Exercise 1. ***Energy loss due to radiation of accelerating charges***

In this exercise, we will look at the radiation of accelerated charges in the relativistic limit. The energy loss associated with this is important especially in particle accelerators. Here we will consider both linear and circular accelerators.

1. Show that the power radiated from an accelerated electron of charge e and mass m is given by
$$P = \frac{2ke^2}{3c^3}\gamma^6[|\dot{\vec{v}}|^2 - |\vec{v}\times\dot{\vec{v}}|^2/c^2], \tag{1}$$
where $k = \frac{1}{4\pi\epsilon_0}$ and $\dot{\vec{v}} = d\vec{v}/dt$.

   To this end consider the Lorentz invariant generalisation of the Larmor's formula
$$P = -\frac{2ke^2}{3m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \tag{2}$$
and rewrite it in terms of its components. Upon using $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$ and after some algebra you should obtain the desired result.

2. Assume now that the acceleration is in the direction of motion as in a linear accelerator. Show that in the relativistic limit as $v \to c$ the power radiated from an accelerated electron can be written as
$$P = \frac{2ke^2}{3m^2c^4}\frac{dE}{dx}\frac{dE}{dt} \tag{3}$$
with dx being the infinitesimal distance travelled by the electron.

   For an energy gain of $10MeV/m$ through acceleration for electrons with $mc^2 = 0.5MeV$, what is the fraction of the radiated power to the energy gain per unit time? Is it significant?

3. For a circular accelerator, the tangential acceleration is negligible relative to the orthogonal acceleration, and so we can write $|\dot{v}| = v\,\omega$ with ω the angular frequency. Starting from Equation 2 show that in this case the power can written as
$$P = \frac{2ke^2c}{3\rho^2}\left(\frac{v}{c}\right)^4\gamma^4 \tag{4}$$
where ρ is the orbit radius and $\omega = v/\rho$.

   Now what is the energy loss during one period as $v \to c$? Calculate this explicitly assuming a typical energy of $E = \gamma mc^2 = 10GeV$ and radius $\rho = 10m$ of an electron-synchrotron. Observe how this is much more significant than the effect of radiation in the case of linear acceleration.

Exercise 2. ***The Lorentz transformation of acceleration***

In this exercise we will find how the acceleration transforms from one frame to another under Lorentz transformations.

Consider a reference frame $\mathcal{S}'$ that moves away from another frame $\mathcal{S}$ with a velocity $\mathbf{v}_{S'}$. In frame $\mathcal{S}'$ there is a particle moving with a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$.

Show that the acceleration transformed into the frame $\mathcal{S}$ has the following components:

$$\mathbf{a}_{\parallel} = \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^3} \quad \mathbf{a}'_{\parallel} \tag{5}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v_{S'}^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}_{S'} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_{\perp} + \frac{\mathbf{v}_{S'}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right) \tag{6}$$

where $\mathbf{a}_{\parallel,\perp}$ are the acceleration parallel/perpendicular to the direction of motion.

Exercise 3. ***Twin paradox revisited***

Consider a spaceship leaving the Earth today. On board there is one of two twins born in 1995. The other remains on Earth. Assume that the spaceship, in its own reference frame, has an acceleration $a' = 30$ m/s^2. It accelerates for 5 years (time in its own frame) in a straight-line path, decelerates at the same rate for 5 years, turns around, accelerates for 5 years and decelerates again for 5 years. When the astronaut returns back on Earth he is 40 years old.

1. How old is the twin that stayed on Earth?
2. How far away from the Earth did the spaceship travel as seen from the Earth's reference frame?
3. Check that during the journey the speed of light has never been surpassed.