Theoretical Physics, Problem Set 8.

FS15 *Hand in: 22.04.15*

1. Lorentz transformations on the celestial sphere

The position of the stars on the celestial sphere is the direction their light comes from. The light of a star comes from different directions w.r.t. two inertial systems (aberration), because they are rotated or moved w.r.t. each other. Consequently, the two observers see the starry sky in a different way. Show: The transformation is Möbius.

A transformation on the extended complex plane $\mathbb{C} \cup \{\infty\}$ is called Möbius if it is of the form

$$z \longmapsto \frac{az+b}{cz+d}, \qquad (a, b, c, d \in \mathbb{C}),$$

with $ad - bc \neq 0$. The composition of Möbius transformations corresponds to the product of the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Two matrices A, B define the same transformation if and only if $B = \lambda A$, $\lambda \neq 0$. The scaling $\det A = 1$ only leaves the choice $\lambda = \pm 1$. Therefore the Möbius group is $\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$.

The plane $\mathbb{C} \cup \{\infty\}$ can be considered as Riemann sphere $S^2 = \{\vec{v} \in \mathbb{R}^3 \,|\, |\vec{v}| = 1\}$ via the stereographic projection. In both descriptions (plane and sphere), a transformation is Möbius if and only if it is circle- and orientation-preserving.

Show that a Lorentz transformation $\Lambda \in L_+^\uparrow$ induces a Möbius transformation $\pm A$ on S^2 when the latter is perceived as the celestial sphere of the direstions of the light. For this purpose, note that Lorentz transformations are characterized by the following properties:

i) Invariance of form of the law of inertia

$$\vec{x} = \vec{b} + \vec{v}t \quad \Longleftrightarrow \quad \vec{x}\,' = \vec{b}\,' + \vec{v}\,' t'\,;$$

ii) Invariance of the speed of light ($c = 1$)

$$|\vec{v}| = 1 \quad \Longleftrightarrow \quad |\vec{v}\,'| = 1\,.$$

Consider the map $S : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\vec{v} \mapsto \vec{v}\,'$, which is induced by Λ on the velocities. Show that S is collinear, even though S, in contrast to $\Lambda : \mathbb{R}^4 \mapsto \mathbb{R}^4$, is not linear. *Hint:* What is the meaning of the property "the points $\vec{v}_1, \vec{v}_2, \vec{v}_3$ lie on a line in $\mathbb{R}^3$" in terms of inertial trajectories in $\mathbb{R}^{1+3}$?

With the help of (ii), conclude that S maps circles in S^2 on circles in S^2. *Hint:* What is such a circle with respect to $S^2 \subset \mathbb{R}^3$?

The result can be summarized in the isomorphism

$$L_+^\uparrow \cong \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}\,.$$

2. Doppler shift and aberration

(a) Let the field $\varphi(\vec{x}, t)$ be a scalar field under Lorentz transformations $x' = \Lambda x$, i.e. $\varphi'(x') = \varphi(\Lambda^{-1}x')$. Show that for a wave $\varphi(\vec{x}, t) = \mathrm{e}^{\mathrm{i}(\vec{k}\vec{x}-\omega t)}$, the frequency and the wave vector form a 4-vector $k \equiv (\omega/c, \vec{k})$.

Henceforth let K and K' be connected via a boost $\Lambda = \Lambda(v\vec{e}_1)$ in 1-direction. Consider light with direction of propagation $\vec{e}$ and frequency ω w.r.t. K.

(b) Let $\vec{e} = \vec{e}_1$. Compute the frequency w.r.t. K' (*Doppler shift*).

(c) Let $\vec{e} = \vec{e}_2$. Determine the angle $\alpha(v)$ between the wave vector and the 2-axis w.r.t. K' (*Aberration*).

3. Calculating with tensors

For a 2-dimensional vector space, let a metric in the basis e_1, e_2 be given by

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The tensor T of type $\binom{0}{2}$ shall be defined by $T(a, b) = 2a^1b^2$, where (a^μ) and (b^μ) are the components of the vectors a and b w.r.t. the basis (e_μ).

(a) Determine the matrices $(T_{\mu\nu}), (T_\mu{}^\nu), (T^\mu{}_\nu), (T^{\mu\nu})$. What is the trace of T?

(b) Consider the basis transformation $\bar{e}_1 = e_1+2e_2$, $\bar{e}_2 = e_1$. Determine the transformation matrices $\Lambda^\mu{}_\nu$, $\Lambda_\mu{}^\nu$, as well as the components $\bar{g}_{\mu\nu}, \bar{T}^{\mu\nu}$.