Theoretical Physics, Problem Set 9.

FS15 *Hand in: 29.04.15*

1. Field of a uniformly moved charge

A point charge e is moving on an inertial trajectory $\vec{x} = \vec{v}t$, $\vec{v} = (v, 0, 0)$. Compute the fields $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$. In which directions is $\vec{E}(\vec{x}, t = 0)$ the strongest and weakest at the same distance $|\vec{x}|$ from the charge respectively?

Hint: Compute first the fields in the rest frame of the particle.

2. Dual field tensor

Define the dual field tensor

$$\mathcal{F}_{\rho\sigma} = \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \tag{1}$$

and the dual current

$$\mathcal{J}_{\nu\rho\sigma} = j^{\mu} \varepsilon_{\mu\nu\rho\sigma} \,,$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor with $\varepsilon_{0123} = +1$ (parity of the permutation $(0123) \mapsto (\mu\nu\rho\sigma)$).

Remark: Here duality is not meant in terms of the dual space, but of the Hodge duality (s. Anhang B, inessential for this exercise). It follows that $\mathcal{F}$ and $\mathcal{J}$ are tensors; alternatively also from Exercise 1.2, whereby $|g|^{1/2} \varepsilon_{\mu\nu\rho\sigma}$, ($g = \det(g_{\mu\nu})$) is a tensor under arbitrary coordinate transformations ($\varepsilon_{\mu\nu\rho\sigma}$ under Lorentz transformations respectively).

i) Express the tensor components $\mathcal{F}_{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$. The duality turns out to be an electric-magnetic one.

ii) Show: The Maxwell equations read

$$\mathcal{F}^{\mu\nu}{}_{,\mu} = 0 \,,$$
$$\mathcal{F}_{\rho\sigma,\mu} + \mathcal{F}_{\mu\rho,\sigma} + \mathcal{F}_{\sigma\mu,\rho} = -\frac{1}{c} \mathcal{J}_{\rho\sigma\mu} \,.$$

Hint: Show

$$\mathcal{F}_{\rho\sigma,\mu} + \mathcal{F}_{\mu\rho,\sigma} + \mathcal{F}_{\sigma\mu,\rho} = F^{\alpha\nu}{}_{,\alpha} \varepsilon_{\mu\nu\rho\sigma} \,. \tag{2}$$

In the $(\vec{E}, \vec{B})$-notation, the left-hand sides of the homogeneous and the inhomogeneous Maxwell equations emerge from each other under $(\vec{E}, \vec{B}) \rightsquigarrow (-\vec{B}, \vec{E})$. The equations above express this symmetry in relativistic notation.