Solid State Theory
Solution 12

Problem 12.1 Cyclotron mass

The cyclotron mass is defined as

$$\omega_c = \frac{eB}{m_H c}. \tag{1}$$

In Lec. 23, we showed that

$$m_H^* = \frac{\partial S}{2\pi \partial \epsilon}, \tag{2}$$

where S is the area enclosed by the cyclotron orbit and ϵ is energy. The semi-classical equations of motion (see Lec. 23), define the cyclotron orbit as the cross-section of the Fermi surface on a plane perpendicular to the magnetic field.
Consider a 3D dispersion relation which can be written as a quadratic form plus a constant

$$\epsilon_k = const. + M_{\alpha\beta} k_\alpha k_\beta. \tag{3}$$

The mass tensor $M_{\alpha\beta} = \hbar^2/2m_{\alpha\beta}$ can always be transformed to a diagonal form and as we are working along the high symmetry axis only this is not a problem.

$$\epsilon_k = const. + M_x k_x^2 + M_y k_y^2 + M_z k_z^2. \tag{4}$$

The equal energy surfaces of this dispersion relation are ellipsoids with semi-principle axes of length

$$k_i(\epsilon) = \sqrt{\frac{\epsilon}{M_i}} \qquad i = \{x, y, z\}. \tag{5}$$

Then for a magnetic field oriented along one of the semi-principle axes (say $\vec{B} = B\hat{z}$) of the ellipsoid induces a cyclotron motion that encloses an area

$$S = \pi k_x(\epsilon) k_y(\epsilon) \propto \frac{\epsilon}{\sqrt{M_x M_y}}. \tag{6}$$

As a result, the cyclotron mass is

$$m_H^* = \frac{\partial S}{2\pi \partial \epsilon} = \frac{\hbar^2}{\sqrt{4 M_x M_y}} = \sqrt{m_x m_y} \tag{7}$$

On the other hand, in 3D the specific heat is given by

$$C_v = \frac{\pi^3}{3} k^2 T N(\epsilon_F), \tag{8}$$

where $N(\epsilon_F)$ is the density of states at the fermi surface. Because the density of states is proportional to the number of states per volume with respect to energy. Since the volume of the ellipsoid fermi surface is proportional to

$$\sqrt{m_x m_y m_z} (\epsilon)^{3/2}.$$

Therefore,

$$C_V \propto \sqrt{m_x m_y m_z}(\epsilon)^{1/2}. \tag{9}$$

Comparing this result to the one for electronic specific heat for a 3D system with isotropic spectrum

$$C_V^{iso} \propto (m^*)^{3/2}(\epsilon)^{1/2},$$

the effective mass for an anisotropic system can be defined as

$$m^*_{(ani)} = (m_x m_y m_z)^{1/3} \tag{10}$$

The important distinction is that the effective mass in the specific heat expression is symmetric with respect to any permutation of indices x, y, z, whereas the effective mass does not have such a symmetry as the magnetic field does not cause any response along its direction.

Problem 12.2 Magnetoresistance in the two-band model

We have carriers in two different bands, the applied electric field is the same but their contribution to the current will be different and given by

$$\mathbf{E} = \frac{1}{\sigma_i}\mathbf{J}_i + \frac{\beta_i}{\sigma_i}\mathbf{H} \times \mathbf{J}_i. \tag{11}$$

Where σ is the conductivity and $R = \beta/\sigma$ is the Hall coefficient. We can invert this by first taking its crossproduct with $\mathbf{H}$ giving:

$$\mathbf{H} \times \mathbf{E} = \frac{1}{\sigma_i}\mathbf{H} \times \mathbf{J}_i - \frac{\beta_i H^2}{\sigma_i}\mathbf{J}_i \tag{12}$$

and then combining both equations to give:

$$\mathbf{J}_i = \frac{1}{1 + \beta_i^2 H^2}\left(\mathbf{E} - \beta_i \mathbf{H} \times \mathbf{E}\right) \tag{13}$$

which can also be inverted back to the original form in a similar fashion.
We now have an expression for the total current

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \tag{14}$$

as a function of applied electric and magnetic field. This can again be inverted to give:

$$\mathbf{E} = \frac{1}{\sigma(1 + \beta^2 H^2)}\left(\mathbf{J} + \beta \mathbf{H} \times \mathbf{J}\right) \tag{15}$$

where

$$\sigma = \frac{\sigma_1}{1 + \beta_1^2 H^2} + \frac{\sigma_2}{1 + \beta_2^2 H^2} \tag{16}$$

and

$$\beta = \frac{1}{\sigma}\left(\frac{\sigma_1 \beta_1}{1 + \beta_1^2 H^2} + \frac{\sigma_2 \beta_2}{1 + \beta_2^2 H^2}\right) \tag{17}$$

For low magnetic field we obtain relatively easily the hall coefficient

$$R = \frac{\sigma_1\beta_1 + \sigma_2\beta_2}{(\sigma_1+\sigma_2)^2} = \frac{\sigma_1^2 R_1 + \sigma_2^2 R_2}{(\sigma_1+\sigma_2)^2}. \tag{18}$$

Finding the magnetoresistance is more tricky as we then have

$$\rho = \frac{1}{\sigma(1+\beta^2 H^2)} \tag{19}$$

which we have to contrast with the $\mathbf{H} = 0$ case

$$\rho_0 = \frac{1}{\sigma_1+\sigma_2}. \tag{20}$$

After some algebraic manipulations we obtain a formula comparing them:

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = \frac{\sigma_1\sigma_2(\beta_1-\beta_2)^2 H^2}{(\sigma_1+\sigma_2)^2 + H^2(\beta_1\sigma_1+\beta_2\sigma_2)^2} \tag{21}$$

The magnetoresistance is a positive quantity that vanishes only if $\beta_1 = \beta_2$ though this must not necessarily mean that the two carriers are the same.