Lecture 23 | Hall effect and magneto-resistance

In the presence of magnetic field the Boltzmann equation in the static case is

$$e\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\frac{\partial f}{\partial \vec{k}} = \frac{f-f_0}{\tau}$$

Substituting $f = f_0 + f_1$ we obtain

$$e\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\left(\frac{\partial f_0}{\partial k} + \frac{\partial f_1}{\partial k}\right) = \frac{f_1}{\tau}$$

Replacing $\frac{\partial f_0(\varepsilon)}{\partial k} = \vec{v}_k \frac{\partial f_0(\varepsilon)}{\partial \varepsilon}$ we obtain

$$e\,\vec{E}\cdot\vec{v}_k\frac{\partial f_0}{\partial \varepsilon} + \underbrace{e\,\frac{[\vec{v}_k\times\vec{B}]\cdot\vec{v}_k}{c}\frac{\partial f_0}{\partial K}}_{=0} = \frac{f_1}{\tau} -$$

$$- e\left(\vec{E} + \frac{[\vec{v}_k\times\vec{B}]}{c}\right)\frac{\partial f_1}{\partial k} \Rightarrow$$

$$e\left(\vec{E}\cdot\vec{v}_k\right)\frac{\partial f_0}{\partial \varepsilon} = \frac{f_1}{\tau} - \frac{e}{c}\left[\vec{v}_k\times\vec{B}\right]\frac{\partial f_1}{\partial \vec{k}}$$

Here we have dropped $\vec{E}\cdot\frac{\partial f_1}{\partial \vec{k}}$ term (as we did in the Lecture 19) since $f_1 \propto E$ and this term would be of higher order

Consider free electrons with $\vec{k} = m\vec{v}$

We will look for solution of the Boltzmann equation in the form

$$f_1 = e\tau(\vec{v}_k \cdot \vec{A})\frac{\partial f_0}{\partial \varepsilon}$$

The vector A should be determined (for $\vec{B}=0$, $\vec{A}=\vec{E}$)

Substituting this Ansatz we obtain

$$\vec{v}\cdot\vec{E} = \vec{v}\cdot\vec{A} - \frac{e\tau}{mc}\vec{A}\cdot[\vec{v}\times\vec{B}]$$

Since this equation is valid for all $\vec{v}$ we can rewrite it as

$$\vec{E} = \vec{A} - \frac{e\tau}{mc}[\vec{H}\times\vec{A}]$$

We can solve it to find $\vec{A}(E)$ but we don't really need to do it.

Indeed electric current is expressed through $\vec{A}$ in exactly the same way as through E, $\vec{j} = \sigma_0\vec{A}$

where σ_0 is zero field conductivity.

As a result we obtain

$$\vec{E} = \frac{\vec{j}}{\sigma_0} - \frac{e\tau}{mc}\frac{[\vec{B}\times\vec{j}]}{\sigma_0}$$

$$\vec{E} = \rho_0\vec{j} + \frac{e\tau}{mc}\rho_0[\vec{j}\times\vec{B}]$$

Here $\rho_0 = \frac{1}{\sigma_0}$ is resistivity for $B=0$

This is the Drude result. Using

$\sigma_0 = \frac{e^2 n\tau}{m}$ we obtain the Hall resistivity $\rho_{xy} = \frac{-B}{nec}$

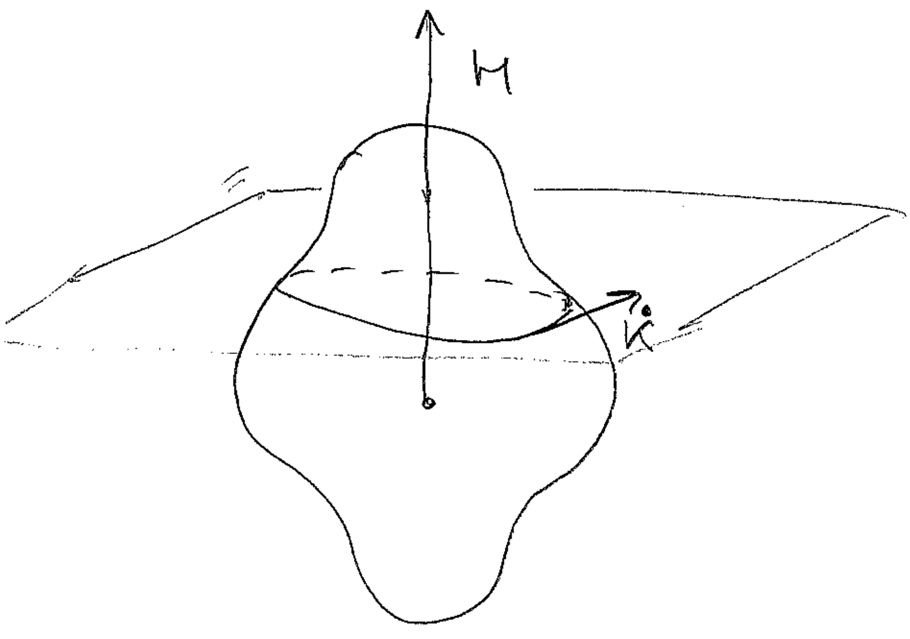
In this approximation magnetoresistance is absent. For spherical spectrum we should replace m by m^*. This will not change the Hall resistivity if $m^* > 0$. But for hole like Fermi surface $m^* < 0$. In the integral for current we replace $d^3k = \int\frac{dS\,d\varepsilon}{|v_k|}$ $= \mathrm{sign}(m^*)\int\frac{dS\,d\varepsilon}{v_k}$. Thus for holes $\rho_{xy} = \frac{B}{nec}$

Semiclassical motion in a uniform magnetic field

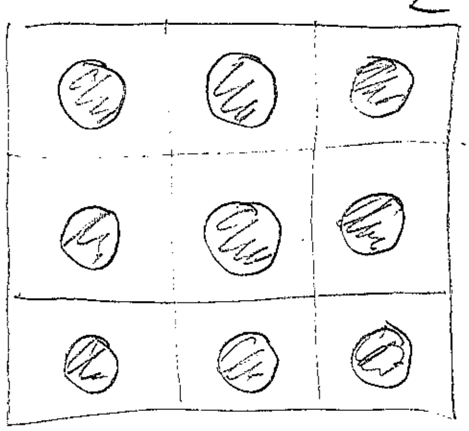
The equation of motion of an electron in a magnetic field is

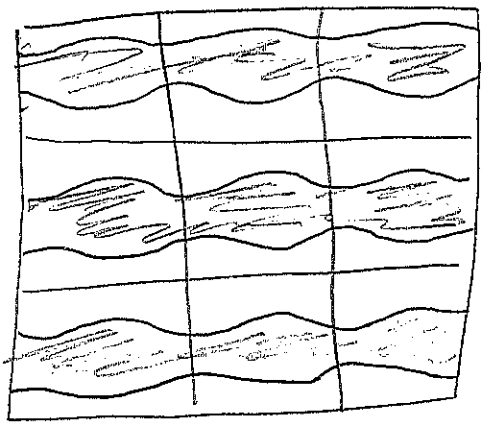
$$\frac{d\vec{k}}{dt} = -\frac{e}{c}[v \times B] = \vec{F}$$

Since the force ortogonal to velocity energy doesn't change. Also momentum component along the field ($B \parallel z$) is conserved. Thus $\vec{k}$ is confined to the orbit defined by the intersection of the Fermi surface with a plane normal to H



One distinguishes closed and open Fermi surfac





For the closed Fermi surfaces all cross-sections are closed orbits. For open Fermi surfaces some cross-sections may be open unbound orbits

Period of motion is given by integration of the equation of motion

$$T = \frac{c}{eB} \oint \frac{dk}{v_\perp}$$

where $\vec{v}_\perp(k)$ is the component of v normal to B at the point $\vec{K}$. The corresponding frequency $\omega_c = \frac{2\pi}{T}$ is called cyclotron frequency.

For the free electrons we have

$$\oint \frac{dk}{v_\perp} = m \oint \frac{dk}{k_\perp} = 2\pi m$$

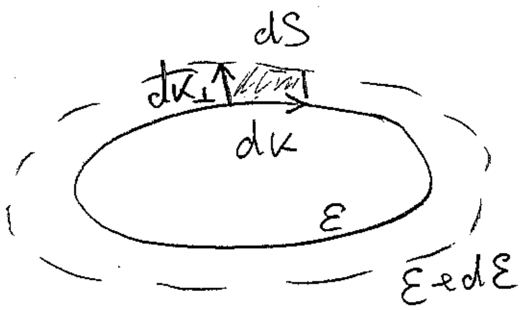
In general situation one defines <u>cyclotron mass</u> m_H^* such that

$$\omega_c = \frac{eB}{m_H^* c}$$

This mass in general is different from other effective masses. In particular electron electron interaction has no effect on it (Kohn's theorem). But even for the simple anisotropic band one can see that the mass entering density of states is different from the cyclotron mass.

The area enclosed by the orbit

$S = \int dk_x dk_y$ can be rewritten as



$$S = \int dk_\perp dk = \int \frac{dk_\perp}{d\mathcal{E}} dk\, d\mathcal{E} = \int d\mathcal{E} \frac{dk}{v_\perp}$$

Thus $T = \frac{c}{eB} \frac{\partial S}{\partial \mathcal{E}}$ and

the cyclotron mass is

$$m_H^* = \frac{1}{2\pi} \frac{\partial S}{\partial \mathcal{E}}$$

If the orbit with $\mathcal{E} = \text{const}$ encircles the states with lower energy then $m^* > 0$ – electron like. If inside the orbit there are states with higher energy then $m^* < 0$ – hole like. Then one can change sign of both charge and mass to go to the hole representation. Cyclotron mass can be defined only for closed orbits.

High field Hall effect

Consider the Boltzmann equation derived on the page 1.

$$e(\vec{E}\cdot\vec{v})\frac{\partial f_0}{\partial \mathcal{E}} = \frac{f_1}{\tau} - \frac{e}{c}\left[\vec{v}\times\vec{B}\right]\frac{\partial f_1}{\partial K}$$

For large enough magnetic field the ratio of the first (collision) term in the r.h.s. is by factor $\frac{1}{\omega_c \tau}$ smaller than the second term. Then it can be neglected. Since spherical symmetry was needed only to derive this term in the high field limit we can obtain general expressions for the arbitrary electron spectrum

Let us consider geometry where $B \parallel z$, $E \parallel x$. Then the Boltzmann equation without collision term is

$$e E_x v_x \frac{\partial f_0}{\partial \mathcal{E}} = eB\left(v_x\frac{\partial f_1}{\partial K_y} - v_y\frac{\partial f_1}{\partial K_x}\right)$$

Let us look for a solution in form

$$f_1 = a\, k_y$$

Substituting it we obtain

$$a = \frac{E_x}{B}\frac{\partial f_0}{\partial \varepsilon}$$

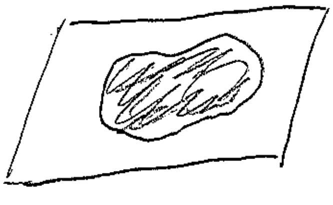
The electric current density along the y axis i

$$j_y = -2e\int \frac{d^3k}{(2\pi\hbar)^3}\, v_y f_1 = -\frac{2eE_x}{B}\int \frac{d^3k}{(2\pi\hbar)^3}\frac{\partial \varepsilon}{\partial k_y} k_y \frac{\partial f_0}{\partial \varepsilon}$$

Replacing $\frac{\partial \varepsilon}{\partial k_y}\frac{\partial f_0}{\partial \varepsilon}$ by $\frac{\partial f_0}{\partial k_y}$ we obtain

$$j_y = -2e\frac{E_x}{B}\int \frac{d^3k}{(2\pi\hbar)^3}\, k_y \frac{\partial f_0}{\partial k_y}$$

Consider closed electron like Fermi surface that lies entirely within the first Brillouin zone. Then f_0 vanishes at the Brillouin zone boundary. Upon partial



integration we transfor the current to $j_y = \frac{2eE_x}{B}\int \frac{d^3k}{(2\pi\hbar)^3} f_0$

The last integral gives the volume inside the Fermi surface divided by $(2\pi\hbar)^3$ and

$$j_y = \sigma_{yx} E_x \quad \text{with} \quad \sigma_{yx} = \frac{n_e e c}{B}$$

or $\sigma_{xy} = -\dfrac{n_e e c}{B}$

where $n_e = \dfrac{2 V_e}{(2\pi\hbar)^3}$, V_e is the volume of the electron like Fermi surface

For the hole like Fermi surface in the expression for the current

$$j_y = -2e \frac{E_x}{B} \int \frac{d^3k}{(2\pi\hbar)^3} \, k_y \frac{\partial f_0}{\partial k_y}.$$

we replace f_0 by $1-\tilde{f}$, where $\tilde{f}$ is the Fermi distribution for holes $\tilde{f}(\varepsilon) = 0$ for $\varepsilon < \varepsilon_F$, $\tilde{f}(\varepsilon) = 1$ for $\varepsilon > \varepsilon_F$

Since in this case $\tilde{f}$ vanishes at the Brillouin zone boundary. we arrive at

$$j_y = -\frac{2eE_x}{B}\int\frac{d^3k}{(2\pi)^3}\,\tilde{f} \quad \text{and}$$

$$\sigma_{xy} = +\frac{n_h e c}{B}$$

where $n_h = \frac{2V_h}{(2\pi\hbar)^3}$, V_h is the volume of the hole like Fermi surface.

In general case we obtain

$$\underline{\sigma_{xy} = -\frac{(n_e - n_h)ec}{B}}$$

This result although coinciding with the Drude result is valid in much more general case. The only assumption we made was that the field is high enough such that the period of the cyclotron orbit is much smaller than a typical collision time.

With $f_1 = a k_y$ one obtains $j_x = 0$ and $\sigma_{xx} = 0$ in this order of approximation.

Our expansion parameter is $\frac{1}{\omega_c \tau}$

In next order there will be contribution to j_x thus $\sigma_{xx} \propto \frac{1}{(\omega_c\tau)^2} \propto \frac{1}{B^2}$

As a result we can write

$$\sigma_{ik} = \begin{pmatrix} \frac{A_{xx}}{B^2} & \frac{A_{xy}}{B} & \frac{A_{xz}}{B} \\ \frac{A_{yx}}{B} & \frac{A_{yy}}{B^2} & \frac{A_{yz}}{B} \\ \frac{A_{zx}}{B} & \frac{A_{zy}}{B} & A_{zz} \end{pmatrix}$$

Motion along the field is not affected by its presence, that's why $\sigma_{zz} = \text{const}$ for $B \to \infty$

Inverting σ_{ik} we obtain the resistivity tensor

$$\rho_{ik} = \begin{pmatrix} b_{xx} & b_{xy}M & b_{xz} \\ b_{yx}M & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix}$$

The Hall resistivity is

$$\rho_{xy} = \frac{-B}{(n_e - n_h)ec}$$

Longitudinal components $\rho_{xx}, \rho_{yy}, \rho_{zz}$ saturate at the finite value for $B \to \infty$

This values are different from ones at zero field

The high field Hall effect can be interpreted in the following way. Equation of motion is

$$\dot{r} = v(k)$$

$$\dot{k} = -e\left(E + \frac{v \times B}{c}\right)$$

Solving second equation for v and substituting to the first one we obtain

$$\dot{r} = -\frac{c\,[\vec{B} \times \dot{\vec{k}}]}{|B|^2} + \frac{c\,[\vec{E} \times \vec{B}]}{|B|^2}$$

Integrating we obtain

$$\vec{r}(t) - \vec{r}(0) = -\frac{c\,[B \times (\vec{k}(t) - \vec{k}(0))]}{B^2} + \frac{c[E \times B]}{|B|^2}$$

The average velocity is given by

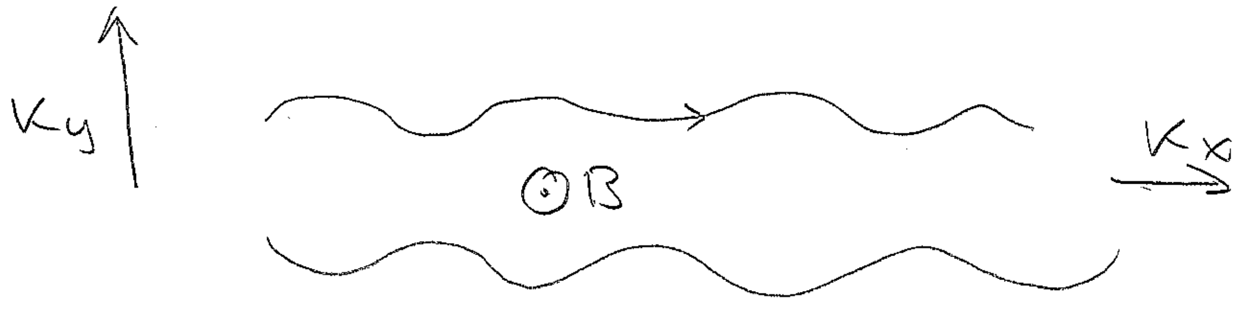
$$\lim_{t\to\infty} \frac{r(t)-r(0)}{t} = \lim_{t\to\infty}\left[-\frac{c\left[B\times(k(t)-k(0))\right]}{B^2 t} + \frac{c\left[E\times B\right]}{|B|^2}\right]$$

For closed orbits $k(t)-k(0)$ is bounded and the first term goes to zero. From the second term multiplying by $-ne$ we obtain

$$\vec{j} = -nec\frac{[\vec{E}\times\vec{B}]}{|B|^2}$$ which reproduces

Hall effect obtained before.

For open orbits result is quite different



In this situation electron can move along the k_x direction. Then $\overline{v_y} = -\frac{c}{eB}\frac{(k_x(t)-k_x(0))}{t} \neq 0$

Then σ_{yy} would not vanish in high field regime

$$\sigma_{ik} = \begin{pmatrix} \frac{A_{xx}}{B^2} & \frac{A_{xy}}{B} & \frac{A_{xz}}{B} \\ \frac{A_{yx}}{B} & A_{yy} & A_{yz} \\ \frac{A_{zx}}{B} & A_{zy} & A_{zz} \end{pmatrix}$$

~~and =~~

$$\text{and } \rho_{ik} = \begin{pmatrix} B^2 b_{xx} & B b_{xy} & B b_{xz} \\ B b_{yx} & b_{yy} & b_{yx} \\ B b_{zx} & b_{zy} & b_{zz} \end{pmatrix}$$

In this case $\rho_{xx} \propto B^2$ grows without limit for $B \to \infty$.

Rotating magnetic field one would have sharp peak in resistivity for E along the direction of the open orbit (B is orthogonal)