Lecture 16

Landau's theory of Fermi liquid

L.D. Landau 1956

In the first 14 lectures we considered electrons as independent particles. In the previous lecture we discussed the Coulomb interaction and found that it is screened at the distances $\lambda_{TF} \sim k_F^{-1}$. Nevertheless at the interelectron distances this interaction is not small. Its contribution to the energy per particle $\sim \frac{e^2}{r_0} \sim \varepsilon_F \frac{r_0}{a_0} \sim \varepsilon_F$.

So this interaction can not be neglected.

It modifies electron spectrum. The suprizing result we are going to find is that even when the distance between electrons is about 2 Å the mean free path between their collisions is much larger. It is about 10^4 Å at room temperature and goes over centimeters at $T \sim 1$ K

The fundamental principle underlying the Fermi liquid theory of Landau is the adiabatic continuity. In the absence of phase transitions a non interacting ground state evolves smoothly or adiabatically into the interacting ground state as the strength of the interaction is increased. The low temperature properties of strongly interacting electrons are the same as those of noninteracting fermions with renormalized parameters.

Concept of quasiparticles

At $T \to 0$ the important energy levels are the weakly excited states, whose energies differ only very little from the energy of the ground state. The character of the energy spectrum in this energy region can be ascertained in some details, by using very general considerations which are valid regardless of the magnitude and character of interaction

A simple example illustrating this concept are phonons. They are lattice vibrations in a crystal. Irrespectively to interaction we have sound waves in the crystals with low energy spectrum $\omega = ck$. Quantising it we get new quasiparticle – phonon. Knowing spectrum we get low temperature properties e.g specific heat. Changing interaction only changes sound velocity c.

Keeping in mind this analogy we assume that the weakly excited states of a system in general case have energy level structure similar to the energy levels of an ideal gas. We assume that these elementary excitations = quasiparticles can be characterized by momentum $\vec{p}$ and energy $\varepsilon(\vec{p})$. In general $\varepsilon(p) \neq \frac{p^2}{2m}$.

These excitations are result of collective interaction of particles of the system and pertain to the system as a whole not to its separate particles

In particular the number of elementary excitations doesn't have to be the same as the total number of particles in the system. Phonons are again a good example.

There are two kinds of spectra — Bose type and Fermi type. In the first case the excitations have integer spin and obey Bose statistics. In the second case the excitations have half integer spin and obey Fermi statistics.

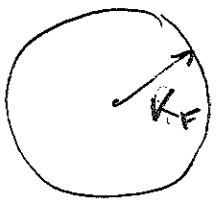
The statistic of elementary excitations doesn't have to be the same as that of the particles that produce the system. Consider again phonons that are bosons no matter whether the crystal is made from fermions or bosons.

To be well defined objects these excitations should have rather long life time $\tau >> \frac{\hbar}{\varepsilon(p)}$

But at low temperatures there are very few of them thus scattering is weak. That's why we call them quasiparticles.

The basic phenomenology of the Fermi liquid is that its spectrum is similar to that of the Fermi gas. Excitations have spin $\frac{1}{2}$ and obey Fermi statistics. The only real Fermi liquid is He^3.

In Fermi gas there is well defined Fermi surface. Excitations are particles ($k > k_F$) and holes ($k < k_F$). They have spin $1/2$ and can appear and disappear only in pairs.



$$k_F = (3\pi^2 n)^{1/3}$$

Close to Fermi surface $\xi = \varepsilon(k) - \mu \approx \frac{k^2}{2m} - \frac{k_F^2}{2m} = v_F(\vec{k} - \vec{k}_F)$

In Fermi liquid with arbitrary interaction between particles we assume that the low energy excitations are similar to those of ideal gas: "particles" and "holes" with momentum bigger or smaller than k_F. Another basic assumption is that $k_F = (3\pi^2 n)^{1/3}$ is the same as for noninteracting system (Luttinger theorem). Although $\varepsilon(k) \neq \frac{k^2}{2m}$ we can still expand $\varepsilon(k) - \mu = \frac{\partial \varepsilon}{\partial \vec{k}} \cdot (\vec{k} - \vec{k}_F) = \vec{v}_F \cdot (\vec{k} - \vec{k}_F)$, $\vec{v}_F = \frac{\vec{k}_F}{m^*}$

Quasiparticle life time

Let us consider quasiparticle decay. In this process particle with energy ε_1 above the Fermi energy scatters by the electron below the Fermi level, ($\varepsilon_2 < \varepsilon_F$ (there are no other electrons). They scatter into two electrons $\varepsilon_3, \varepsilon_4$ that are in <u>unoccupied</u> states thus above the Fermi energy.

$$\varepsilon_1 > \varepsilon_F, \quad \varepsilon_2 < \varepsilon_F, \quad \varepsilon_3 > \varepsilon_F, \quad \varepsilon_4 > \varepsilon_F$$

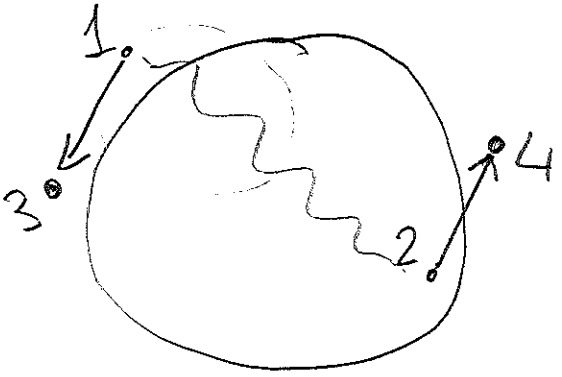
In addition energy conservation requires

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 .$$

If $\varepsilon_1 = \varepsilon_F$ then $\varepsilon_2, \varepsilon_3, \varepsilon_4$ should also lie at the Fermi surface to satisfy the conditions above. Thus the allowed wave vectors should be at the Fermi surface which has zero volume. To find the scattering rate we should integrate over momenta thus we get zero or <u>infinite life time</u> for electron at the Fermi surface at $T = 0$

If ε_1 is a little bigger than ε_F same phase space become available. The second electron can be within the energy layer of thickness $|\varepsilon_1 - \varepsilon_F|$ below the Fermi surface and $\varepsilon_3, \varepsilon_4$ within the shell of the same width above it. If we know the energies ε_2 and ε_3, then ε_4 is fixed by the energy conservation. Thus integrating over these three energy state gives a scattering rate $\propto \underline{(\varepsilon_1 - \varepsilon_F)^2}$.

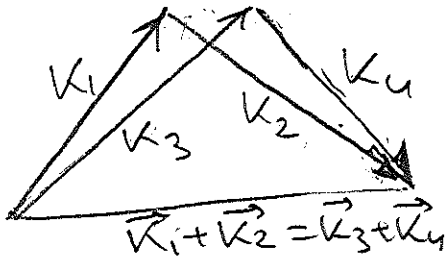
A bit more careful analysis including momenta is the following. The scattering process



Momentum conservation

$$\vec{K}_1 + \vec{K}_2 = \vec{K}_3 + \vec{K}_4$$

$K_1 > K_F$, $K_2 < K_F$, $K_3 > K_F$, $K_4 > K_F$



Scattering probability is

$$W \propto \int \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\, d^3K_2\, d^3K_3$$

K_4 is determined by the momentum conservation law. Planes of $(\vec{K}_1,\vec{K}_2)$ and $(\vec{K}_3,\vec{K}_4)$ do not in general coincide. The angle between $\vec{K}_3$ and $\vec{K}_4$ is given by the energy conservation. Integrating over it removes the δ-function. Thus we have to integrate over $|K_2|$ and $|K_3|$.

If K_1 is close to K_F then all the other K_i are also close to K_F. Then all of them will roughly form isosceles triangles and will have the same angle with $\vec{K}_1+\vec{K}_2$. Then from $\vec{K}_1+\vec{K}_2=\vec{K}_3+\vec{K}_4$ we obtain

$K_3 \approx K_1+K_2-K_4$. Since $K_4 > K_F$ then $K_3 < K_1+K_2-K_F$. But $K_3 > K_F$ as well, so

$K_1+K_2-K_F > K_F \Rightarrow K_2 > 2K_F-K_1 \Rightarrow$

$0 > K_2-K_F > K_F-K_1$,

$0 < K_3-K_F < (K_1-K_F)+(K_2-K_F)$

Integrating $\int d\vec{k}_3 d\vec{k}_4 = \frac{(\vec{k}_1 - \vec{k}_F)^2}{2}$

Thus $\frac{1}{\tau} \propto W \propto (\vec{k}_1 - \vec{k}_F)^2 \propto (\varepsilon_1 - \varepsilon_F)^2$.

The overall prefactor can be constructed from the dimensional reasoning. There are interaction strength V_{int} and the Fermi energy ε_F. But for a liquid volume of which is determined by the interaction strength density is such that the kinetic and interaction energy are about the same. Thus

$$\frac{\hbar}{\tau} \simeq \frac{(\varepsilon_1 - \varepsilon_F)^2}{\varepsilon_F}$$

For $T > 0$ typical quasiparticle energies are $\sim T$. Then the scattering rate is

$\frac{\hbar}{\tau} \simeq \frac{T^2}{\varepsilon_F}$ and the mean free path

$\ell \simeq r_0 \frac{\varepsilon_F^2}{T^2}$. For $\varepsilon_F \sim eV$, $T \sim 1K$

we get $\ell \sim 1cm$

Quasiparticle energy $\varepsilon(p)$ is defined as variational derivative of the total energy with respect to the quasiparticle distribution function

$$\delta E = 2V \int \varepsilon(k)\, \delta n(k)\, \frac{d^3k}{(2\pi)^3}$$ (2 comes from spin)

Quasiparticles obey Fermi statistics

$$n(\varepsilon) = n_F(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{T}}+1}$$ (ε in general depends on n)

$$\frac{\delta\, \varepsilon(\vec{k})}{\delta\, n_{\vec{k}'}} = f(\vec{k},\vec{k}') \quad \Rightarrow$$

$$\varepsilon(\vec{k}) = \varepsilon^0(\vec{k}) + \int f(\vec{k},\vec{k}')\, \delta n_{k'}\, \frac{d^3k'}{(2\pi)^3}$$

$f(\vec{k},\vec{k}') = f(\vec{k}',\vec{k})$ is the Landau function that characterizes interaction of two quasiparticles close to the Fermi surface.

$$\xi(k) = \varepsilon(k) - \mu = \vec{v}_F(\vec{k}-\vec{k}_0).$$

$\vec{v}_F = \frac{\vec{k}_0}{m^*}$, m^* is the effective mass

It is this mass that gives the specific heat

$$C = \left(\frac{k_F m^*}{3\hbar^3}\right) T$$

The expression is the same as for the Fermi gas since distribution function of excitations is the same.

We can relate m^* with the Landau function using Galilei invariance. Because of it the flux of particles is equal to the flux of quasiparticles

Total momentum of particles is

$$\vec{K} = \int \vec{\kappa}\, n_\kappa \frac{d^3\kappa}{(2\pi)^3}$$

The quasiparticle flux is $\int \vec{v}\, n_\kappa \frac{d^3\kappa}{(2\pi)^3}$ where $\vec{v}$ is velocity of quasiparticle $\vec{v} = \nabla_\kappa \varepsilon(\kappa)$. Multiplying by the bare particle mass ($\Sigma p = m \Sigma v$) we obtain

$$\vec{K} = \int \vec{\kappa}\, n_\kappa (d^3\kappa) = m \int \nabla_\kappa \varepsilon\, n_\kappa (d^3\kappa)$$

Varying this expression with respect to δn

and using $\delta\varepsilon = \int f(k,k')\,\delta n_{k'}\,\frac{d^3k'}{(2\pi)^3}$

we obtain

$$\int \frac{k}{m}\,\delta n\,(d^3k) = \int \nabla_k \varepsilon\,\delta n (d^3k) + \int n\,\delta n\,\nabla_k f(k,k')(d^3k)(d^3k')$$

Integrating second term by parts and interchanging $k \leftrightarrow k'$ we obtain

$$\frac{\vec{k}}{m} = \frac{\partial \varepsilon}{\partial \vec{k}} - \int f(\vec{k},\vec{k}')\,\nabla_{k'} n_{k'}\,(d^3k')$$

Since n_k is a step function

$$\nabla_{k'} n_{k'} = -\frac{k'}{|k'|}\,\delta(k'-k_F)$$

we obtain

$$\frac{1}{m^*} = \frac{1}{m} - \frac{k_F}{(2\pi)^3}\int f(\theta)\cos\theta\, d\Omega$$

θ is the angle between $\vec{k}, \vec{k}'$