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Quantum Information Processing
Solutions 2.

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We are looking for a method to measure polarisation encoded pairs of photonic qubits in the Bell-basis:

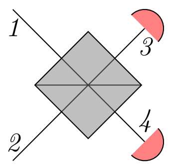
$$\begin{aligned} \left|\Psi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|H\right\rangle_1\left|V\right\rangle_2 \pm \left|V\right\rangle_1\left|H\right\rangle_2\right] \\ \left|\Phi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|H\right\rangle_1\left|H\right\rangle_2 \pm \left|V\right\rangle_1\left|V\right\rangle_2\right]. \end{aligned} \tag{1}$$

It has been shown that it is impossible to perform a complete Bell-state measurement of entangled photon pairs using only linear optics, if they are only entangled in a single degree of freedom.

Exercise 1. *Houng-Ou-Mandel interferometer*

The state of a photonic mode μ can be represented in second quantised form using bosonic operators $\hat{b}^\dagger_\mu$ that satisfy the usual commutation relations (distinct modes commute). Note that different polarisations behave as distinct modes. For example the state $\left|H\right\rangle_1\left|V\right\rangle_2$ is then written as $\hat{b}^\dagger_{1H}\hat{b}^\dagger_{2V}\left|0\right\rangle$, where $\left|0\right\rangle$ is the electromagnetic vacuum.

The input modes 1 and 2 and the output modes 3 and 4 of a 50:50 beam splitter are connected via the boundary conditions of Maxwell's Equations:



$$\begin{aligned} \hat{b}^\dagger_{1j} &= \frac{1}{\sqrt{2}}\left(\hat{b}^\dagger_{3j} + \hat{b}^\dagger_{4j}\right) \\ \hat{b}^\dagger_{2j} &= \frac{1}{\sqrt{2}}\left(\hat{b}^\dagger_{3j} - \hat{b}^\dagger_{4j}\right) \end{aligned} \qquad j = H, V \tag{2}$$

(a) *Which of the four Bell states can you detect from the signal on the two detectors on ports 3 and 4, if each of the two photons of a maximally entangled pair is sent onto one input port?*

Hint: The detectors cannot distinguish polarisation, nor can they count the photons. They just click if there was at least one photon, otherwise they remain silent.

(b) *Given additional detectors and polarising beam splitters that transmit one polarisation and reflect the other, how many Bell states could you detect unambiguously?*

Solution.

(a) The joint detector output has four possible outcomes, for each detector either to click or not to click, depending if there is at least one photons in their respective modes or not. Thus, by expressing the Bell states (1) in second quantisation in terms of photon number states of the output modes, the detection operator will be diagonal, and we can read off the possible outcomes. Using the beamsplitter relations (2), we find

$$\left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left[\hat{b}^{\dagger}_{1H}\hat{b}^{\dagger}_{2V} \pm \hat{b}^{\dagger}_{1V}\hat{b}^{\dagger}_{2H}\right]|0\rangle \tag{S.1}$$

$$= \frac{1}{2\sqrt{2}}\left[\left(\hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{3V} - \hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{4V} + \hat{b}^{\dagger}_{3V}\hat{b}^{\dagger}_{4H} - \hat{b}^{\dagger}_{4H}\hat{b}^{\dagger}_{4V}\right)\right. \tag{S.2}$$

$$\left.\pm\left(\hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{3V} + \hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{4V} - \hat{b}^{\dagger}_{3V}\hat{b}^{\dagger}_{4H} - \hat{b}^{\dagger}_{4H}\hat{b}^{\dagger}_{4V}\right)\right]|0\rangle \tag{S.3}$$

$$= \begin{cases} \text{"}+\text{"}: & \frac{1}{\sqrt{2}}\left[\boxed{\hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{3V}} - \boxed{\hat{b}^{\dagger}_{4H}\hat{b}^{\dagger}_{4V}}\right]|0\rangle \\ \text{"}-\text{"}: & \frac{1}{\sqrt{2}}\left[-\boxed{\hat{b}^{\dagger}_{3H}\hat{b}^{\dagger}_{4V}} + \boxed{\hat{b}^{\dagger}_{3V}\hat{b}^{\dagger}_{4H}}\right]|0\rangle \end{cases} \tag{S.4}$$

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left[\hat{b}^{\dagger}_{1H}\hat{b}^{\dagger}_{2H} \pm \hat{b}^{\dagger}_{1V}\hat{b}^{\dagger}_{2V}\right]|0\rangle \tag{S.5}$$

$$= \frac{1}{2\sqrt{2}}\left[\left(\boxed{\hat{b}^{\dagger 2}_{3H}} - \boxed{\hat{b}^{\dagger 2}_{4H}}\right) \pm \left(\boxed{\hat{b}^{\dagger 2}_{3V}} - \boxed{\hat{b}^{\dagger 2}_{4V}}\right)\right]|0\rangle \tag{S.6}$$

Obviously, we will always have at least one click. However, while only $|\Psi^-\rangle$ leads to a coincidence on both detectors, all other three states will either lead to a single click on detector 3 or a single click on detector 4. Thus, using the basic Hong-Ou-Mandel interference, we can only distinguish one of the Bell states.

(b) From (S.4) and (S.6), we see that the $|\Phi\rangle$ states are characterised by co-polarised photon pairs in a single output, while $|\Psi^+\rangle$ is the only state with cross-polarised photon pairs in a single output. Thus, by replacing each detector by a polarising beam splitter (PBS) and two detects on it's outputs, the $|\Phi\rangle$ states would still bunch on a single detector, while $|\Psi^+\rangle$ would lead to a coincidence between two detectors on a PBS of a single output of the main beam splitter.

Therefore, we can now reliably detect two out of four of the Bell states.

It has been shown, that using linear optics only, one cannot deterministically detect all four Bell-states, but using the help of additional anxillary entangled photons, one can reach non-deterministic Bell measurements that get arbitrarily close to 100 % detection efficiency.

Exercise 2. *Hyperentanglement-assisted deterministic Bell-measurement*

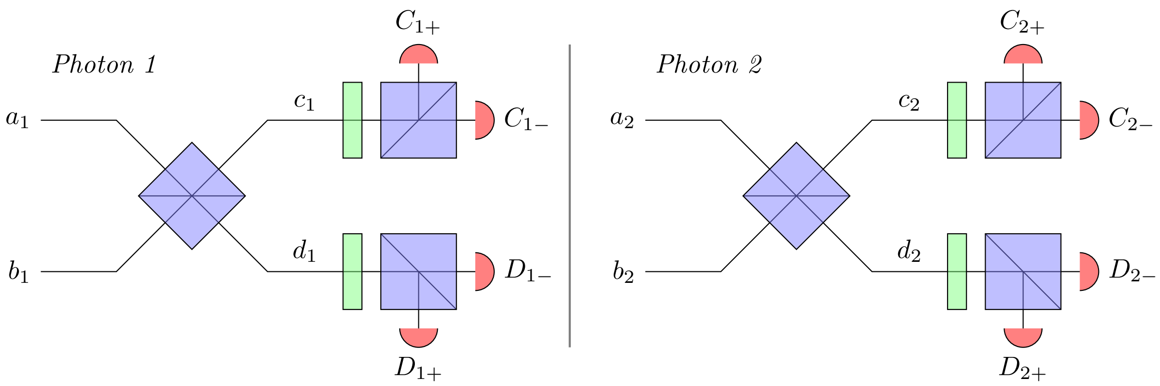
If a photon pair is entangled in more than one degree of freedom, one speaks of hyperentangled photons. Such photons are generated for example by spontaneous down conversion in nonlinear crystals. Here, we look at photon pairs simultaneously entangled in polarisation and mode.

Besides their polarisation, each photon furthermore has the option to be either in mode a or b, and besides their joint polarisation state, they shall be entangled in their mode degree of freedom. Precisely, the two-photon state shall be

$$|\Pi\rangle \otimes \frac{1}{\sqrt{2}}\left[|a\rangle_1 |b\rangle_2 + |b\rangle_1 |a\rangle_2\right], \tag{3}$$

where $|\Pi\rangle$ is any two-photon polarisation state that we want to measure in the Bell-basis.

Each photon is analysed individually using the following scheme:



Here, blue squares represent polarising beam splitters (PBS) that transmit photons in $|V\rangle$ and reflect photons in $|H\rangle$ polarisation. Green rectangles represent half-wave plates (HWP) oriented at $\pi/8$ angle, such that $|H\rangle \to \frac{1}{\sqrt{2}}[|H\rangle + |V\rangle]$ and $|V\rangle \to \frac{1}{\sqrt{2}}[|H\rangle - |V\rangle]$. Red half circles represent single-photon counters.

(a) *How does the initial PBS in each analyser act on the photon's polarisation and mode degrees of freedom? If you would consider those two as separate qubits, how would you describe the operation performed by the PBS?*

(b) *If you would place detectors at c_1, d_1, c_2 and d_2 instead of the HWP and further PBS, how could you partition the input states based on the detector signals?*

(c) *Consider a single photon in mode c_1. What polarisation state in mode c_1 is identified by a click in detector C_{1+}, and by a click in C_{1-}?*

(d) *If you would input polarisation entangled photons at c_1 and c_2 (skipping the initial PBS), how could you partition the Bell states from the detector signal now?*

(e) *Would this scheme allow for deterministic entanglement swapping of a polarisation qubit?*

Hint: Since each photon is analysed separately from the other, there will only ever be one photon, and we don't need to work in second quantisation.

Solution.

(a) According to the description of the PBS, it will map the photon states as follows:

$$\begin{aligned} |H\rangle|a\rangle &\to |H\rangle|c\rangle \\ |H\rangle|b\rangle &\to |H\rangle|d\rangle \\ |V\rangle|a\rangle &\to |V\rangle|d\rangle \\ |V\rangle|b\rangle &\to |V\rangle|c\rangle \end{aligned} \qquad \text{or as operator:} \quad \begin{bmatrix} |H\rangle|c\rangle \\ |H\rangle|d\rangle \\ |V\rangle|c\rangle \\ |V\rangle|d\rangle \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \langle H|\langle a| \\ \langle H|\langle b| \\ \langle V|\langle a| \\ \langle V|\langle b| \end{bmatrix} + \text{h.c.} \quad \text{(S.7)}$$

If we identify the spatial mode $|a\rangle$ with $|c\rangle$ and $|b\rangle$ with $|d\rangle$, and consider the polarisation degrees of freedom and the mode degree of freedom as individual qubits, then the PBS performs a CNOT gate between them.

(b) We need to apply the beam-splitter operator (S.7) to the input states (3). Specifically, we intend to expand the polarisation degrees of freedom in terms of the Bell states (1). Let us evaluate

$$
\begin{aligned}
|H\rangle|H\rangle \otimes \frac{1}{\sqrt{2}}\left[|a\rangle_1|b\rangle_2+|b\rangle_1|a\rangle_2\right] &\rightarrow |H\rangle|H\rangle \otimes \frac{1}{\sqrt{2}}\left[|c\rangle_1|d\rangle_2+|d\rangle_1|c\rangle_2\right], \\
|V\rangle|V\rangle \otimes \frac{1}{\sqrt{2}}\left[|a\rangle_1|b\rangle_2+|b\rangle_1|a\rangle_2\right] &\rightarrow |V\rangle|V\rangle \otimes \frac{1}{\sqrt{2}}\left[|d\rangle_1|c\rangle_2+|c\rangle_1|d\rangle_2\right], \\
|H\rangle|V\rangle \otimes \frac{1}{\sqrt{2}}\left[|a\rangle_1|b\rangle_2+|b\rangle_1|a\rangle_2\right] &\rightarrow |H\rangle|V\rangle \otimes \frac{1}{\sqrt{2}}\left[|c\rangle_1|c\rangle_2+|d\rangle_1|d\rangle_2\right], \\
|V\rangle|H\rangle \otimes \frac{1}{\sqrt{2}}\left[|a\rangle_1|b\rangle_2+|b\rangle_1|a\rangle_2\right] &\rightarrow |V\rangle|H\rangle \otimes \frac{1}{\sqrt{2}}\left[|d\rangle_1|d\rangle_2+|c\rangle_1|c\rangle_2\right].
\end{aligned}
\tag{S.8}
$$

We can deduce that, the polarisation states with even parity will lead to "crossed" detection events with one c detector and d detector firing for the two photons, the polarisation states with odd parity will lead to "parallel" detection events, with either both c detectors or both d detectors firings. Thus, from the pattern of the photon distribution among the c and d detectors, we can deduce the parity, but not the relative phase of the photon pair.

(c) The HWP rotates the polarisation by 45°, essentially allowing the following PBS to perform polarisation analysis in the diagonal basis:

$$\left|D^+\right\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle+|V\rangle\right] \qquad \rightarrow |H\rangle \qquad \rightarrow \text{click on } C_{1+} \tag{S.9}$$

$$\left|D^-\right\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle-|V\rangle\right] \qquad \rightarrow |V\rangle \qquad \rightarrow \text{click on } C_{1-} \tag{S.10}$$

(d) We rewrite the polarisation degree of freedom in the diagonal basis:

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\left[\left|D^+\right\rangle_1\left|D^+\right\rangle_2-\left|D^-\right\rangle_1\left|D^-\right\rangle_2\right] \tag{S.11}$$

$$\left|\Psi^-\right\rangle = -\frac{1}{\sqrt{2}}\left[\left|D^+\right\rangle_1\left|D^-\right\rangle_2-\left|D^-\right\rangle_1\left|D^+\right\rangle_2\right] \tag{S.12}$$

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\left[\left|D^+\right\rangle_1\left|D^+\right\rangle_2+\left|D^-\right\rangle_1\left|D^-\right\rangle_2\right] \tag{S.13}$$

$$\left|\Phi^-\right\rangle = -\frac{1}{\sqrt{2}}\left[\left|D^+\right\rangle_1\left|D^-\right\rangle_2+\left|D^-\right\rangle_1\left|D^+\right\rangle_2\right] \tag{S.14}$$

Clearly, this time, the relative phase distinguishes the detection pattern: Relative phase "+" leads to "parallel" detection events, and relative phase "−" leads to "crossed" detection events. Together with the conclusions from (b), we can now complete the detection table:

Photon 1 / Photon 2	C_{1+}	C_{1-}	D_{1+}	D_{1-}
C_{2+}	$\|\Psi^+\rangle$	$\|\Psi^-\rangle$	$\|\Phi^+\rangle$	$\|\Phi^-\rangle$
C_{2-}	$\|\Psi^-\rangle$	$\|\Psi^+\rangle$	$\|\Phi^-\rangle$	$\|\Phi^+\rangle$
D_{2+}	$\|\Phi^+\rangle$	$\|\Phi^-\rangle$	$\|\Psi^+\rangle$	$\|\Psi^-\rangle$
D_{2-}	$\|\Phi^-\rangle$	$\|\Phi^+\rangle$	$\|\Psi^-\rangle$	$\|\Psi^+\rangle$

(e) The basic idea of entanglement swapping is based on the fact that the measurement of two qubits in the Bell-basis projects them into an entangled state. If both of these qubits were entangled with two external qubits, then the projection will "swap" the entanglement onto the external qubits. In order to apply this technique, the measured qubits must share modal entanglement beforehand. Therefore, this scheme is in general not applicable.