Lecture 21

Anderson Localization

Let us consider non-interacting electrons in random potential $U(r)$. The Schrödinger equation is

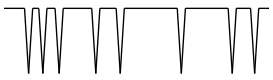
$$-\frac{\hbar^2}{2m}\nabla^2\psi + U(r)\psi(r) = E\psi.$$

This is continuous version of the problem. We can discuss also the lattice formulation.

$$H = \sum_i \varepsilon_i a_i^+ a_i + \sum_{ij} t_{ij} a_i^+ a_j.$$

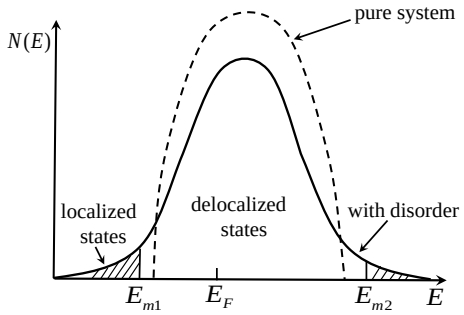
If site energies ε_i are random disorder is called diagonal. This is the case for the Anderson model. If t_{ij} is random it is non-diagonal disorder.

The Lifschitz model corresponds to positional disorder, the same wells, but randomly spaced.

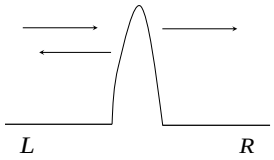


Consider for simplicity the Anderson model with diagonal disorder of the width W : $\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2 = W^2$. P. W. Anderson (1958) argued that for sufficiently large W/t all the states in the system are localized.
How to formulate localization condition?
Assume at $t = 0$ a particle was on the site i. This is not an eigenfunction of the Hamiltonian and will change in time. Assume we solve the Schrödinger equation.
If $\lim\limits_{t\to\infty} |\psi_i(t)|^2 = 0$ then we have delocalized state.
If $\lim\limits_{t\to\infty} |\psi_i(t)|^2 > 0$ we call this state localized.

General picture



Energies E_{m1} and E_{m2} which separate the regions of localized and delocalized states are called mobility edges. For strong disorder $E_{m1} \to E_{m2}$ and all the states are localized. When the Fermi energy crosses the mobility edge we have the Metal Insulator transition.
In one dimension all the states are localized!
Consider randomly distributed barriers



$$\psi_R = te^{ikx}$$
$$\psi_L = e^{ikx} + re^{-ikx}$$
$$|t|^2 + |r|^2 = 1.$$

Reflection amplitude r has an arbitrary phase dependent on the position of impurity x_0, $r \propto \exp(2ikx_0)$. In general the wave function on the right of the barrier $\psi_R = u_n \exp(ikx) + v_n \exp(-ikx)$ is related to that on the left $\psi_L = u_{n-1} \exp(ikx) + v_{n-1} \exp(-ikx)$ by the transfer matrix equation.

$$\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} 1/t_n & r_n^*/t_n^* \\ r_n/t_n & 1/t_n^* \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

The transmission and reflection coefficients for a system of N barriers are given by the product of transfer matrices

$$\begin{pmatrix} 1 \\ r \end{pmatrix} = \prod_{n=1}^{N} \begin{pmatrix} 1/t_n & r_n^*/t_n^* \\ r_n/t_n & 1/t_n^* \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix}$$

For two barriers

$$\frac{1}{t} = \frac{1}{t_1 t_2} + \frac{r_1^* r_2}{t_1^* t_2}$$

Since phases of r_i are random we obtain $\langle \frac{1}{t} \rangle = \frac{1}{t_1 t_2}$ or in general case $\langle t \rangle = \prod t_n$. Thus transmission is multiplicative function. This means that on average multiple reflections cancel out and only transmission of the unreflected waves contributes to the total transmission. As a result the wave function falls off exponentially with the distance - localization. According to the Landauer formula

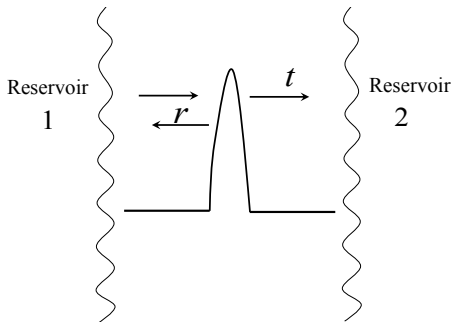
$$R = \frac{\pi\hbar}{e^2} \left| \frac{r}{t} \right|^2, \quad \frac{\pi\hbar}{e^2} = 12.9k\Omega$$

and resistance of the one dimensional system grows exponentially with length

$$R(L) = \frac{\pi\hbar}{e^2} \exp(L/l_c),$$

where l_c is the localization length.

Landauer formula



We have given current I and calculate the difference in electron densities on the sides of the barrier which should be compensated by the external potential V
On the left $\psi = \exp(ik_F x) + r\exp(-ik_F x)$.
On the right $\psi = t\exp(ik_F x)$.
Thus electron density on the left is $n_L = (j_0 + j_R)/v_F e$, where j_0 and j_R describe the incident and the reflected current densities respectively. Similarly on the right $n_R = j_T/v_F e$. Note that $j_R = |r|^2 j_0$, $j_T = |t|^2 j_0$. Density difference $n_L - n_R = \delta n = N(E)eV$. In 1d $N(E) = 2/\pi\hbar v_F$, thus $V = (\pi/2)(\hbar/e^2)j_0(1 + |r|^2 - |t|^2) = \pi\hbar|r|^2 j_0/e^2$. Since transport current is $I = j_0|t|^2$ one gets Landauer formula $R = (\pi\hbar/e^2)|r/t|^2$.

Thouless approach

Let us start with the Einstein relation for conductivity

$$\sigma = e^2 N(E_F) D.$$

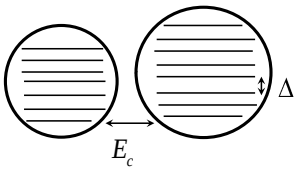
where D is the diffusion coefficient, $D = v_F l/d$, d is the dimensionality of the space.

$$\sigma = \frac{ne^2\tau}{m} = e^2 N(E_F) D.$$

As we understood from the 1d case resistivity or conductivity are not very good quantities and one should better talk about the total conductance. Consider a "block" with the sizes L. Its conductance is $G = \sigma L^{d-2} = (e^2/\hbar) N(E_F) L^d (\hbar D/L^2)$,

$$G = \frac{e^2}{\hbar}\frac{E_c}{\Delta},$$

where Δ is level spacing in the block L^d : $\Delta = N(E_F)L^d$ and E_c is the Thouless energy $E_c = (\hbar D/L^2) = \hbar/\tau$ - inverse diffusive time to traverse the block. E_c determines sensitivity of the energy levels to the change of boundary conditions, $E_c = \pi^2 |\partial^2 E/\partial\phi^2|$, where ϕ is the phase difference at the boundary. Let us consider levels in the adjacent blocks



The coupling between blocks is $\sim E_c$ and the spacing between the energy levels $\sim \Delta$. Thus it looks like a tight binding model with hoping $\sim E_c$ and disorder $W \sim \Delta$. If $E_c \gg \Delta$ then levels are well mixed and we have usual Ohm's law. If $E_c \ll \Delta$ then there is very little mixing and we have localization. Thus if the wire has resistance $\geq \hbar/e^2 \approx 4k\Omega$ then the states are localized and $R \propto \exp(L/L_c)$, Localization length in this case is $L_c \sim l(Sk_F^2)$, where S is the cross section of the wire.

Scaling theory of localization

E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan (1979).
Let us write conductance as

$$G(L) = \frac{e^2}{\hbar}\frac{E_c}{\Delta} = \frac{e^2}{\hbar}g(L).$$

g(L) is dimensionless conductance. Both E_c, Δ and thus g depend on the block size L. The scaling hypothesis assumes that g is the only quantity which determines the behaviour of the system when its size is changed.

$$g(bL) = f(b, g(L)).$$

Let us rewrite it in the differential form, $b = 1 + \alpha$ with $\alpha \ll 1$. Then in lowest orders in α we have

$$g(L) = f(1, g(L)), \qquad \alpha L g'(L) = \alpha \left.\frac{\partial f}{\partial b}\right|_{b=1}.$$

Dividing the second equation by g and introducing $\beta(g) = \left(\partial f/\partial b\right)|_{b=1}/g$ we obtain

$$\frac{d\ln g}{d\ln L} = \beta(g).$$

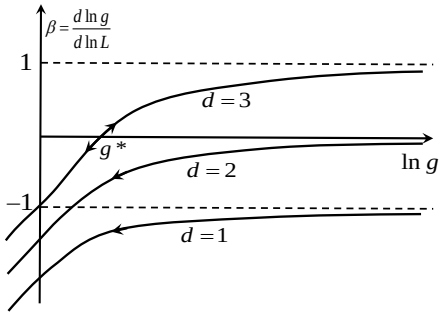
The function $\beta(g)$ is called the Gell-Mann - Low function. For large conductance we expect to have the Ohm's law, $G = \sigma L^{d-2}$. Then for large g

$$\beta(g) \approx d - 2; \qquad g \to \infty.$$

For small g we expect localization, $g \propto \exp(-L/L_c)$, thus

$$\beta(g) \approx \ln g + \text{const}; \qquad g \to 0.$$

From these asymptotics on large and small g we can guess the general dependence of $\beta(g)$ and the renormalization group (RG) flow.



Negative $\beta(g)$ means that with increasing size $g \to 0$ and we have localization. This is the case for 1d and 2d. In 3d situation is different. We have *unstable fixed point* $g^* : \beta(g^*) = 0$. For $g < g^*$ we have localization, for $g > g^*$ with increase of the system size we go to the usual Ohm's law. If we assume that close to g^*

$$\beta(g) = \frac{1}{\nu}\left(\frac{g}{g^*} - 1\right) \approx \frac{1}{\nu} \ln \frac{g}{g^*}$$

and integrate the RG equation from L_0, g_0 we obtain

$$g \approx g^* \left(\frac{g_0}{g^*}\right)^{(L/L_0)^{1/\nu}}.$$

Since localization appears due to the interference effects from multiple scattering we expect, that it doesn't show at the scales lower than the mean free path l. Thus it is naturally to put $L_0 \sim l$. If we change some parameter x (e.g. impurity concentration or pressure) such that $g_0 = g^*(1 + x)$ then

$$g \approx g^*(1 + x)^{(L/L_0)^{1/\nu}} \approx g^* \exp\left(x(L/l)^{1/\nu}\right).$$

For small $x < 0$ conductance g falls off exponentially and we can extract the localization length $L_c \simeq l x^{-\nu}$. For $x > 0$ we should match the growth of g with the Ohm's law $ge^2/\hbar \simeq \sigma L$ at $g \simeq g^*$ which gives

$$\sigma \simeq \frac{e^2}{\hbar}\frac{g^*}{L_c} \simeq \frac{e^2}{\hbar}\frac{g^*}{l}x^\nu.$$