of the equation which requires the argument of the delta function to vanish:

$$\begin{aligned} 0 &= (t - t')^2 - |\vec{x} - \vec{v}t'|^2 \\ &= \delta t^2 - \left|\vec{R} + \vec{v}\delta t\right|^2 \\ &= \delta t^2(1 - v^2) - 2Rv\cos\theta\delta t - R^2, \end{aligned} \tag{812}$$

where θ is the angle formed by $\vec{R}$ and $\vec{v}$:

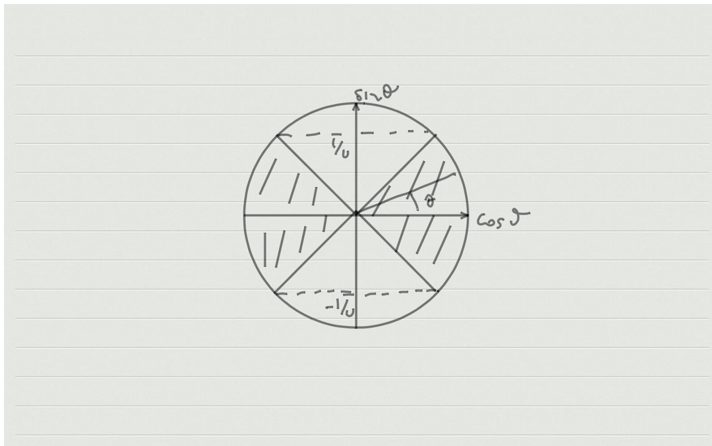
$$\vec{R}\cdot\vec{v} = Rv\cos\theta.$$

The discriminant of the binomial in Eq. 812 is

$$\Delta = 4R^2v^2\cos^2\theta + 4(1 - v^2)R^2 = 4R^2v^2\left[\frac{1}{v^2} - \sin^2\theta\right]. \tag{813}$$

For $v < 1$, the discriminant is positive and the delta-constraint has always a real solution. In our case $v > 1$, we have a solution only when

$$-\frac{1}{v} < \sin\theta < \frac{1}{v} \tag{814}$$



Outside this interval, there are no solutions for the delta-function constraint and the potential is zero. The solutions of the binomial equation are:

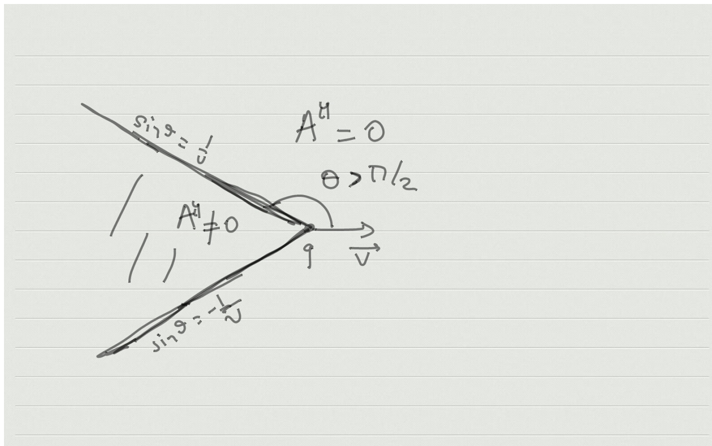
$$\delta t = \frac{Rv}{v^2 - 1}\left[-\cos\theta \pm \sqrt{\frac{1}{v^2} - \sin^2\theta}\right] \tag{815}$$

For $v > 1$, the square root in the bracket is smaller in magnitude than the first term $-\cos\theta$ in the bracket. The Theta-function constraint $\Theta(x^0 > x'^0)$requires that $\delta t > 0$. This can only occur for

$$\cos\theta < 0 \rightsquigarrow \theta > \frac{\pi}{2} \tag{816}$$

In summary, the vector potential is non-zero only in a cone trailing the charged particle, defined by the angle

$$\theta > \frac{\pi}{2}, \quad -\frac{1}{v} < \sin\theta < \frac{1}{v} \tag{817}$$



(see Visualisation of Cherenkov radiation) Computing the value of the potential inside the cone is straightforward, yielding:

$$A^{\mu} = \frac{(q, q\vec{v})}{2\pi R}\sqrt{1 - v^2 \sin^2\theta} \tag{818}$$

The gradient of the vector potential is very large as we move from the inner side of the cone, where the potential is non-zero, to the outside of the cone where the potential is zero and gives value to a large value of the electromagnetic field across the cone boundary. This is the Cherenkov radiation associated with the superluminal motion of charged particles in a dielectric material.

Recently, a lot of excitement was created due to a mistaken experimental measurement of the OPERA collaboration which gave a velocity for neutrinos bigger than the speed of light in the vacuum travelling from CERN in Geneva to their neutrino detector in Gran Sasso in Italy. Theorists raised skepticism about this measurement which was not fitting expectations of Cherenkov radiation phenomena. Although neutrinos do not have electric charge, they do have "weak-force" charge and they are expected to slow down as they travel by emitting Cherenkov "weak-radiation". The neutrinos detected in Gran Sasso were measured to be more energetic, as if they were not losing energy during their travel in contradiction to expectations. The measurement turned out to be wrong and when repeated correctly the velocity of the neutrinos was consistent with $v < c$. (see superluminal neutrinos)

11 Scattering

Consider an electric charge $+q$ which is stricken by an incident electromagnetic field with a certain frequency ω. An electromagnetic force is then exerted on the charge which accelerates it. Consequently, the electric charge will emit radiation. In this Section, we will study the characteristics of the scattered radiation by the accelerated charge. We will distinguish two cases. In one (Thomson scattering), the charge is free as it happens inside a conductor or plasma. In the second (Rayleigh scattering), the charge is bound inside an atom or a molecule.

11.1 Thomson scattering

Consider a free charge q in its rest frame $\vec{v} = 0$ (or, with a very small non-relativistic velocity $v \approx 0$) stricken by an electromagnetic field:

$$\vec{E} = \hat{e} E \Re e^{i(\omega t - \vec{k} \cdot \vec{x})} \tag{819}$$

The force acting on the charge, placed at the position $\vec{x} = 0$, is

$$\vec{F} = q\vec{E} \tag{820}$$

and the acceleration of the charge will be:

$$\dot{\vec{v}} = \frac{\vec{F}}{m} = \hat{e}\frac{qE}{m}\Re e^{i\omega t} = \hat{e}\frac{qE}{m}\cos(\omega t) \tag{821}$$

Due to its acceleration, the charge emits radiation with a power angular distribution given by Larmor's formula:

$$\frac{dP_{\text{rad.}}}{d\Omega} = \frac{q^2}{16\pi^2}\dot{\vec{v}}^2 \sin^2\Theta = \frac{q^4}{16\pi^2 m^2}\sin^2\Theta E^2 \cos^2(\omega t). \tag{822}$$

where Θ is the angle between the direction of the acceleration $\hat{e}$ and the direction $\hat{n}$ of the emitted radiation. Averaging over time, we have

$$\frac{1}{T}\int_0^T dt \cos^2(\omega t) = \frac{1}{2}, \quad T = \frac{2\pi}{\omega}. \tag{823}$$

and thus

$$\left\langle \frac{dP_{\text{rad.}}}{d\Omega} \right\rangle = \frac{q^4}{32\pi^2 m^2} E^2 \sin^2\Theta. \tag{824}$$

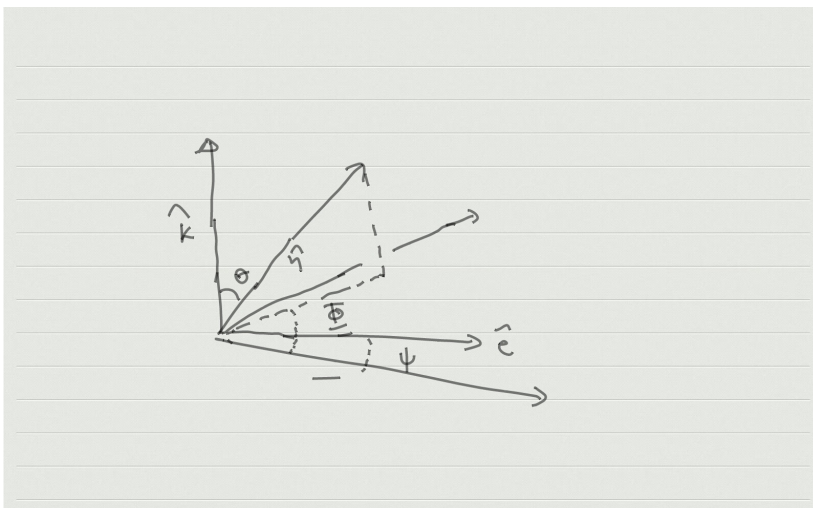
The average incoming flux of radiation is the time-average of the incoming Poynting vector:

$$\langle S \rangle = \left\langle E^2 \cos^2(\omega t) \right\rangle = \frac{E^2}{2}. \tag{825}$$

The differential cross-section is defined as the ratio of the outgoing radiation power per unit solid angle divided by the incoming flux of energy:

$$\frac{d\sigma}{d\Omega} \equiv \frac{\left\langle \frac{dP_{\text{rad.}}}{d\Omega} \right\rangle}{\langle S \rangle} = \frac{q^4}{16\pi^2 m^2} \sin^2 \Theta \tag{826}$$

It has units of a surface. It is experimentally better to find the cross-section distribution on the angle θ formed by the propagation direction $\hat{k}$ of the incident electromagnetic field and the direction of the outgoing radiation.



We can write:

$$\sin^2 \Theta = 1 - \sin^2 \theta \cos^2(\phi - \psi). \tag{827}$$

Averaging over the azimuthal angle ψ of the polarisation vector, we have

$$\frac{1}{2\pi} \int_0^{2\pi} d\psi \cos^2(\phi - \psi) = \frac{1}{2}. \tag{828}$$

Thus, the differential cross-section for unpolarised incoming photon beam is:

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{16\pi^2 m^2} \frac{1 + \cos^2 \theta}{2} \tag{829}$$

The total cross-section is obtained by integrating the differential cross-section over all solid angles:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{q^4}{16\pi^2 m^2}\frac{8\pi}{3} = \frac{8\pi}{3} r_q^2 \tag{830}$$

where

$$r_q \equiv \frac{q^2}{4\pi\epsilon_0 mc^2} \tag{831}$$

For an electron, the Thomson radius is:

$$r_e \approx 2.8210^{-13} cm. \tag{832}$$

Notice that the cross-section is independent of the frequency of the incoming electromagnetic wave. Our result for the cross-section receives significant quantum corrections for frequencies $\hbar\omega \sim mc^2$.

11.2 Rayleigh scattering

We now examine the scattering of an electromagnetic wave on an electric charge which is bound in an atom or molecule. Then, the equation of motion of the charge is given by solving

$$\frac{qE}{m} = \ddot{x} + \gamma \dot{x} + \omega_0^2 x \tag{833}$$

with a solution:

$$\vec{x} = \frac{\frac{q\vec{E}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}. \tag{834}$$

The acceleration is then:

$$\dot{\vec{v}} = \ddot{\vec{x}} = \frac{\frac{q\ddot{\vec{E}}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega} = \frac{q\vec{E}}{m}\frac{-\omega^2}{\omega_0^2 - \omega^2 + i\gamma\omega} \tag{835}$$

Unlike Thomson scattering where the electron is free, the acceleration is now frequency dependent. We can write

$$\dot{\vec{v}} = \dot{\vec{v}}_{\text{Thomson}} \frac{-\omega^2}{\omega_0^2 - \omega^2 + i\gamma\omega}, \tag{836}$$

where

$$\dot{\vec{v}}_{\text{Thomson}} = \frac{q\vec{E}}{m}, \tag{837}$$

the acceleration of the charge if it were free. The calculation of the cross-section proceeds identically as in Thomson scattering. We find

$$\sigma = \sigma_{\text{Thomson}} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \tag{838}$$

with

$$\sigma_{\text{Thomson}} \equiv \frac{8\pi}{3} r_q^2. \tag{839}$$

For $\omega \ll \omega_0$, the process is known as Rayleigh scattering and the cross-section has a characteristic ω^4 frequency dependence:

$$\sigma_{\text{Rayleigh}} = \sigma_{\text{Thomson}} \frac{\omega^4}{\omega_0^4} \tag{840}$$

High frequencies are scattered more than lower frequencies. Rayleigh scattering takes place in the atmosphere. Sunlight contains all visible light frequencies, however the blue light (highest visible frequency) scatters the most. This explains why the light we see away from the sun line of sight is more blue while it is more yellow or red (during sunset/sunrise) when looking in the direction of the sun. (see A video on blue sky)

12 Lagrangian formalism of Electrodynamics

In classical mechanics, we can obtain equations of motion by requiring that an action remains stationary under small variations of the physical degrees of freedom around their physical values. This is a convenient form to cast laws of physics, since the action is a scalar quantity and it is guaranteed to remain invariant at all reference frames. We have

$$\delta S = 0 \tag{841}$$

where for a system with a finite number of degrees of freedom,

$$S = \int L dt \tag{842}$$

and the Lagrangian is

$$L = \sum_i L_i \tag{843}$$

L_i corresponds (for typical systems) to the difference of the kinetic and potential energy.

For the electromagnetic field, energy is stored in a continuous space. Then, we should introduce a *Lagrangian density* $\mathcal{L}$ where

$$L_i \to d^3\vec{x}\mathcal{L}. \tag{844}$$

The action becomes a four-dimensional integral over the Lagrangian density:

$$S = \int dt \int d^3\vec{x}\mathcal{L} = \int d^4x\mathcal{L}. \tag{845}$$

The principle of minimum action reads:

$$\delta \int d^4x\mathcal{L} = 0. \tag{846}$$

12.1 Maxwell equations as Euler-Lagrange equations

For the electromagnetic field, we assume a Lagrangian density which is a function of the four-vector potential A^μ and its first derivatives [3]:

$$\mathcal{L} = \mathcal{L}(A^\mu, \partial^\nu A^\mu). \tag{847}$$

[3] We do not need that the Lagrangian density depends on second or higher order derivatives since Maxwell equations are second order differential equations. From our experience with classical mechanics, we know that higher derivatives in $\mathcal{L}$ lead to third or higher order differential equations as equations of motion.

Then

$$\delta S = 0 \rightsquigarrow \int d^4x \left\{ \frac{\delta S}{\delta A^\mu(x)} \delta A^\mu(x) + \frac{\delta S}{\delta \partial^\nu A^\mu(x)} \delta \partial^\nu A^\mu(x) \right\} = 0 \quad (848)$$

Recall that

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y) \quad (849)$$

and

$$\frac{\delta G(f(x))}{\delta f(y)} = \frac{\partial G}{\partial f(x)} \delta(x-y). \quad (850)$$

Thus,

$$\begin{aligned} 0 &= \delta S \\ \rightsquigarrow 0 &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial A^\mu} \delta A^\mu(x) + \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \delta(\partial^\nu A^\mu(x)) \right] \\ \rightsquigarrow 0 &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial A^\mu} \delta A^\mu(x) + \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \partial^\nu (\delta A^\mu(x)) \right] \end{aligned} \quad (851)$$

Performing integration by parts and dropping the total derivative term, we obtain:

$$\int d^4x \left[\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \right] \delta A^\mu(x) = 0. \quad (852)$$

The above must be satisfied for arbitrary small variations $\delta A^\mu(x)$. This can happen if:

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = 0. \quad (853)$$

Consider now the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\nu A^\nu \quad (854)$$

Euler-Lagrange equations (Eq 853) give (**exercise**) Maxwell equations:

$$\partial_\mu F^{\mu\nu} = J^\nu. \quad (855)$$

12.1.1 Gauge fixing

We can include a gauge choice within the Lagrangian formalism for the electromagnetic field. Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\nu A^\nu - \frac{\xi}{2}(\partial_\mu A^\mu)^2 \tag{856}$$

Euler-Lagrange equations yield:

$$\partial^2 A^\mu - (1-\xi)\partial^\mu(\partial \cdot A) = J^\mu. \tag{857}$$

Choosing $\xi = 1$ produces Maxwell equations in the Lorentz gauge.

If we promote ξ into a field, the corresponding Euler-Lagrange equation yields:

$$\frac{\partial \mathcal{L}}{\partial \xi} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu \xi)} = 0 \rightsquigarrow (\partial \cdot A)^2 = 0 \rightsquigarrow \partial \cdot A = 0, \tag{858}$$

which is the Lorentz gauge-fixing condition.

References

[1] Feynman Lectures in Physics, Vol.2

[2] J.D Jackson, Classical Electrodynamics

[3] The Classical Theory of Fields, Fourth Edition: Volume 2, by L D Landau (Author), E.M. Lifshitz

[4] S. Weinberg, Gravitation and Cosmology