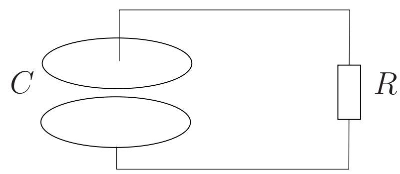
Theoretical Physics, Problem set 5.

FS15 *Hand in: 25.03.13*

1. Energy flow during discharge of a capacitor

Discuss qualitatively the energy flow (Poynting vector) during the slow discharge of a capacitor via a resistor.



2. Satellite on a leash

On a mission of the Space Shuttle (1996) a satellite was marooned via a tensioning rope of the length of 20 km. The rope was conducting and separated from the surrounding dilute plasma (ionized gas) by a sheath. Explain why there was a current in the rope. Compute the electromotive force along the rope in terms of magnitude. Where does the involved energy come from?

Hint: The radius of the earth is about 6'400 km. The magnetic field near the earth amounts to $\sim 40\mu$T in SI units (1 Tesla =1 Vs/m^2).

Remark: In the lecture Heaviside-Lorentz units are used. To make a transition to SI units, one has to replace

$$1 \rightsquigarrow \varepsilon_0^{-1}\,, \qquad c^{-2} \rightsquigarrow \mu_0$$

in Coulomb's and Ampère's law of force, (1.1) and (2.1) respectively. Instead of distributing c^{-2} to equal parts to the field (2.2) and the force (2.3), μ_0 in the SI system is defined to be only in (2.2). Since the force is defined purely mechanical, it follows from (2.3) that $c^{-1}e_{\mathrm{HL}}\vec{B}_{\mathrm{HL}} = e_{\mathrm{SI}}\vec{B}_{\mathrm{SI}}$. In particular, for the Lorentz force it follows $[e(\vec{v}/c)\wedge\vec{B}]_{\mathrm{HL}} = [e\vec{v}\wedge\vec{B}]_{\mathrm{SI}}$.

3. Complete and partially polarized light

A monochromatic wave in propagation direction $\vec{e}_3$ is, in complex notation, of the form

$$\begin{aligned} \vec{B} &= \vec{e}_3 \wedge \vec{E}\,, \qquad & \vec{E}(\vec{x},t) &= \vec{E}(t - \vec{e}_3\cdot\vec{x}/c)\,,\\ \vec{E}(t) &= \vec{E}_0 \mathrm{e}^{-\mathrm{i}\omega_0 t}\,, \qquad & \vec{E}_0 &= (E_1, E_2, 0)\,. \end{aligned} \tag{1}$$

The polarization of the wave is described by the complex vector

$$\underline{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \in \mathbb{C}^2, \tag{2}$$

which contains four real degrees of freedom.

A measurement of polarization first of all filters the wave, namely in the direction of a particular polarization $\underline{\varepsilon}$, $((\underline{\varepsilon},\underline{\varepsilon}) = 1)$, i.e. $\underline{E} \rightsquigarrow \underline{E}' = (\underline{\varepsilon},\underline{E})\underline{\varepsilon}$. Subsequently the intensity

$$I = \frac{c}{2}(\underline{E}',\underline{E}') = \frac{c}{2}|(\underline{\varepsilon},\underline{E})|^2$$

is measured. If the polarization is changed by a phase, $\underline{E} \rightsquigarrow \mathrm{e}^{\mathrm{i}\varphi}\underline{E}$, only the physical field $\mathrm{Re}\,\vec{E}(t)$ is delayed which has no influence on the measurements. Thus there remain three degrees of freedom which have to be expressed in an experimentally useful way.

A *quasi*-monochromatic wave is described by the replacement $\underline{E} \rightsquigarrow \underline{E}(t)$ in (2), where the corresponding amplitude $\vec{E}_0(t)$ in (1) is now

- changing slowly on the time scale $2\pi/\omega_0$ (period); the characteristic time scale of $\vec{E}_0(t)$ is called coherence time τ.
- changing rapidly on the time scale of the measurements. The amplitudes $\vec{E}_0(t_i)$, $(i = 1, 2)$ turn out to be uncorrelated if $t_2 - t_1 \gg \tau$.

Regard $\underline{E}$ as a random variable. We denote mean values by $\langle \cdot \rangle$. The strictly monochromatic wave corresponds to the deterministic special case of a pure polarization.

The goal of the exercise is to emphasize the following statement: The polarization of light is described by the matrix

$$S = \begin{pmatrix} \langle E_1 \overline{E}_1 \rangle & \langle E_1 \overline{E}_2 \rangle \\ \langle E_2 \overline{E}_1 \rangle & \langle E_2 \overline{E}_2 \rangle \end{pmatrix} = S^*\,, \qquad \text{d.h. } S = \langle \underline{E}\, \underline{E}^* \rangle, \tag{3}$$

where $\underline{E}^* = (\overline{E}_1, \overline{E}_2)$.

i) Show that S is of the form

$$S = s_0\sigma_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 \equiv s_0 1_2 + \vec{s} \cdot \vec{\sigma}, \tag{4}$$

where $s_i \in \mathbb{R}$ (four stokes parameters), $\sigma_0 = 1_2$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices). *Hint:* The 2×2-matrices $S = S^*$ form a real vector space. What is its dimension?

ii) Express s_i using the mean values $\langle E_j \overline{E}_k \rangle$. *Hint:* $(S, T) = \mathrm{tr}(ST)/2$ is a scalar product. Further $\sigma_i^2 = 1_2$ and $\sigma_1\sigma_2 = \mathrm{i}\sigma_3$ (and cyclic).

iii) Show that in the case of a pure polarization ($\vec{E}_0$ fix, mean values not necessary)

$$|\vec{s}| = s_0. \tag{5}$$

This corresponds to the mentioned three degrees of freedom. *Hint:* Compute S^2 in this case an take the trace.

iv) Show that generally

$$|\vec{s}| \leq s_0\,.$$

Hint: Average over (5); alternatively show: $S \geq 0$ and the eigenvalues of S are $s_0 \pm |\vec{s}|$.

v) Find the meaning of s_0 and s_i/s_0, $(i = 1, 2, 3)$. What does $\vec{s} = 0$ mean? *Hint:* The eigenvalues of σ_i are ± 1, $(i = 1, 2, 3)$. What are the eigenvectors $\vec{e}_\pm^{(i)}$? Express s_i using the coefficients $\alpha_\pm^{(i)}$ of the decomposition $\underline{E} = \alpha_+^{(i)} \vec{e}_+^{(i)} + \alpha_-^{(i)} \vec{e}_-^{(i)}$. Use that the trace does not depend on the basis.

Remark: In optics the matrices σ_i are defined a bit different. Here the Pauli-matrices, which are common in quantum mechanics, are used.