# Theoretical Physics, Problem Set 13.

*FS15* *Hand in: 27.05.15*

## 1. Stability of the hydrogen atom

The Hamilton operator of the hydrogen atom is

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{|\vec{x}|}$$

on $L^2(\mathbb{R}^3)$. Its classical counterpart is the energy as a function of $(\vec{x}, \vec{p})$, given by the expression on the right-hand side.

Classically, the energies are not bounded from below; quantum mechanically they however are. As a heuristic reason for this stability there is often quoted the uncertainty principle of Heisenberg: an electron at a distance $r$ from the kernel has a momentum of order $\Delta p \sim \hbar/r$, and hence energy $\sim \hbar^2/(2mr^2) - e^2/r$. The latter has, as a function of $r > 0$, a minimum: namely for $r = a_0$ with the Bohr radius $a_0 = \hbar^2/me^2$.

i) The consideration like that is of no use: by $\langle p_i^2 \rangle_\psi \geq \langle p_i^2 \rangle_\psi - \langle p_i \rangle_\psi^2 = \langle (\Delta p_i)^2 \rangle_\psi$, and the same for $x_i$, we can deduce

$$\langle \vec{p}^2 \rangle_\psi \langle \vec{x}^{\,2} \rangle_\psi \geq \sum_{i=1}^{3} \langle p_i^2 \rangle_\psi \langle x_i^2 \rangle_\psi \geq \frac{3\hbar^2}{4}$$

and hence

$$\langle H \rangle_\psi \geq \frac{3\hbar^2}{8m} \frac{1}{\langle \vec{x}^{\,2} \rangle_\psi} - e^2 \Big\langle \frac{1}{|\vec{x}|} \Big\rangle_\psi \,.$$

But (show!) there are states $|\psi\rangle$, for which the right-hand side is arbitrary large and negative.

*Hint:* Choose $\psi(\vec{x})$ as a superposition of two wave functions: one away from the kernel and one very near.

ii) A better uncertainty principle with respect to the above is the Hardy inequality

$$-\Delta \geq \frac{1}{4\,\vec{x}^{\,2}} \tag{1}$$

(in the sense of quadratic forms, i.e. quantum mechanical expectation values). Show with it: $H \geq -C$ for a $C > 0$.

*Hint:* $\vec{p}^2 = -\hbar^2 \Delta$.