Theoretical Physics, Problem Set 4.

FS15 *Hand in: 18.03.15*

1. Electrostatic energy in an external field

Let there be a charge density $\rho(\vec{x})$ in a neighbourhood of $\vec{x} = 0$. Furthermore, let there be an external potential $\varphi(\vec{x})$, which is nearly constant in this neighbourhood and whose sources are located outside. Show that the electrostatic energy of the former in the field $\vec{E} = -\vec{\nabla}\varphi$ of the latter can be expressed as follows:

$$W = e\varphi(0) - \vec{p} \cdot \vec{E}(0) - \frac{1}{6} \sum_{i,j=1}^{3} Q_{ij} \frac{\partial E_j}{\partial x_i}(0) + \dots \,, \qquad (1)$$

where $e = \int d^3x\, \rho(\vec{x})$ is the total charge, $\vec{p} = \int d^3x\, \vec{x}\rho(\vec{x})$ is the dipole moment, and $Q_{ij} = \int d^3x\, (3x_i x_j - \vec{x}^2\delta_{ij})\rho(\vec{x})$ is the quadrupole moment.

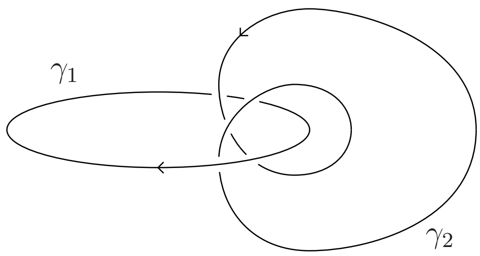
Hint: Expand the potential φ inside $W = \int d^3x \rho(\vec{x})\varphi(\vec{x})$ around $\vec{x} = 0$ using the Taylor series.

2. Gauss' linking number

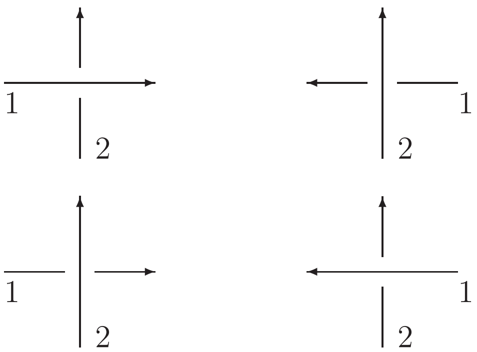
Gauss' linking number of two closed curves γ_1, γ_2 is defined as

$$n(\gamma_1, \gamma_2) = \int_{\gamma_2} \int_{\gamma_1} \frac{(d\vec{s}_2 \wedge d\vec{s}_1) \cdot \vec{r}}{4\pi r^3}$$

where $\vec{r}$ has the same meaning as in Ampères' law of force (2.1). To show: $n(\gamma_1, \gamma_2)$ is an integer describing how often one curve is winded around the other ($n(\gamma_1, \gamma_2) = n(\gamma_2, \gamma_1)$).



More precisely: Project the two curves on a plane; let the following be



positive

crossings.

negative

Then $n(\gamma_1, \gamma_2) = (n_+ - n_-)/2$, where $n_\pm$ is the number of positive/negative crossings.

Hint: Setting $I_1 = I_2 = c = 1$, we have $n(\gamma_1, \gamma_2) = \int_{\gamma_2} \vec{B}_1 \cdot d\vec{s}_2$. Use the theorem of Stokes to show that this expression is invariant under deformations of γ_2. The same holds for decompositions such as the one pictured below. Use deformations and decompositions to simplify the linkings as much as possible.