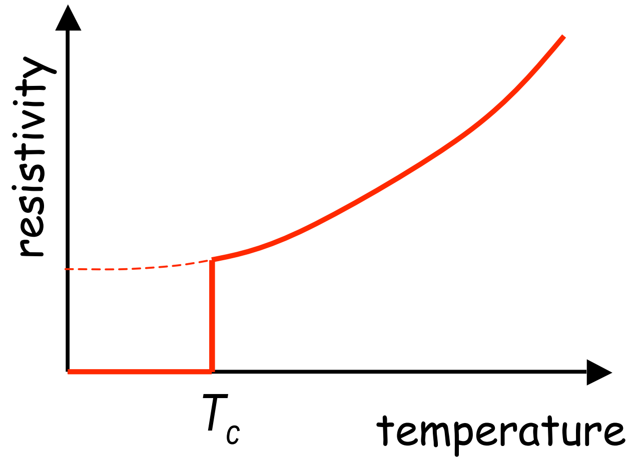
Unconventional Superconductivity

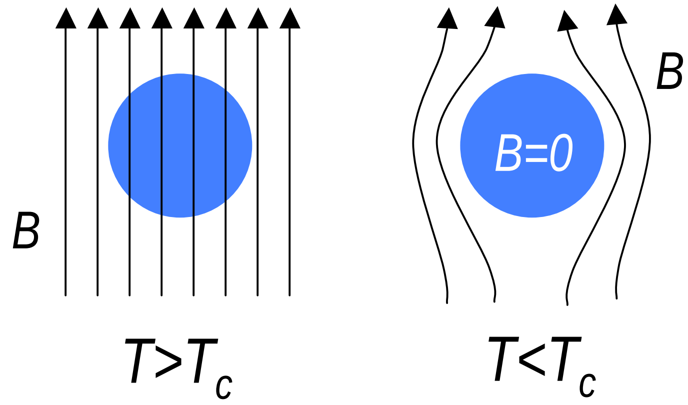
Introduction

Superconductivity

Electrical resistance (1911)



Field expulsion (1933)
Meissner-Ochsenfeld effect



Superconductivity as a thermodynamic phase

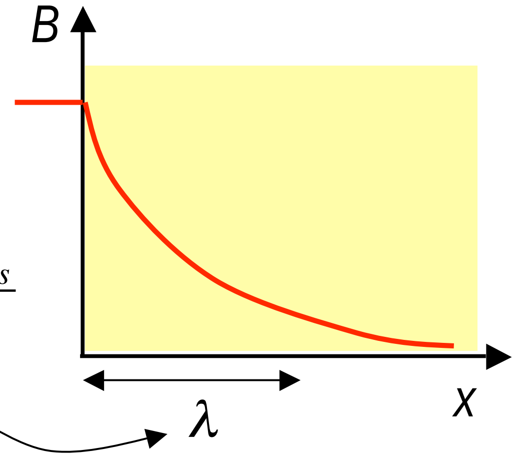
London theory (1935)

$$\left.\begin{aligned}\nabla \times \lambda^2 \vec{j} &= -\vec{B} \\ \nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j}\end{aligned}\right\} \Rightarrow \boxed{\nabla^2 \vec{B} = \lambda^{-2}\vec{B}}$$

density of superconducting electrons

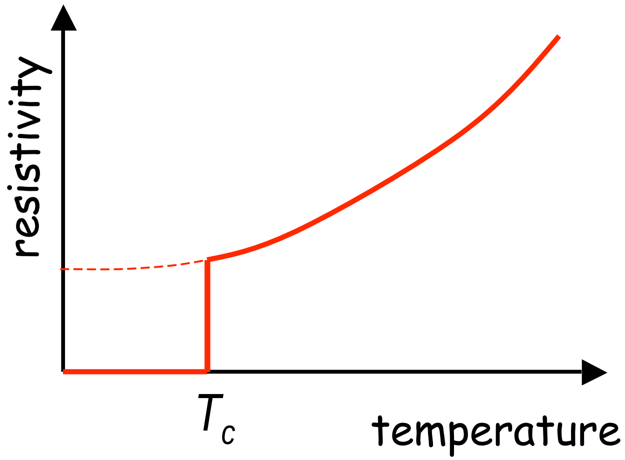
$$\lambda^{-2} = \frac{4\pi e^2 n_s}{mc^2}$$

London penetration depth

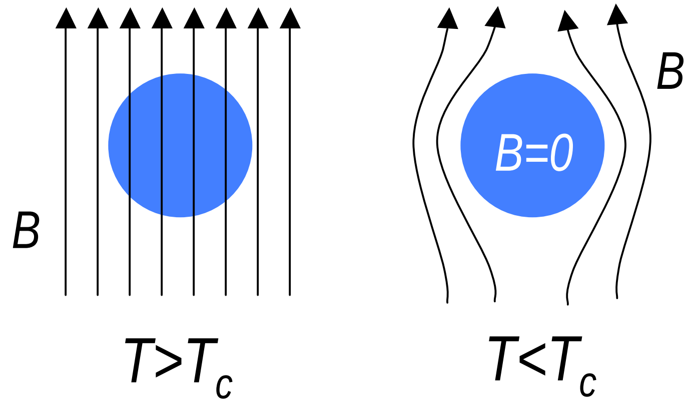


Superconductivity

Electrical resistance (1911)



Field expulsion (1933)
Meissner-Ochsenfeld effect



Superconductivity as a thermodynamic phase

Order parameter: $\Psi(\vec{r}) = |\Psi(\vec{r})| e^{i\varphi(\vec{r})}$ condensate with a broken *U(1)*-gauge symmetry

$$F\left[\Psi,\vec{A}\right] = \int d^3r \left[a(T)|\Psi|^2 + b|\Psi|^4 + K\left|\vec{D}\Psi\right|^2 + \frac{1}{8\pi}\left(\vec{\nabla}\times\vec{A}\right)^2 \right]$$

Ginzburg-Landau theory (1950)

minimal coupling $\vec{D} = \vec{\nabla} + i\frac{2e}{\hbar c}\vec{A}$

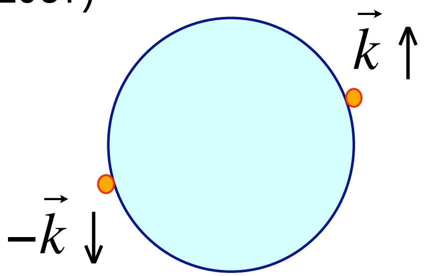
Conventional superconductivity

Order parameter $\Psi(\vec{r}) = |\Psi(\vec{r})| e^{i\varphi(\vec{r})}$ structureless complex condensate wave function

Microscopic origin: Coherent state of Cooper pairs

$$|\psi\rangle = \prod_{|\vec{k}|\le k_F} \{u_{\vec{k}} + v_{\vec{k}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow}\} |0\rangle = \left(\prod_{|\vec{k}|\le k_F} u_{\vec{k}}\right) \exp\left(\sum_{|\vec{k}|\le k_F} \frac{v_{\vec{k}}}{u_{\vec{k}}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow}\right)$$

Bardeen-Cooper-Schrieffer (1957)



pairs of electrons diametral on Fermi surface; vanishing total momentum

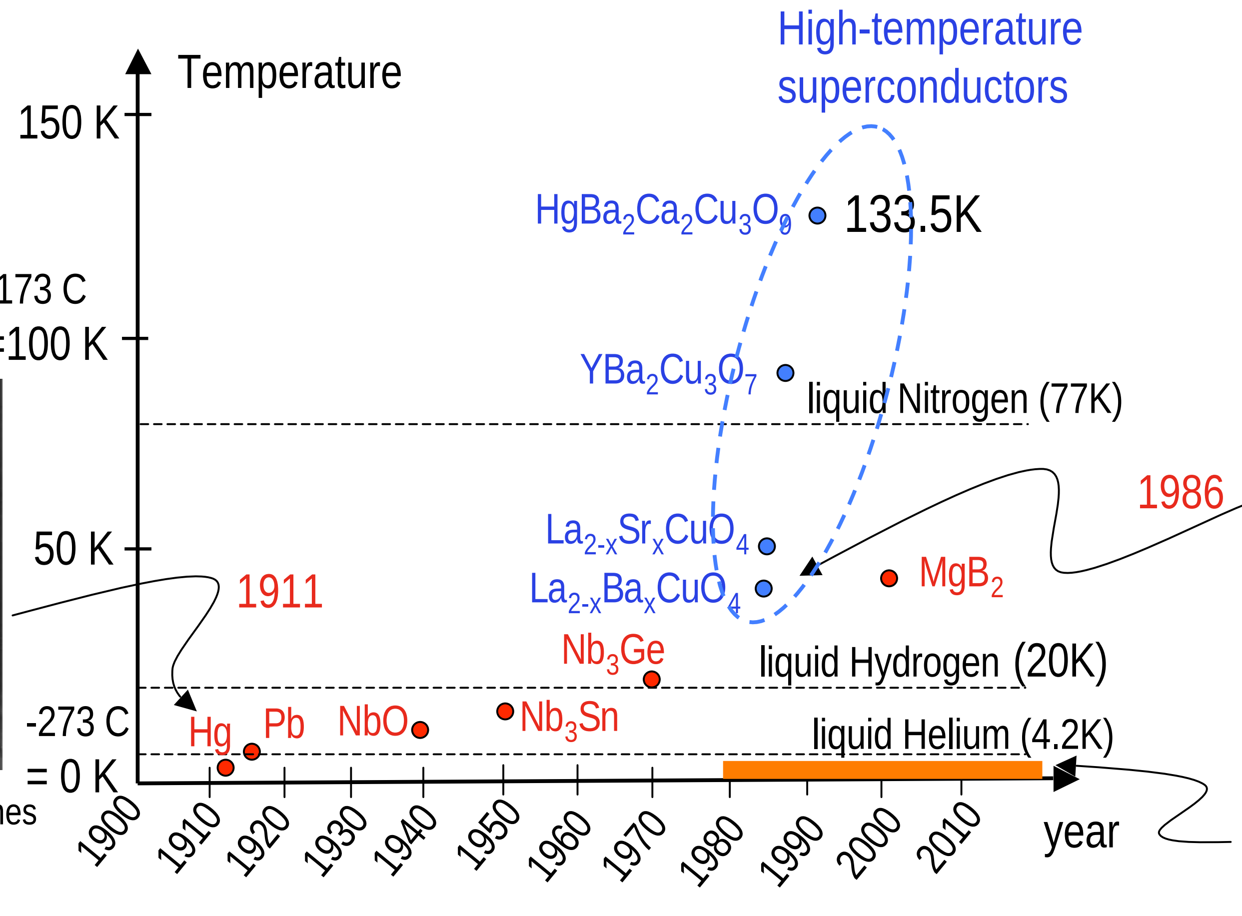
$$\Psi_{\vec{k}} = \langle\psi| c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} |\psi\rangle = u_{\vec{k}} v_{\vec{k}}$$

violation of *U(1)*-gauge symmetry

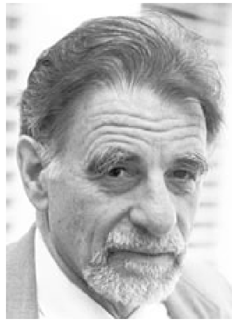
$$c_{\vec{k}\uparrow} \to c_{\vec{k}\uparrow} e^{i\alpha} \Rightarrow \Psi_{\vec{k}} \to \Psi_{\vec{k}} e^{i2\alpha}$$

Conventional $\Psi_{\vec{k}} = \Psi$ independent of $\vec{k}$

The unsteady rise of T_c



H. Kamerlingh-Onnes



K.A. Müller

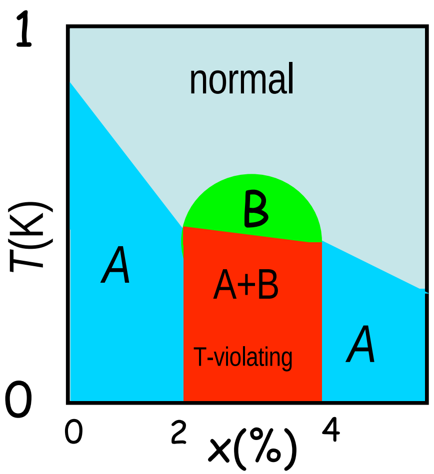
J.G. Bednorz

The novel superconductors

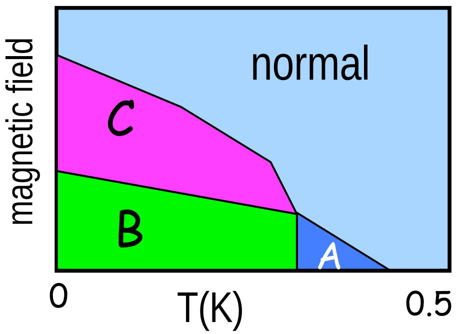
Heavy Fermion superconductors:

$CeCu_2Si_2$ Steglich et al. (1979)

$U_{1-x}Th_xBe_{13}$ Ott et al. (1983)

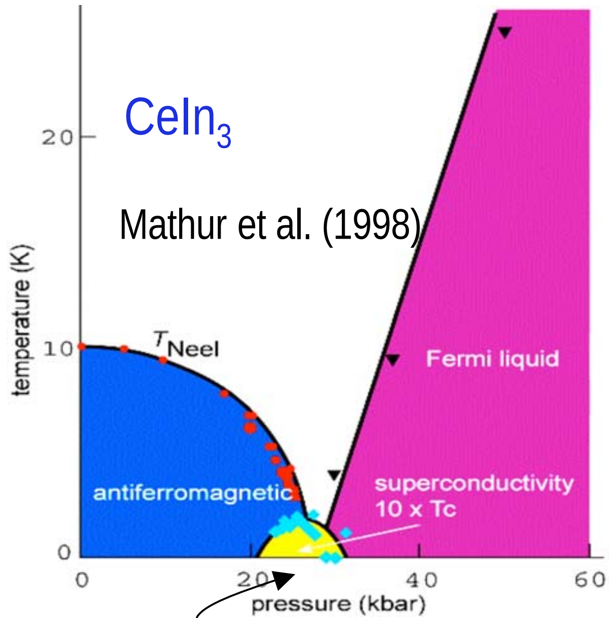


UPt_3 Stewart et al. (1984)



$CeIn_3$

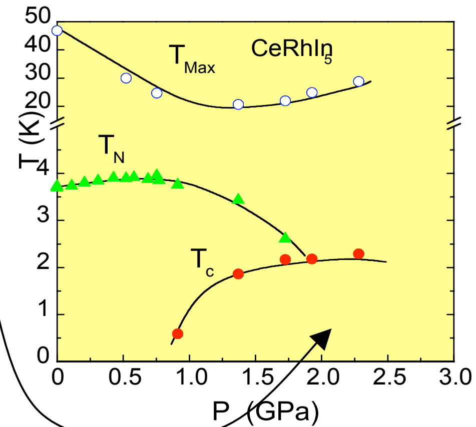
Mathur et al. (1998)



Quantum
Critical point

AF ⟷ PM

$CeRhIn_5$ Thompson et al. (2001)



The novel superconductors

High-temperature superconductors

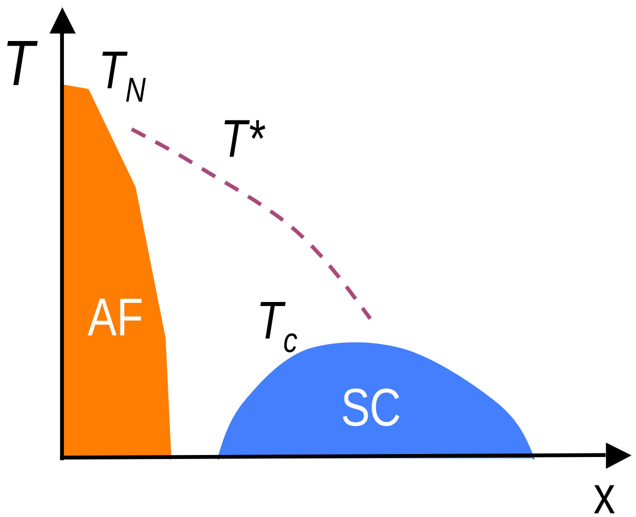
Layered perovskite cooper-oxides

Müller & Bednorz (1986)

$La_{2-x}Sr_xCuO_4$ T_c=45K

$YBa_2Cu_3O_{6+\delta}$ T_c=92K

$HgBa_2Ca_2Cu_3O_9$ T_c=133.5K

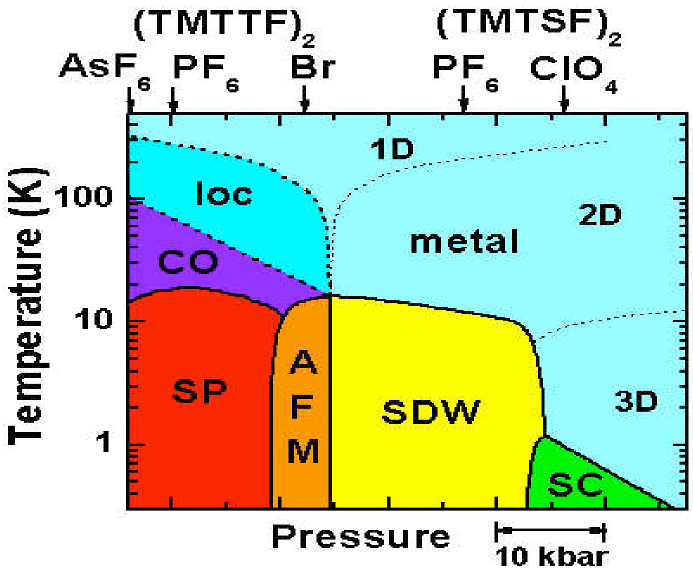


Organic superconductors

Jerome, Bechtgard et al (1980)

$(TMTSF)_2M$ ($M=PF_6$, SbF_6, ReO_4,…) T_c ~ 1K

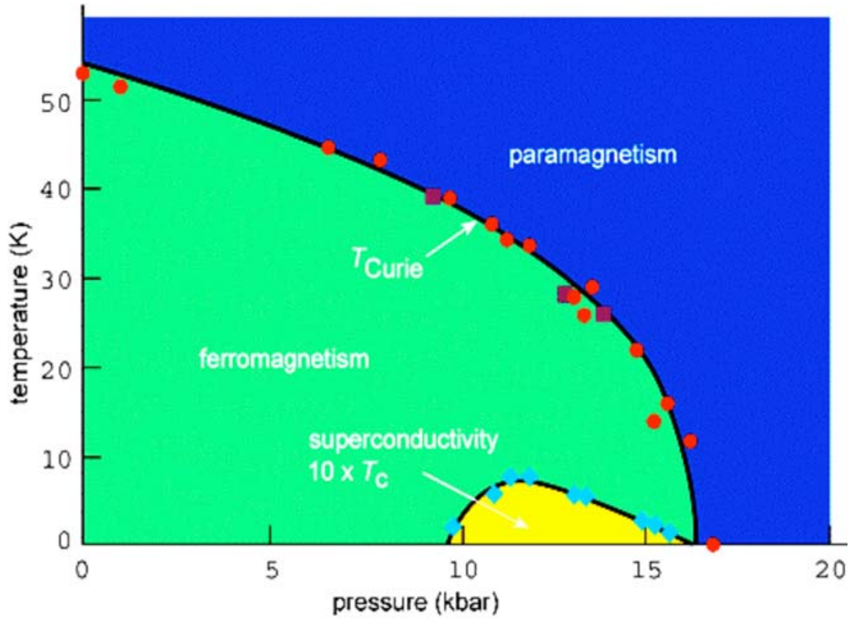
$(BEDT\text{-}TTF)_2M$ ….. T_c ~ 10K



The novel superconductors

Ferromagnetic superconductors:

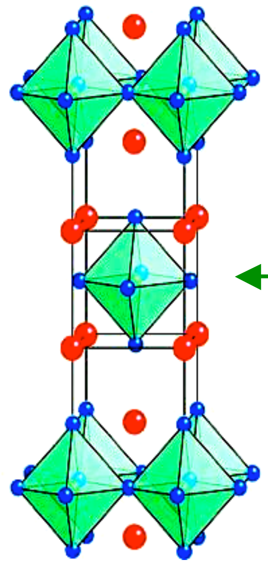
UGe_2 Saxena et a. (2000)



$ZrZn_2$ Pfleiderer et al. (2001)

Superconductivity within
the ferromagnetic phase

Sr_2RuO_4



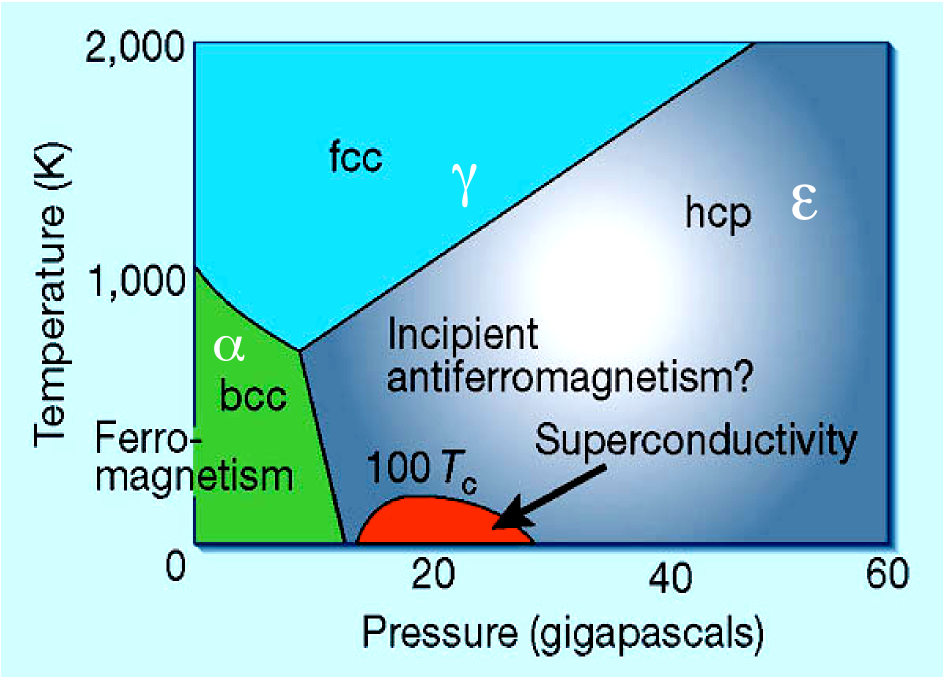
some similarities with
high-T_c superconductors,

but T_c = 1.5 K

spin-triplet superconductor

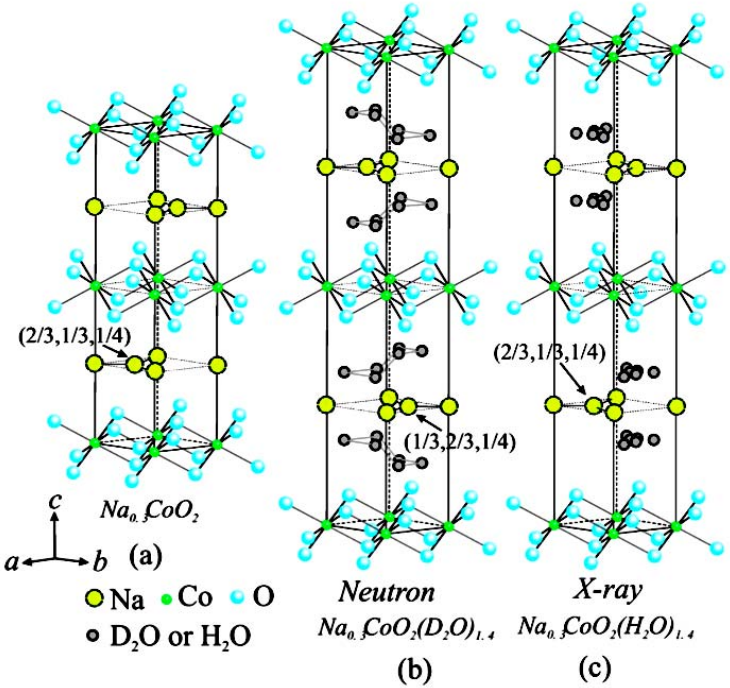
The novel superconductors - under extreme conditions

Iron under pressure



Shimizu et al. Nature 412, 316 (2001)

Hydrated Na_xCoO_4



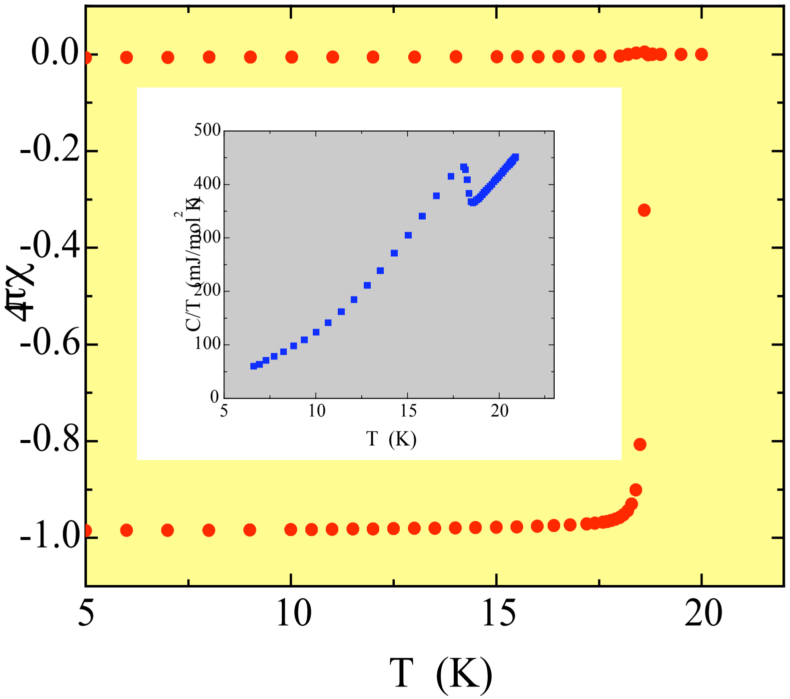
Layered structure: triangular

Superconductivity in a frustrated electron system $T_c \sim 5$ K

Takada et al., Nature 422, 53 (2003)

The novel superconductors

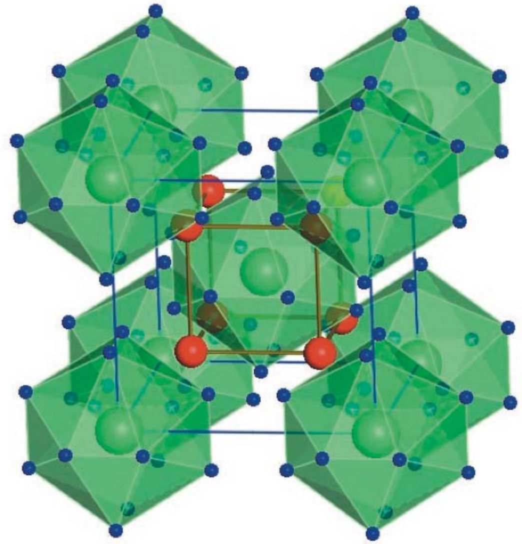
Time-dependent superconductivity



$PuCoGa_5$ T_c = 18 K

Thompson et al. (Los Alamos)

Skutterudite



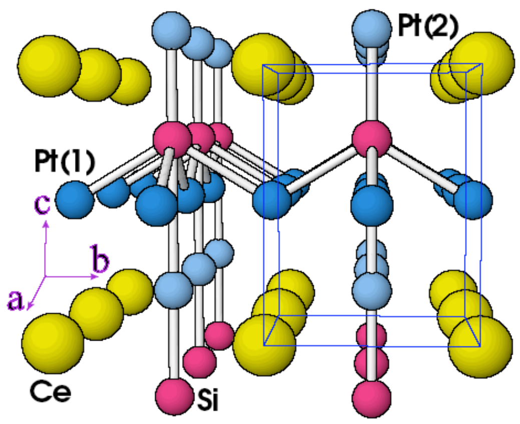
$PrOs_4Sb_{12}$ T_c = 1.8 K

Bauer et al. PRB 65, R100506 (2002)

Multiple phases

The novel superconductors - no inversion symmetry

No paramagnetic limiting

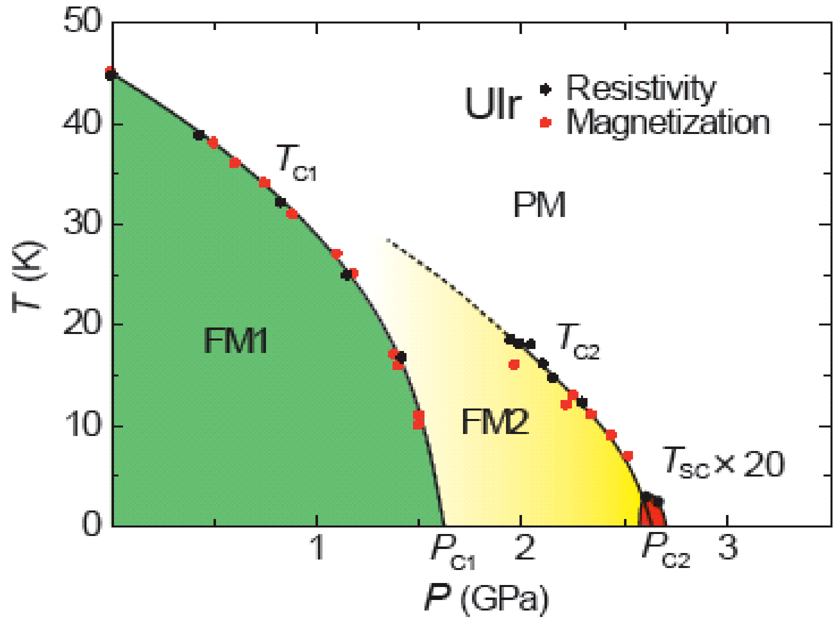


$CePt_3Si$ T_c=0.8 K

H_{c2} exceeds drastically the paramagnetic limit

Bauer et al. PRL 92, 027003 (2004)

Ferromagnetic quantum phase transition



UIr T_c=0.15 K

Akazawa et al. J.Phys. Condens. Matter 16, L29 (2004)

Bardeen-Cooper-Schrieffer

Microscopic theory of superconductivity

BCS mean field theory

simple model:

$$\mathcal{H} = \underbrace{\sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s}}_{\text{band energy}} + \underbrace{g \sum_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}}_{\text{pairing interaction}}$$

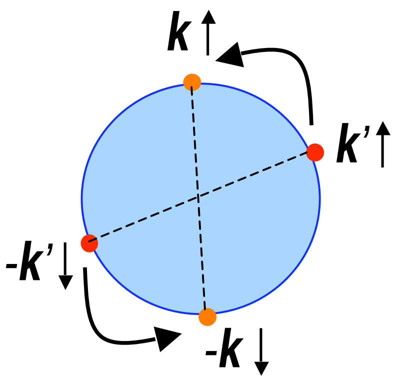
band energy:

$$\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu = \frac{\hbar^2}{2m}(\vec{k}^2 - k_F^2)$$

pairing interaction:

$$U(\vec{r} - \vec{r}') = g\delta^{(3)}(\vec{r} - \vec{r}')$$

attractive contact Interaction $g < 0$



$$V(\vec{q} = \vec{k} - \vec{k}') = \int d^3r \, U(\vec{r}) e^{i\vec{q}\cdot\vec{r}} = g = V_{\vec{k},\vec{k}'}$$

consider only scattering between zero-momentum electron pairs of opposite spin (spin singlet)

BCS mean field theory

simple model:

$$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}$$

decoupling of interaction term by means of

mean fields:

$$\rho_{\vec{q}} = \sum_{\vec{k},s} \langle c^{\dagger}_{\vec{k}+\vec{q}s} c_{\vec{k}s} \rangle$$

particle density

$$\vec{S}_{\vec{q}} = \sum_{\vec{k}} \sum_{s,s'} \langle c^{\dagger}_{\vec{k}s} \vec{\sigma}_{ss'} c_{\vec{k}s'} \rangle$$

spin density

$$b_{\vec{k}} = \langle c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} \rangle$$

BCS - “off diagonal”

BCS mean field theory

simple model: $$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}$$

replace: $$c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} = b^*_{\vec{k}} + \{c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}}\}\,, \quad c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} = b_{\vec{k}} + \{c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} - b_{\vec{k}}\}$$

mean field Hamiltonian:

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} \{b^*_{\vec{k}'} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + b_{\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}} b_{\vec{k}'}\}$$

$$= \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \sum_{\vec{k}} \left\{\Delta^* c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - \Delta^* b_{\vec{k}}\right\}$$

with $$\Delta^* = -g \sum_{\vec{k}'} b^*_{\vec{k}'}\,, \qquad \Delta = -g \sum_{\vec{k}'} b_{\vec{k}'}$$

BCS mean field theory

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} \{ b^*_{\vec{k}'} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + b_{\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}} b_{\vec{k}'} \}$$

$$= \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \sum_{\vec{k}} \left\{ \Delta^* c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - \Delta^* b_{\vec{k}} \right\}$$

find quasiparticle states with $\frac{\partial}{\partial t}\gamma^\dagger_{\vec{k}} = i[\mathcal{H}_{mf}, \gamma^\dagger_{\vec{k}}] = E_{\vec{k}} \gamma^\dagger_{\vec{k}}$

Bogolyubov-transformation

$$\begin{aligned} c_{\vec{k}\uparrow} &= u^*_{\vec{k}} \gamma_{\vec{k}1} + v_{\vec{k}} \gamma^\dagger_{\vec{k}2} \\ c^\dagger_{-\vec{k}\downarrow} &= -v^*_{\vec{k}} \gamma_{\vec{k}1} + u_{\vec{k}} \gamma^\dagger_{\vec{k}2} \end{aligned} \qquad |u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$$

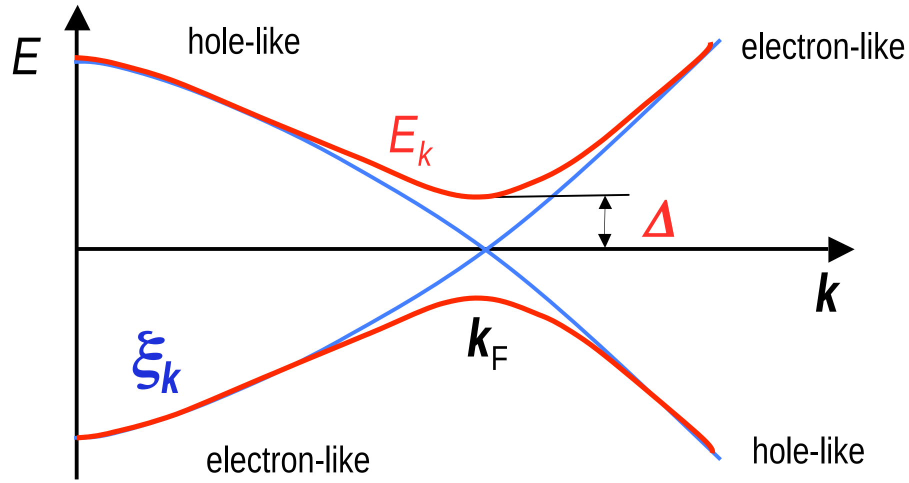
➡ quasiparticle energy $E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + \Delta^2}$

➡ $$\mathcal{H} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] + \sum_{\vec{k}} E_{\vec{k}} (\gamma^\dagger_{\vec{k}1}\gamma_{\vec{k}1} + \gamma^\dagger_{\vec{k}2}\gamma_{\vec{k}2})$$

Quasiparticle Spectrum

$$\mathcal{H} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] + \sum_{\vec{k}} E_{\vec{k}} (\gamma^\dagger_{\vec{k}1}\gamma_{\vec{k}1} + \gamma^\dagger_{\vec{k}2}\gamma_{\vec{k}2})$$

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + \Delta^2}$$



quasiparticle excitation gap: Δ

condensation energy gain due to gap

Self-consistence equation:

$$\Delta = -g\sum_{\vec{k}} b_{\vec{k}} = -g\sum_{\vec{k}} u^*_{\vec{k}} v_{\vec{k}} [1 - f(E_{\vec{k}})]$$

$$= -g\sum_{\vec{k}} \frac{\Delta}{2E_{\vec{k}}} \tanh\left(\frac{E_{\vec{k}}}{k_B T}\right)$$

Fermi distribution function

$$f(E) = \frac{1}{1 + e^{E/k_B T}}$$

solution only for $g < 0$ attractive

critical temperature

continuous transition (2nd order) ➡ linearized gap equation

$$T \to T_c \qquad \Leftrightarrow \qquad \Delta \to 0$$

$$\Delta = -g\Delta \sum_{\vec{k}} \frac{1}{2\xi_{\vec{k}}} \tanh\left(\frac{\xi_{\vec{k}}}{2k_B T}\right)$$

$$1 = -g\sum_{\vec{k}} \frac{1}{2\xi_{\vec{k}}} \tanh\left(\frac{\xi_{\vec{k}}}{2k_B T}\right) = -g\int d\xi \frac{N(\xi)}{2\xi} \tanh\left(\frac{\xi}{2k_B T_c}\right)$$

N(ξ): electron density of states

Interaction with characteristic energy scale

cutoff

$$1 = -gN(0)\int_{-\epsilon_c}^{\epsilon_c} \frac{d\xi}{2\xi} \tanh\left(\frac{\xi}{2k_B T_c}\right) = -gN(0)\ln\left(\frac{1.14\epsilon_c}{k_B T_c}\right)$$

constant density of states between $-\varepsilon_c$ and $+\varepsilon_c$

$$k_B T_c = 1.14\epsilon_c e^{-1/|g|N(0)}$$

Zero-temperature

Gap at *T=0:* $$1 = -gN(0)\int_0^{\epsilon_c} \frac{d\xi}{\sqrt{\xi^2+\Delta^2}} = -gN(0)\sinh^{-1}\frac{\epsilon_c}{\Delta}$$

$$\Rightarrow \quad \Delta \approx 2\epsilon_c e^{-1/|g|N(0)} = 1.764 k_B T_c$$

Condensation energy at *T=0:* $E_{cond} = E_s - E_n$ energy gain relative to normal state

$$E_{cond} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] = -\frac{1}{2}N(0)|\Delta|^2$$

depends on density of states at the Fermi surface and the gap magnitude

weak-coupling approach

Zero-temperature

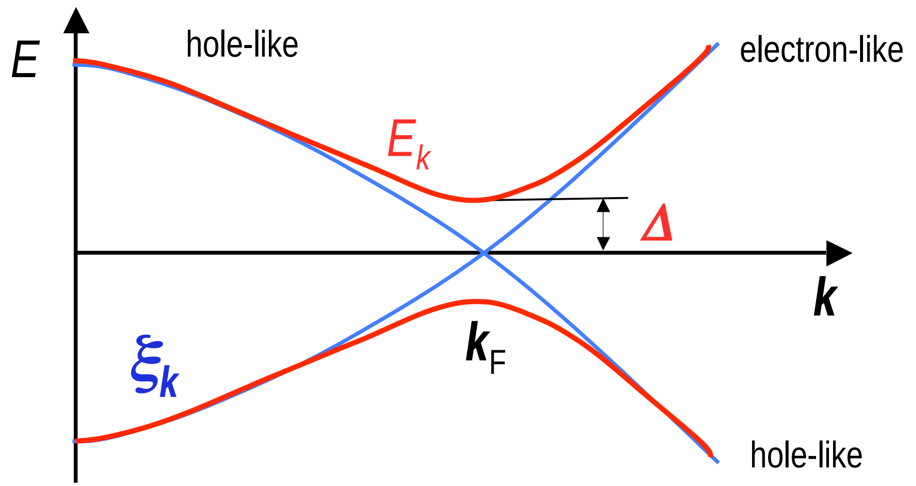
Gap at T=0: $$1 = -gN(0)\int_0^{\epsilon_c}\frac{d\xi}{\sqrt{\xi^2+\Delta^2}} = -gN(0)\sinh^{-1}\frac{\epsilon_c}{\Delta}$$

$$\Rightarrow \quad \Delta \approx 2\epsilon_c e^{-1/|g|N(0)} = 1.764 k_B T_c$$

Condensation energy at T=0: $E_{cond} = E_s - E_n$ energy gain relative to normal state

$$E_{cond} = -\frac{1}{2}N(0)|\Delta|^2$$

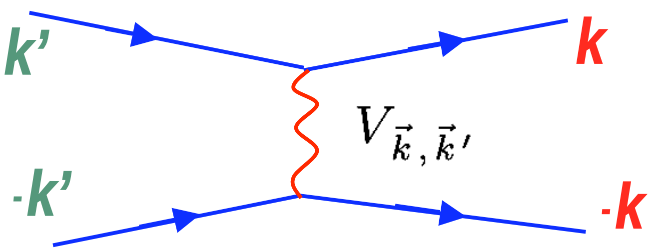
modification of the quasiparticle spectrum

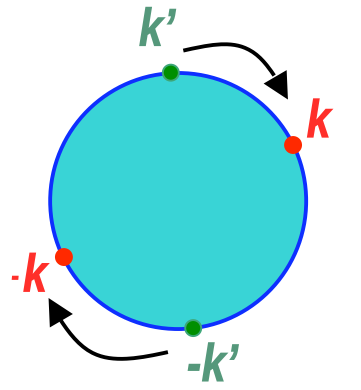


Pairing interaction

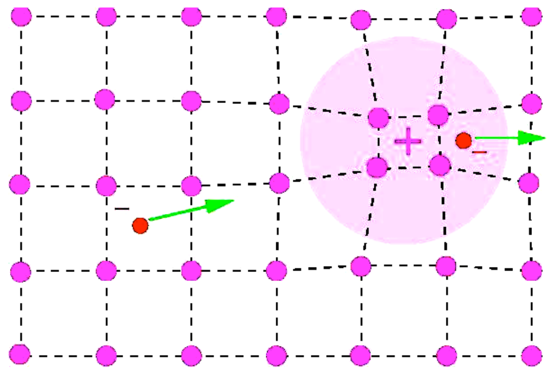
Cooper pair formation (bound state of 2 electrons) needs attractive interaction

$$\mathcal{H}_{pair} = \frac{1}{2\Omega} \sum_{\vec{k},\vec{k}'} \sum_{s,s'} V_{\vec{k},\vec{k}'} c^\dagger_{\vec{k},s} c^\dagger_{-\vec{k},s'} c_{-\vec{k}',s'} c_{\vec{k}',s}$$





electron phonon interaction:



electrons polarize their environment

renormalized Coulomb interaction

$$V_{\vec{k},\vec{k}'} = \frac{4\pi e^2}{q^2} \Rightarrow V_{\vec{k},\vec{k}'} = \frac{4\pi e^2}{q^2 \varepsilon(\vec{q},\omega)}$$

electron-phonon versus Coulomb interaction

Polarization effects:

$$\frac{1}{\varepsilon(\vec{q},\omega)} \approx \frac{q^2}{q^2+k_{TF}^2} + \frac{q^2}{q^2+k_{TF}^2}\frac{\omega_{\vec{q}}^2}{\omega^2-\omega_{\vec{q}}^2}$$

with $k_{TF}^2 = \frac{6\pi e^2 n_e}{\epsilon_{\mathrm{F}}}$ Thomas-Fermi screening length $\lambda_{TF} = k_{TF}^{-1} \sim 5 - 10\text{Å}$

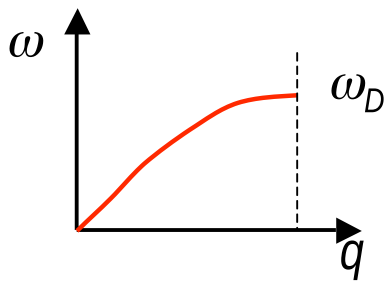
$$V_{\vec{k},\vec{k}'} = \frac{4\pi e^2}{q^2\varepsilon(\vec{q},\omega)} = \underbrace{\frac{4\pi e^2}{q^2+k_{TF}^2}}_{\text{renorm.Coulomb}} + \underbrace{\frac{4\pi e^2}{q^2+k_{TF}^2}\frac{\omega_q^2}{\omega^2-\omega_q^2}}_{\text{electron-phonon}}$$

$$\vec{q} = \vec{k} - \vec{k}'$$

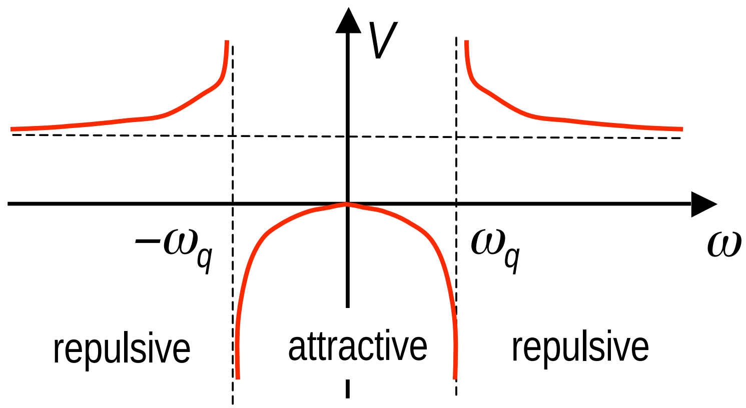
$$\omega = \epsilon_{\vec{k}} - \epsilon_{\vec{k}'}$$

phonon spectrum

$$\omega_q = sq$$

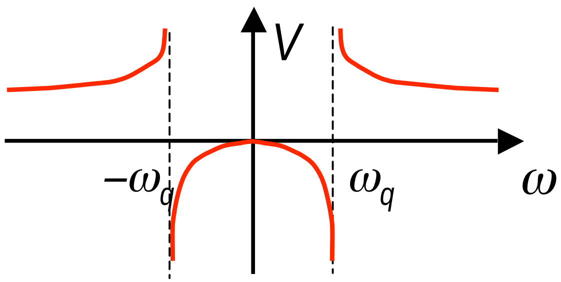


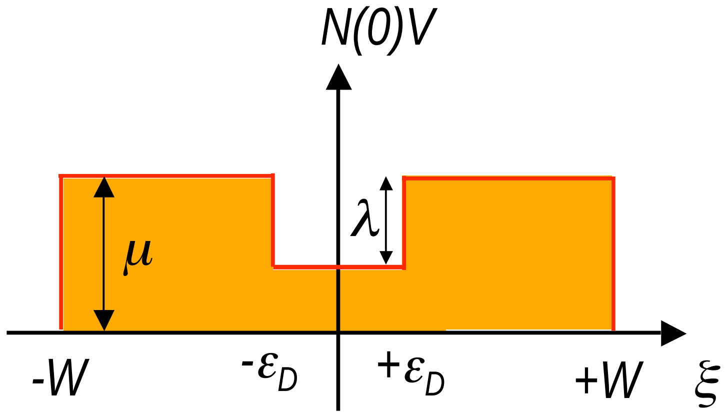
Debye frequency: characteristic energy scale



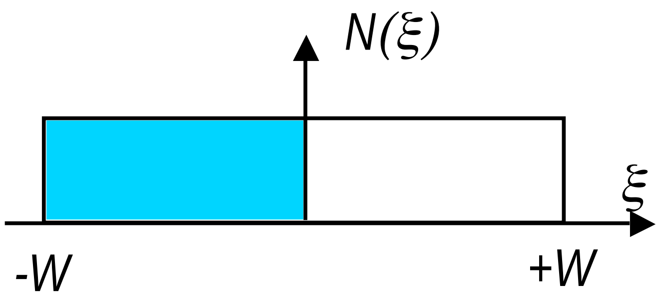
Poor-man's model

Anderson & Morel (1962)





poor man's electron band:



band width: $2W$

constant density of states: $N(\xi) = N(0)$

poor man's interaction:

$$V_{\mathbf{k},\mathbf{k}'} = V(\xi,\xi') = V_C + V_{ep}$$

- repulsive part

$$N(0)V_C = \begin{cases} \mu & |\xi,\xi'| < W \\ 0 & \text{otherwise} \end{cases}$$

- attractive part

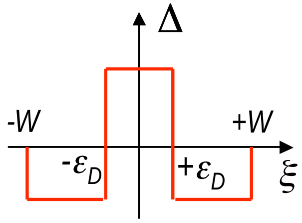
$$N(0)V_{ep} = \begin{cases} -\lambda & |\xi,\xi'| < \varepsilon_D \\ 0 & \text{otherwise} \end{cases}$$

Poor-man's model

Anderson & Morel (1962)

linearized self-consistent gap equation:

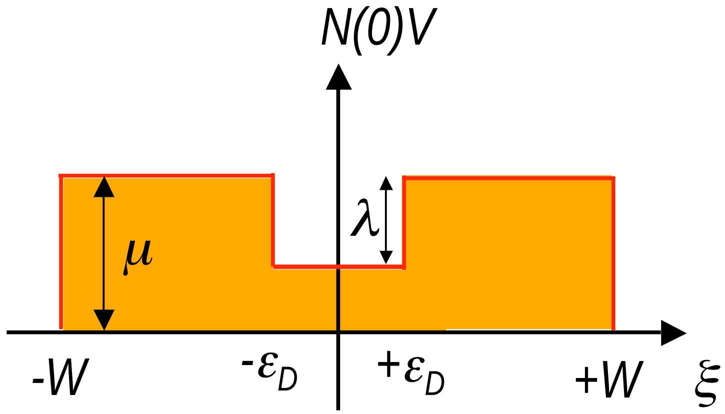
$$\Delta(\xi) = -N_0 \int d\xi' \tilde{V}(\xi,\xi') \frac{\tanh(\beta\xi'/2)}{\xi'} \Delta(\xi')$$



$$\Delta(\xi) = \begin{cases} \Delta_1 & |\xi| < \epsilon_D \\ \Delta_2 & \epsilon_D < |\xi| < W \end{cases}$$

$$\begin{aligned} \Delta_1 &= (\lambda-\mu)\Delta_1 \int_0^{\epsilon_D} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} - \mu\Delta_2 \int_{\epsilon_D}^{W} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} \\ &= (\lambda-\mu)\Delta_1 \ln(1.14\epsilon_D/k_BT) - \mu\Delta_2 \ln(W/\epsilon_D) \end{aligned}$$

$$\begin{aligned} \Delta_2 &= -\mu\Delta_1 \int_0^{\epsilon_D} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} - \mu\Delta_2 \int_{\epsilon_D}^{W} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} \\ &= -\mu\Delta_1 \ln(1.14\epsilon_D/k_BT) - \mu\Delta_2 \ln(W/\epsilon_D) \end{aligned}$$



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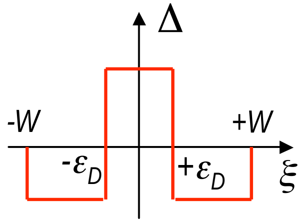

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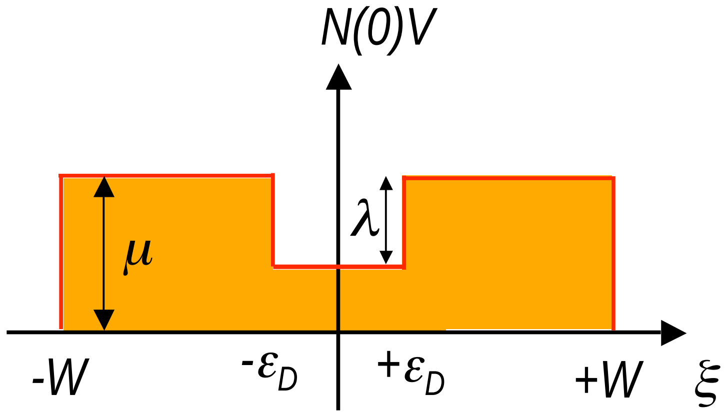
$$\begin{aligned}\Delta_2 &= -\mu\Delta_1 \int_0^{\epsilon_D} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} - \mu\Delta_2 \int_{\epsilon_D}^{W} d\xi' \frac{\tanh(\beta\xi'/2)}{\xi'} \\ &= -\mu\Delta_1 \ln(1.14\epsilon_D/k_B T) - \mu\Delta_2 \ln(W/\epsilon_D)\end{aligned}$$



transition temperature T_c

$$k_B T_c = 1.14\epsilon_D \exp\left(-\frac{1}{\lambda-\mu^*}\right)$$

renormalized Coulomb repulsion

$$\mu^* = \frac{\mu}{1+\mu\ln(W/\epsilon_D)}$$

Retardation effect:

Coulomb	fast	$\dfrac{W}{\varepsilon_D}$
electron-phonon	slow	

➡ $T_c \neq 0$ even if $\lambda < \mu$

Retardation effect:

weak-coupling regime $\lambda << 1$

$$k_B T_c = 1.14 \epsilon_D \exp\left(-\frac{1}{\lambda - \mu^*}\right)$$

strong-coupling regime $\lambda > 1$

$$k_B T_c = 0.7 \epsilon_D \exp\left(-\frac{1.04(1+\lambda)}{\lambda - \mu^*(1+0.62\lambda)}\right)$$

Eliashberg, McMillan (68)

renormalized Coulomb repulsion

$$\mu^* = \frac{\mu}{1 + \mu \ln(W/\epsilon_D)}$$

Important: $\frac{W}{\varepsilon_D} \sim \frac{T_F}{T_D} >> 1$

Metallic strongly correlated electron systems

small energy scales: T_F
small band widths: W

⬇

strong effect of Coulomb repulsion

handy-cap for electron-phonon mediated superconductivity

When Coulomb repulsion is too strong for electron-phonon induced pairing

Alternative ways to superconductivity

Alternative ways to Cooper pairing

Coulomb and electron-phonon interaction very short-ranged (λ_{TF}) *"contact interaction"*

Bound Cooper pair wavefunction:

$$\psi(\vec{r}, s; \vec{r}', s') = f(|\vec{r} - \vec{r}'|)\chi(s, s')$$

with $f(r \to 0) \neq 0$

relative angular momentum *l=0*
important for "contact interaction"

How to avoid Coulomb repulsion?

higher-angular momentum pairing

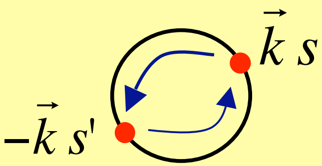
$l > 0 \rightarrow f(r \to 0) \propto r^l$

"contact interaction" not effective

Symmetry of pairs of identical electrons:

$$\Psi_{ss'}(\vec{k}) = \langle \hat{c}_{\vec{k}s} \hat{c}_{-\vec{k}s'} \rangle = \underbrace{\Phi(\vec{k})}_{\text{orbital}} \underbrace{\chi(s, s')}_{\text{spin}}$$

wave function totally antisymmetric under particle exchange



$\vec{k} \to -\vec{k} \qquad s \leftrightarrow s'$

even parity: $l = 0,2,4,\ldots$ (even), $S=0$ singlet (odd)

odd parity: $l = 1,3,5,\ldots$ (odd), $S=1$ triplet (even)

Requirements for the formation of Cooper pairs

Anderson's Theorems (1959,1984)

Cooper pair formation with P=0 relies on symmetries which guarantee degenerate partner electrons

- Spin singlet pairing: time reversal symmetry

$$\left|\vec{k}\uparrow\right\rangle \qquad T\left|\vec{k}\uparrow\right\rangle = \left|-\vec{k}\downarrow\right\rangle$$

harmful:
magnetic impurities
ferromagnetism
paramagnetic limiting

- Spin triplet pairing: time reversal & inversion symmetry

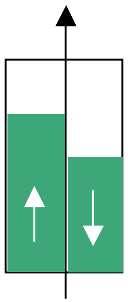
$$\left|\vec{k}\uparrow\right\rangle \qquad I\left|\vec{k}\uparrow\right\rangle = \left|-\vec{k}\uparrow\right\rangle \qquad T\left|\vec{k}\uparrow\right\rangle = \left|-\vec{k}\downarrow\right\rangle \qquad IT\left|\vec{k}\uparrow\right\rangle = \left|\vec{k}\downarrow\right\rangle$$

harmful: crystal structure without inversion center

Paramagnetic limiting: *lack of time reversal symmetry*

Zeeman splitting of Fermi surfaces exceeds the gap magnitude

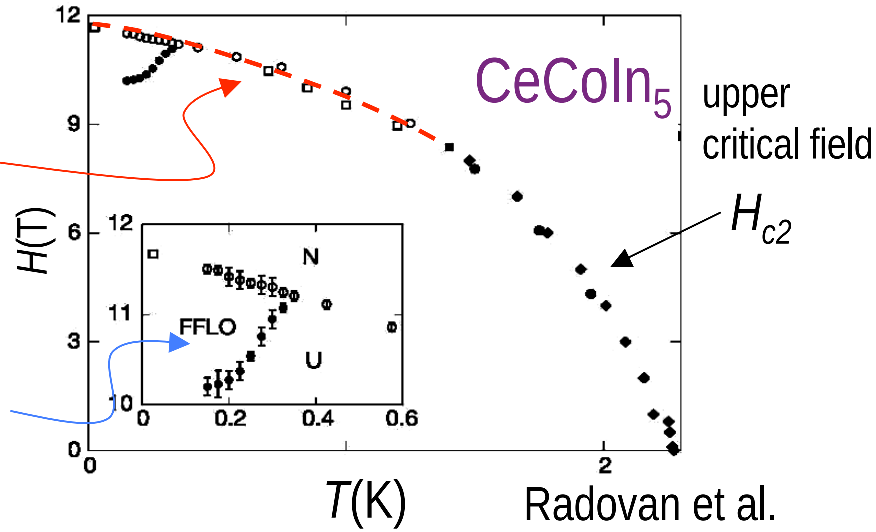
→ No singlet pairing possible



$\updownarrow$ $\mu_B H > \Delta$

Paramagnetic suppression
1^{st} order transition

modulated Fulde-Ferrel-Larkin-Ovchinikov phase



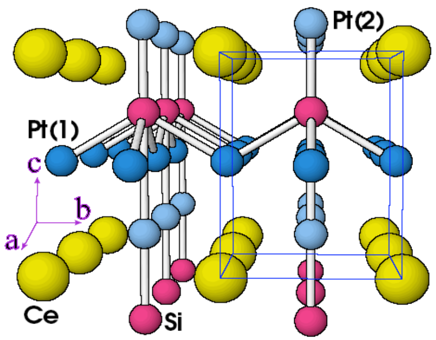
Radovan et al.

Antisymmetric spin-orbit coupling: *lack of inversion symmetry*

Crystal structure without an inversion center

e.g. $CePt_3Si$

no mirror plane for $z \longrightarrow -z$



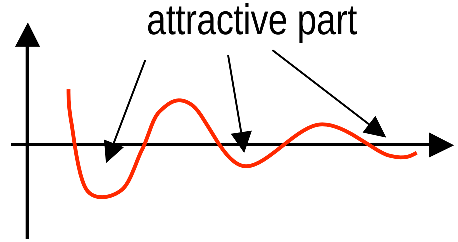
Bauer et al.

Alternative mechanism for Cooper pairing

Pairing from purely repulsive interactions: Kohn & Luttinger (1965)

screened Coulomb potential in metal has long-ranged oscillatory tail (sharp Fermi edge)

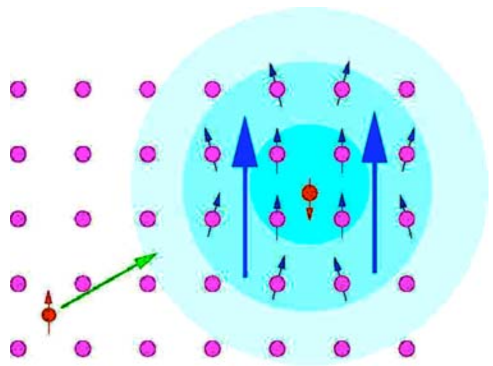
Friedel oscillations: $V(r) \propto r^{-3} \cos(2k_F r)$



pairing in high-angular momentum channel $l > 0$ $T_c/T_F \sim \exp\{-(2l)^4\}$

very low !

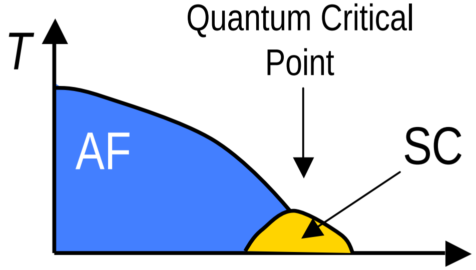
Pairing by magnetic fluctuations: Berk & Schrieffer (1966)



easily spin polarizable medium

longer ranged interaction

T_c reasonable for higher angular momentum pairing

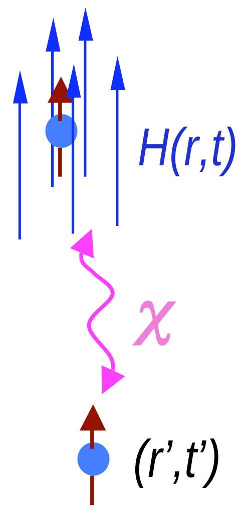


Spin fluctuation exchange mechanism

Exchange interaction: $\mathcal{H}_{ex} = \int d^3r d^3r' U \delta(\vec{r} - \vec{r}')\rho_{\uparrow}(\vec{r})\rho_{\downarrow}(\vec{r}')$

spin-induced local "magnetic field" $\vec{H}(\vec{r}, t) = -\frac{I}{\mu_B \hbar}\vec{S}(\vec{r}, t)$

$I = U/\Omega$



induced spin polarization:

dynamical spin susceptibility

$$\vec{S}(\vec{r}', t') = \mu_B \int d^3r\, dt\, \chi(\vec{r}' - \vec{r}, t' - t)\, \vec{H}(\vec{r}, t)$$

Spin density-spin density interaction:

$$\mathcal{H}_{sf} = -\frac{I^2}{2\hbar^2}\int d^3r\, d^3r' \; \{\chi(\vec{r} - \vec{r}', t - t') - \chi(\vec{r}' - \vec{r}, t' - t)\}\, \vec{S}(\vec{r}, t) \cdot \vec{S}(\vec{r}', t')$$

simplified spin fluctuation exchange model

Spin fluctuation exchange mechanism

effective pairing interaction:

$$\mathcal{H}'_{sf} = \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1s_2s_3s_4} c^{\dagger}_{\vec{k},s_1} c^{\dagger}_{-\vec{k},s_2} c_{-\vec{k}',s_3} c_{\vec{k}',s_4}$$

$$V_{\vec{k},\vec{k}';s_1s_2s_3s_4} = -\frac{I^2}{4} Re\chi(\vec{k}-\vec{k}',\omega = \varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'})\,\vec{\sigma}_{s_1s_4}\cdot\vec{\sigma}_{s_2s_3}$$

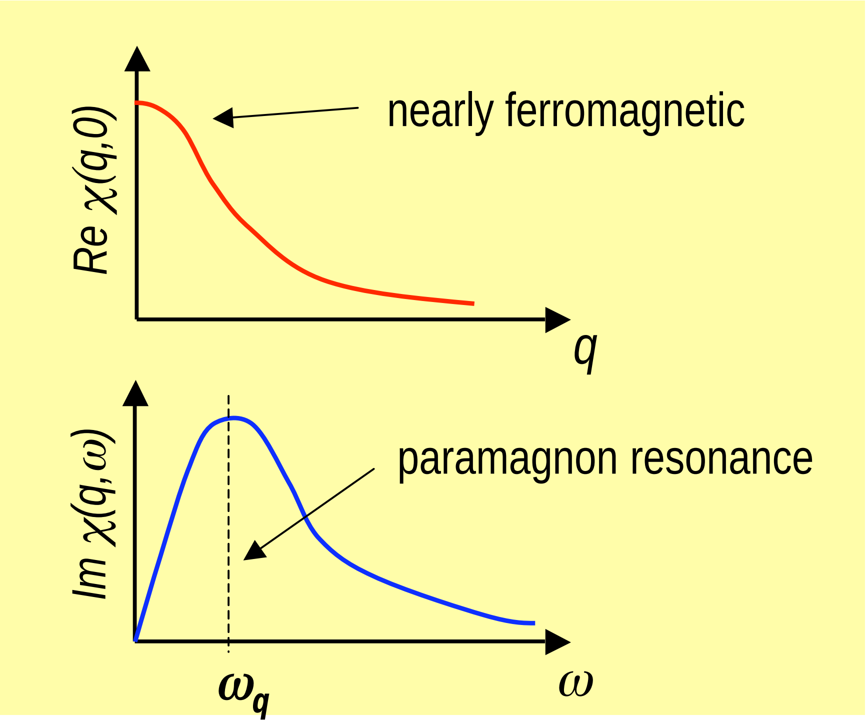
dynamical spin susceptibility:

$$\chi(\vec{q},\omega) = \frac{\chi_0(\vec{q},\omega)}{1 - I\chi_0(\vec{q},\omega)} \quad \text{RPA}$$

for isotropic electron gas:

$$\chi_0(\vec{q},\omega) \approx N(0)\left(1 - \frac{\vec{q}^2}{12k_F^2} + i\frac{\pi}{2}\frac{\omega}{v_F|\vec{q}|}\right)$$

$q << 2k_F, \quad \omega << \varepsilon_F$



Spin fluctuation exchange mechanism

effective pairing interaction:

$$\mathcal{H}'_{sf} = \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} c^{\dagger}_{\vec{k},s_1} c^{\dagger}_{-\vec{k},s_2} c_{-\vec{k}',s_3} c_{\vec{k}',s_4}$$

$$V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} = -\frac{I^2}{4} Re\chi(\vec{k}-\vec{k}', \omega = \varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}) \vec{\sigma}_{s_1 s_4} \cdot \vec{\sigma}_{s_2 s_3}$$

Cooper spin channels:

$$V^s_{\vec{k},\vec{k}'} = \frac{3I^2}{4} Re\chi(\vec{k}-\vec{k}', \omega = \varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'})$$ ***S=0*** spin singlet

$$V^t_{\vec{k},\vec{k}'} = -\frac{I^2}{4} Re\chi(\vec{k}-\vec{k}', \omega = \varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'})$$ ***S=1*** spin triplet

$|k-k'| \ll k_F$ *S=0*: repulsive *S=1*: attractive

Spin fluctuation exchange mechanism

Pairing for spin triplet *l=1 (p-wave):*

angular structure of gap function

$\Delta_k = \Delta\, g_k$

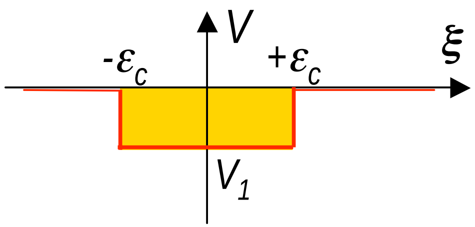
$$g^{\alpha}_{\vec{k}} = \begin{cases} \frac{1}{\sqrt{2}k_F}(k_x + ik_y) & \alpha = +1 \\ \frac{k_z}{k_F} & \alpha = 0 \\ \frac{1}{\sqrt{2}k_F}(k_x - ik_y) & \alpha = -1 \end{cases}$$

Projected effective interaction:

$$V(\xi,\xi') = -\frac{I^2}{4\Omega}\sum_{\vec{k},\vec{k}'} g^{\alpha}_{\vec{k}}\chi(\vec{k}-\vec{k}',\omega=0)g^{\alpha}_{\vec{k}'}\delta(\xi-\xi_{\vec{k}})\delta(\xi'-\xi_{\vec{k}'}) \approx \begin{cases} V_1 & |\xi|,|\xi'| < \epsilon_c \\ 0 & \text{otherwise} \end{cases}$$

$$V_1 = -\frac{I}{12}\frac{IN(0)}{(1-IN(0))^2}$$

$$k_B T_c = 1.14\epsilon_c e^{-1/\lambda_s} \qquad \lambda_s = N(0)V_1$$



characteristic energy: paramagnon spectrum

$$\epsilon_c = \frac{8}{\pi IN(0)}(1-IN(0))E_F$$

Spin fluctuation exchange mechanism

Stoner instability criterion:

$$IN(0) = 1$$

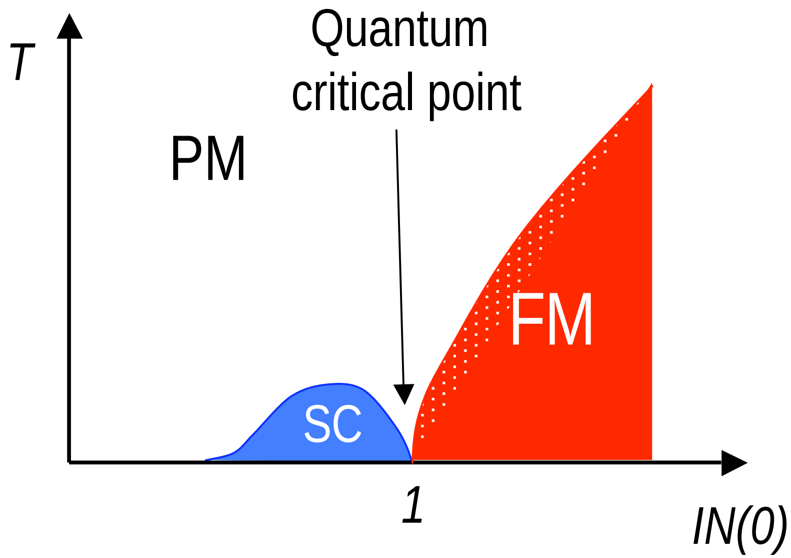
Quantum phase transition

Paramagnet ⟶ *Ferromagnet*

$V_1 \longrightarrow \infty$

$\varepsilon_c \longrightarrow 0$

$\xi_{FM} \longrightarrow \infty$ FM correlation length



more detailed analysis: Monthoux & Lonzarich (1999- …)

$$V_1 = -\frac{I}{12}\frac{IN(0)}{(1 - IN(0))^2}$$

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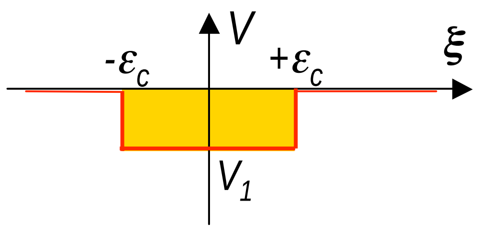


characteristic energy: paramagnon spectrum

$$\epsilon_c = \frac{8}{\pi IN(0)}(1 - IN(0))E_F$$

Generalized BCS theory
New aspects

Generalized formulation of the BCS mean field theory

BCS Hamiltonian:

$$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + \frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1s_2s_3s_4} c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} c_{-\vec{k}'s_3} c_{\vec{k}'s_4}$$

Mean field Hamiltonian:

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \frac{1}{2} \sum_{\vec{k},s_1,s_2} \left[\Delta_{\vec{k},s_1s_2} c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} + \Delta^*_{\vec{k},s_1s_2} c_{\vec{k}s_1} c_{-\vec{k}s_2} \right]$$

$$-\frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1s_2s_3s_4} \langle c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} \rangle \langle c_{-\vec{k}'s_3} c_{\vec{k}'s_4} \rangle$$

Self-consistence equations:

$$\Delta_{\vec{k},ss'} = - \sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4} \langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4} \rangle$$

$$\Delta^*_{\vec{k},ss'} = - \sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's} \langle c^\dagger_{\vec{k}'s_1} c^\dagger_{-\vec{k}'s_2} \rangle$$

gap: 2x2-matrix

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k}\uparrow\uparrow} & \Delta_{\vec{k}\uparrow\downarrow} \\ \Delta_{\vec{k}\downarrow\uparrow} & \Delta_{\vec{k}\downarrow\downarrow} \end{pmatrix}$$

Self-consistent gap equation

Bogolyubov transformation → Quasiparticle spectrum

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2} \qquad |\Delta_{\vec{k}}|^2 = \frac{1}{2}\mathrm{tr}\left(\hat{\Delta}_{\vec{k}}^{\dagger}\hat{\Delta}_{\vec{k}}\right)$$

Note: quasiparticle gap is ***k***-dependent

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k}\uparrow\uparrow} & \Delta_{\vec{k}\uparrow\downarrow} \\ \Delta_{\vec{k}\downarrow\uparrow} & \Delta_{\vec{k}\downarrow\downarrow} \end{pmatrix}$$

Self-consistence equation:

$$\Delta_{\vec{k},s_1s_2} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';s_1s_2s_3s_4} \frac{\Delta_{\vec{k}',s_4s_3}}{2E_{\vec{k}}} \tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right)$$

Structure of the gap function

Gap function: 2x2 matrix in spin space

$$\Delta_{\vec{k},ss'} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4}\langle c_{\vec{k}'s_3}c_{-\vec{k}'s_4}\rangle$$

$$\Delta^*_{\vec{k},ss'} = -\sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's}\langle c^\dagger_{\vec{k}'s_1}c^\dagger_{-\vec{k}'s_2}\rangle$$

$$\langle c_{-\vec{k}s_1}c_{\vec{k}s_2}\rangle = \phi(\vec{k})\chi_{s_1s_2}$$

orbital spin

$$\phi(\vec{k}) = \phi(-\vec{k}) \quad \Leftrightarrow \quad \chi_{s_1s_2} = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

even parity, spin singlet

$$\phi(\vec{k}) = -\phi(-\vec{k}) \quad \Leftrightarrow \quad \chi_{s_1s_2} = \begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{cases}$$

odd parity, spin triplet

$$\Delta_{\vec{k},s_1s_2} = -\Delta_{-\vec{k},s_2s_1} = \begin{cases} \Delta_{-\vec{k},s_1s_2} = -\Delta_{\vec{k},s_2s_1} & \text{even} \\ -\Delta_{-\vec{k},s_1s_2} = \Delta_{\vec{k},s_2s_1} & \text{odd} \end{cases}$$

Structure of the gap function

Gap function: 2x2 matrix in spin space

$$\Delta_{\vec{k},ss'} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4} \langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4} \rangle$$

$$\Delta^*_{\vec{k},ss'} = -\sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's} \langle c^\dagger_{\vec{k}'s_1} c^\dagger_{-\vec{k}'s_2} \rangle$$

Even parity spin singlet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k},\uparrow\uparrow} & \Delta_{\vec{k},\uparrow\downarrow} \\ \Delta_{\vec{k},\downarrow\uparrow} & \Delta_{\vec{k},\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} 0 & \psi(\vec{k}) \\ -\psi(\vec{k}) & 0 \end{pmatrix} = i\hat{\sigma}_y \psi(\vec{k})$$

represented by scalar function $\psi(\vec{k}) = \psi(-\vec{k})$ even $E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + |\psi(\vec{k})|^2}$

Odd parity spin triplet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} -d_x(\vec{k}) + i d_y(\vec{k}) & d_z(\vec{k}) \\ d_z(\vec{k}) & d_x(\vec{k}) + i d_y(\vec{k}) \end{pmatrix} = i\left(\vec{d}(\vec{k}) \cdot \hat{\vec{\sigma}}\right)\hat{\sigma}_y$$

represented by vector function $\vec{d}(\vec{k}) = -\vec{d}(-\vec{k})$ odd $E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + |\vec{d}(\vec{k})|^2}$

Structure of the gap function

$$\Delta_{\vec{k},ss'} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4} \langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4} \rangle$$

$$\Delta^*_{\vec{k},ss'} = -\sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's} \langle c^\dagger_{\vec{k}'s_1} c^\dagger_{-\vec{k}'s_2} \rangle$$

Gap function: 2x2 matrix in spin space

Even parity spin singlet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k},\uparrow\uparrow} & \Delta_{\vec{k},\uparrow\downarrow} \\ \Delta_{\vec{k},\downarrow\uparrow} & \Delta_{\vec{k},\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} 0 & \psi(\vec{k}) \\ -\psi(\vec{k}) & 0 \end{pmatrix} = i\hat{\sigma}_y \psi(\vec{k})$$

represented by scalar function $\psi(\vec{k}) = \psi(-\vec{k})$ even $E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + |\psi(\vec{k})|^2}$

Odd parity spin triplet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} -d_x(\vec{k}) + i d_y(\vec{k}) & d_z(\vec{k}) \\ d_z(\vec{k}) & d_x(\vec{k}) + i d_y(\vec{k}) \end{pmatrix} = i\left(\vec{d}(\vec{k}) \cdot \hat{\vec{\sigma}}\right)\hat{\sigma}_y$$

spin configuration $d_x(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle) - d_y i(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) + d_z(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ ⟺ " $\vec{d} \perp \vec{S}$ "

Transition temperature

Pairing interaction:

$$V_{\vec{k},\vec{k}';s_1s_2s_3s_4} = J^0_{\vec{k},\vec{k}'}\hat{\sigma}^0_{s_1s_4}\hat{\sigma}^0_{s_2s_3} + J_{\vec{k},\vec{k}'}\hat{\vec{\sigma}}_{s_1s_4}\cdot\hat{\vec{\sigma}}_{s_2s_3}$$

density-density spin-spin

Self-consistence equation:

even parity spin singlet

$$\psi(\vec{k}) = -\sum_{\vec{k}'}\underbrace{(J^0_{\vec{k},\vec{k}'} - 3J_{\vec{k},\vec{k}'})}_{=v^s_{\vec{k},\vec{k}'}}\frac{\psi(\vec{k}')}{2E_{\vec{k}'}}\tanh\left(\frac{E_{\vec{k}'}}{2k_BT}\right)$$

$T \to T_c$

$$-\lambda\psi(\vec{k}) = -N(0)\langle v^s_{\vec{k},\vec{k}'}\psi(\vec{k}')\rangle_{\vec{k}',FS}$$

odd parity spin triplet

$$\vec{d}(\vec{k}) = -\sum_{\vec{k}'}\underbrace{(J^0_{\vec{k},\vec{k}'} + J_{\vec{k},\vec{k}'})}_{=v^t_{\vec{k},\vec{k}'}}\frac{\vec{d}(\vec{k}')}{2E_{\vec{k}'}}\tanh\left(\frac{E_{\vec{k}'}}{2k_BT}\right)$$

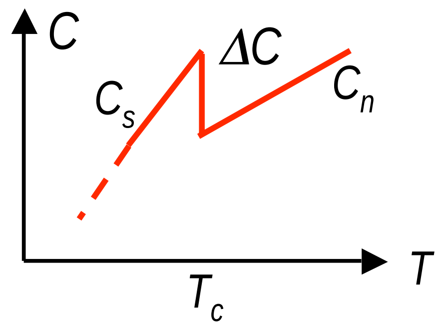
$T \to T_c$

$$-\lambda\vec{d}(\vec{k}) = -N(0)\langle v^t_{\vec{k},\vec{k}'}\vec{d}(\vec{k}')\rangle_{\vec{k}',FS}$$

eigenvalue λ ➡ $k_BT_c = 1.14\epsilon_c e^{-1/\lambda}$

Some thermodynamic properties

Specific heat discontinuity at $T=T_c$



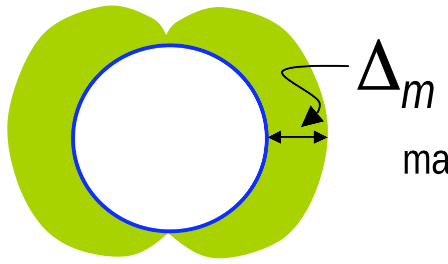
2[nd] order phase transition → discontinuity of specific heat

Entropy and specific heat:

$$S = -\frac{2k_B}{\Omega}\sum_{\vec{k}}\left\{f(E_{\vec{k}})\ln(f(E_{\vec{k}})) + (1-f(E_{\vec{k}}))\ln(1-f(E_{\vec{k}}))\right\} \quad \Rightarrow$$

$$C = T\frac{dS}{dT} = -\frac{2}{\Omega}\sum_{\vec{k}} E_{\vec{k}}\frac{df(E_{\vec{k}})}{dT} = -\frac{2N(0)}{T}\int_{-\infty}^{+\infty} d\xi \left\langle \frac{\partial f(E_{\vec{k}})}{\partial E_{\vec{k}}} E_{\vec{k}}^2 - \frac{T}{2}\frac{\partial|\Delta_m(T)|^2}{\partial T}|g_{\vec{k}}|^2 \right\rangle_{\vec{k},FS}$$

Gap anisotropy: $|\Delta_{\vec{k}}|^2 = \Delta_m^2 |g_{\vec{k}}|^2$



Specific heat discontinuity:

$$\left.\frac{\Delta C}{C_n}\right|_{T=T_c} = \left.\frac{C-C_n}{C_n}\right|_{T=T_c} = 1.43\frac{\langle|g_{\vec{k}}|^2\rangle_{\vec{k},FS}^2}{\langle|g_{\vec{k}}|^4\rangle_{\vec{k},FS}} \leq 1.43$$

"universal value"

weak coupling

Low-temperature properties

thermodynamics is dominated by the excited quasiparticles

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2} \qquad |\Delta_{\vec{k}}|^2 = \frac{1}{2}\mathrm{tr}\left(\hat{\Delta}_{\vec{k}}^{\dagger}\hat{\Delta}_{\vec{k}}\right) \qquad \Delta_{\mathbf{k}} = \Delta_m g_{\mathbf{k}}$$

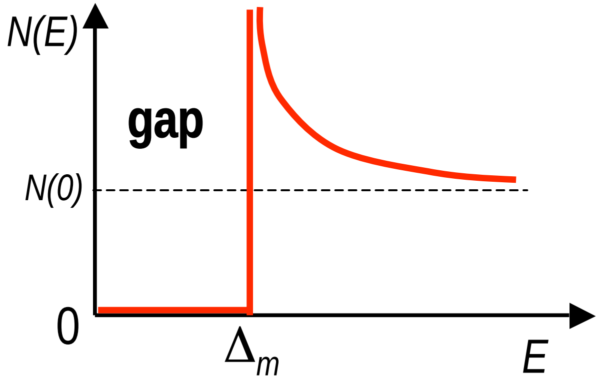
key quantity: *density of states*

$$N(E) = \frac{2}{\Omega}\sum_{\vec{k}} \delta(E_{\vec{k}} - E)$$

$$N(E) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi}\int d\xi\,\delta\left(\sqrt{\xi^2 + |\Delta_m g_{\vec{k}}|^2} - E\right) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi}\frac{E}{\sqrt{E^2 - |\Delta_m g_{\vec{k}}|^2}}$$

Isotropic gap function: $\Delta_{\mathbf{k}} = \Delta_m = \text{const.}$

$$N(E) = N(0)\begin{cases} 0 & |E| < \Delta_m \\ \dfrac{E}{\sqrt{E^2 - |\Delta_m|^2}} & \Delta_m \geq |E|^2 \end{cases}$$



Low-temperature properties

thermodynamics is dominated by the excited quasiparticles

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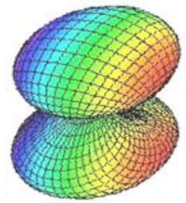
key quantity: *density of states* $N(E) = \frac{2}{\Omega}\sum_{\vec{k}} \delta(E_{\vec{k}} - E)$

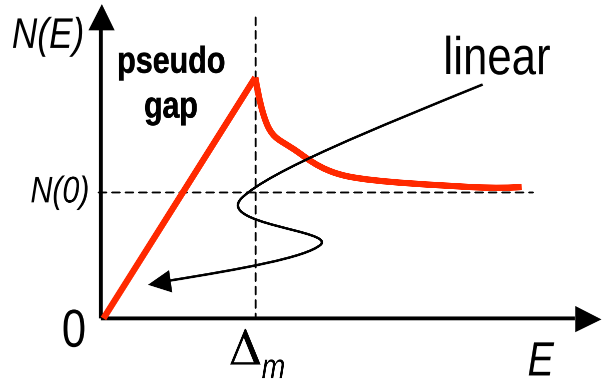
$$N(E) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi}\int d\xi\, \delta\left(\sqrt{\xi^2 + |\Delta_m g_{\vec{k}}|^2} - E\right) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi} \frac{E}{\sqrt{E^2 - |\Delta_m g_{\vec{k}}|^2}}$$

Anisotropic gap function: $\Delta_{\vec{k}} = \Delta_m \cos\theta$



line node

$$N(E) = N(0)\frac{E}{\Delta_m}\begin{cases} \frac{\pi}{2} & |E| < \Delta_m \\ \arcsin\left(\frac{\Delta_m}{E}\right) & \Delta_m \geq |E| \end{cases}$$



Low-temperature properties

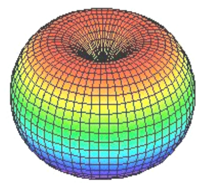
thermodynamics is dominated by the excited quasiparticles

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2} \qquad |\Delta_{\vec{k}}|^2 = \frac{1}{2}\mathrm{tr}\left(\hat{\Delta}^{\dagger}_{\vec{k}}\hat{\Delta}_{\vec{k}}\right) \qquad \Delta_{\boldsymbol{k}} = \Delta_m g_{\boldsymbol{k}}$$

key quantity: *density of states*
$$N(E) = \frac{2}{\Omega}\sum_{\vec{k}} \delta(E_{\vec{k}} - E)$$

$$N(E) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi}\int d\xi\,\delta\left(\sqrt{\xi^2 + |\Delta_m g_{\vec{k}}|^2} - E\right) = N(0)\int \frac{d\Omega_{\vec{k}}}{4\pi}\frac{E}{\sqrt{E^2 - |\Delta_m g_{\vec{k}}|^2}}$$

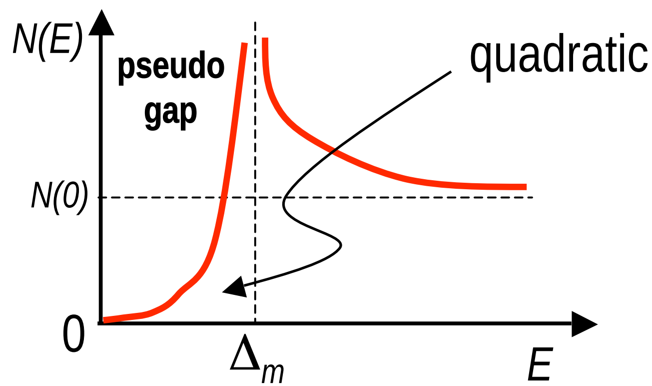
Anisotropic gap function:
$$\Delta_{\vec{k}} = \Delta_m \sin\theta$$



point nodes

$$N(E) = N(0)\frac{E}{\Delta_m}\ln\left|\frac{1 + \frac{E}{\Delta_m}}{1 - \frac{E}{\Delta_m}}\right|$$

$N(E) = A\,E^2$ for $E \ll \Delta_m$



Low-temperature properties

Specific heat: restricted to quasiparticle contributions

$$C(T) = \frac{2}{\Omega}\sum_{\vec{k}} E_{\vec{k}} \frac{df(E_{\vec{k}})}{dT} = \int dE\; N(E)\; E\; \frac{df(E)}{dT} = \int dE\; N(E)\; \frac{E^2}{k_B T^2} \frac{1}{4\cosh^2(E/2k_B T)}$$

- **Isotropic gap function:** activated behavior with a real gap (semiconductor-like)

$$C(T) \approx N(0) k_B \left(\frac{\Delta_m}{k_B T}\right)^2 \sqrt{2\pi k_B T \Delta_m}\, \boxed{e^{-\Delta_m/k_B T}}$$

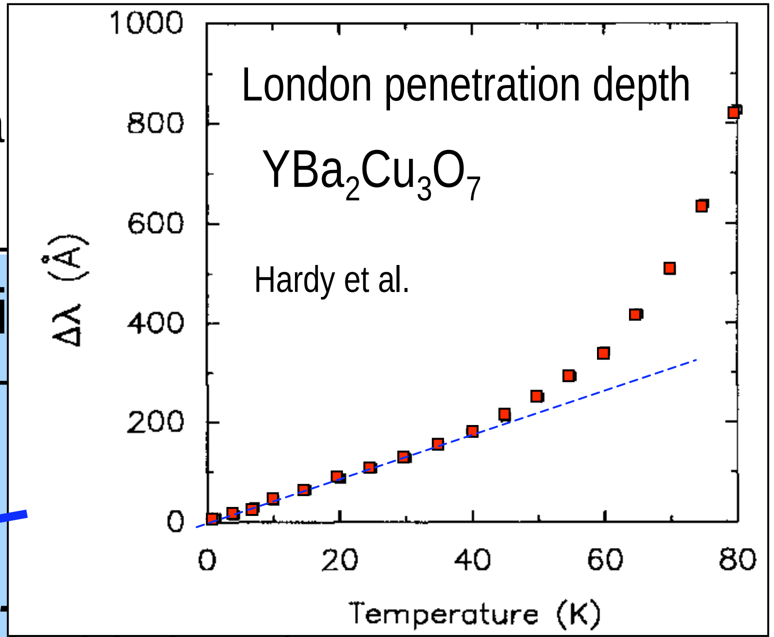
- **Anisotropic gap functions:** contributions from “subgap states” ⇒ powerlaws

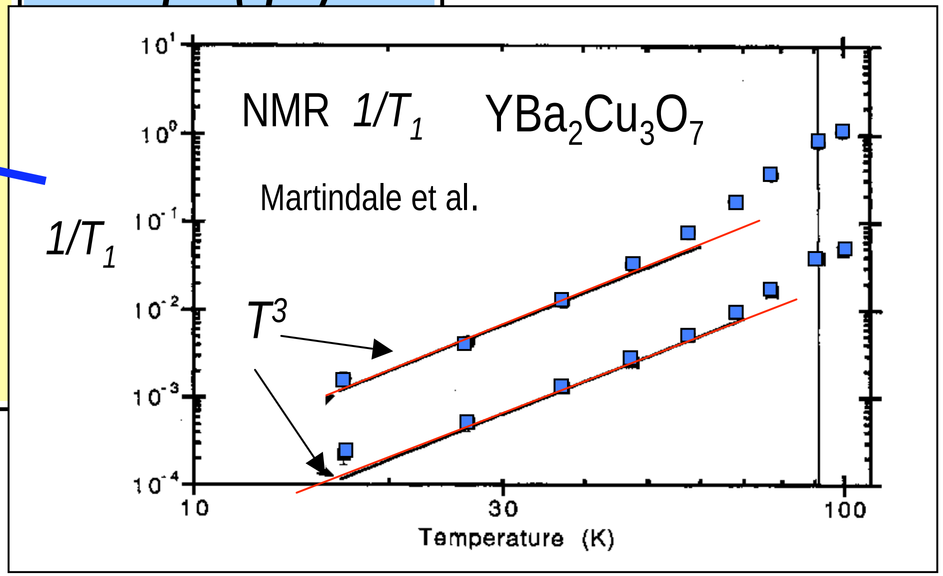
$$C(T) = \int dE\; \underbrace{N(E)}_{\propto E^n} \frac{E^2}{k_B T^2} \frac{1}{4\cosh^2(E/2k_B T)} \propto \boxed{T^{n+1}} \begin{cases} T^2 & \textit{line nodes} \\ T^3 & \textit{point nodes} \end{cases}$$

Low-temperature properties

powerlaws in other quantities depending on ga

quantity	line nodes	poi
specific heat C(T)	T^2	
London penetration depth $\lambda(T)$	T (T^3)	
NMR $1/T_1$	T^3	
heat conductivity $\kappa(T)$	T^2	





high-temperature superconductors
with line nodes in the gap

Other characteristic properties

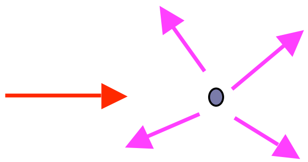
Impurity scattering - Anderson's theorem (1959)

Pure metals:

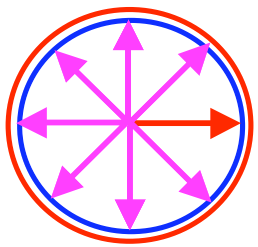
electron momentum well defined

Dirty metals:

impurity scattering (non-magnetic)



momentum averaging over the Fermi surface



Interference effects for Cooper pairs

$$\langle \Psi(\vec{k}) \rangle_{\vec{k},FS} = \Psi_0$$

- conventional pairing: *l = 0* isotropic

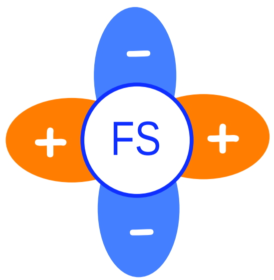
$$\Psi_0 \neq 0$$

momentum average harmless

Anderson's theorem

for non-magnetic impurities

- unconventional pairing: *l > 0* anisotropic



$$\Psi_0 = 0$$

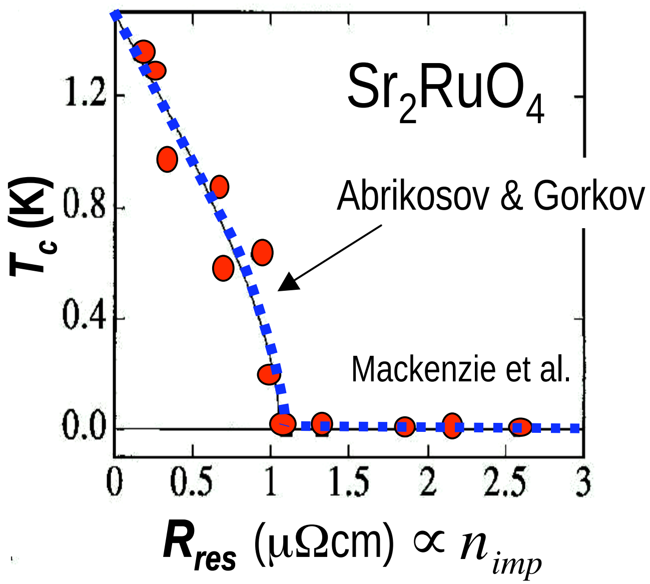
Momentum average destructive interference

→ **Suppression of superconductivity**

Impurity scattering - Anderson's theorem (1959)

Suppression of T_c

with increasing impurity concentration



mean free path: $l = v_F \tau$

life time: $\tau \propto 1/n_{imp}$

$T_c \rightarrow 0$ $\quad k_B T_{c0} \sim \frac{\hbar}{\tau}$

only clean samples are superconducting

Interference effects for Cooper pairs

$$\langle \Psi(\vec{k}) \rangle_{\vec{k},FS} = \Psi_0$$

- conventional pairing: *l=0* isotropic

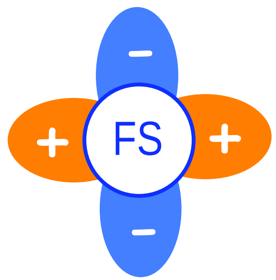
$\Psi_0 \neq 0$

momentum average harmless

Anderson's theorem

for non-magnetic impurities

- unconventional pairing: *l>0* anisotropic



$\Psi_0 = 0$

Momentum average destructive interference

→ **Suppression of superconductivity**

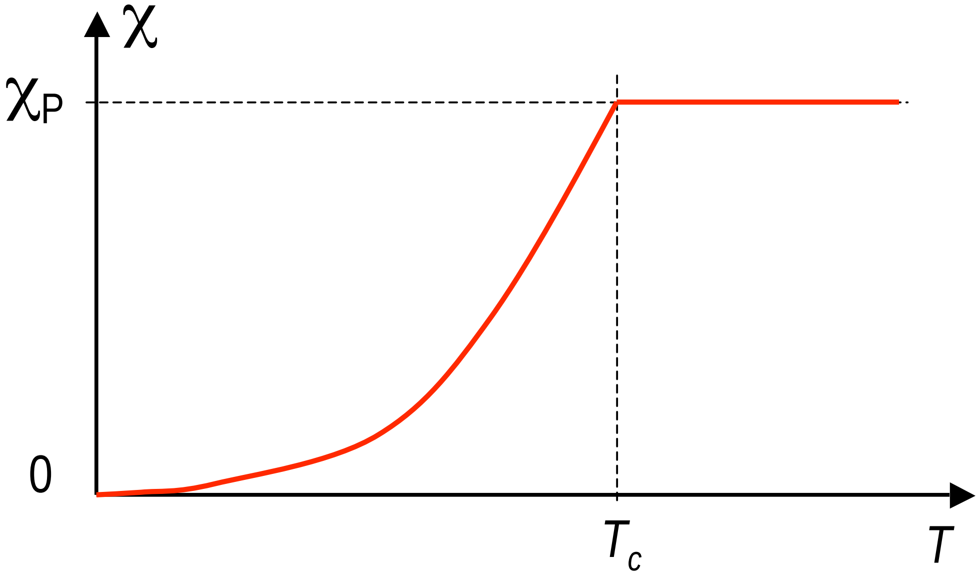
Spin susceptibility

Spin singlet pairing: Spin polarization is pair-breaking

$$\chi(T) = \frac{M(T)}{H} = 2\mu_B^2 N(0) \int \frac{d\Omega_{\vec{k}}}{4\pi} \int \frac{d\xi}{4k_B T \cosh^2(E_{\vec{k}}/2k_B T)}$$

$$= \chi_P \int \frac{d\Omega_{\vec{k}}}{4\pi} Y(\hat{k};T) = \chi_P Y(T)$$

Yosida function



$$\chi_P = 2\mu_B N(0)$$

Pauli spin susceptibility

suppression of spin susceptibility due to the gapped quasiparticle spectrum

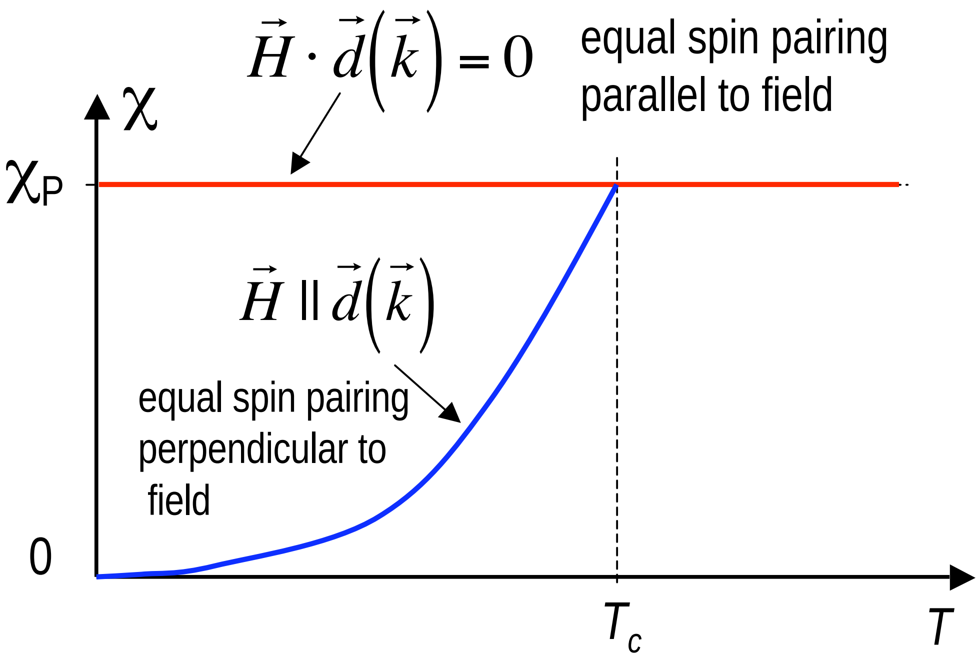
Spin susceptibility

Spin triplet pairing:

Spin polarization is not always pair-breaking

$$\chi_{\mu\nu}(T) = \chi_P \int \frac{d\Omega_{\vec{k}}}{4\pi} \left\{ \delta_{\mu\nu} - Re \frac{d_\mu(\vec{k})^* d_\nu(\vec{k})}{|\vec{d}(\vec{k})|^2} (1 - Y(\hat{k};T)) \right\}$$

Yosida function



$$\chi_P = 2\mu_B N(0)$$

Pauli spin susceptibility

Equal spin pairing:
pairing with parallel spins in the same direction for all directions of ***k***

Coherence Factor - transition probabilities

Nuclear magnetic resonance

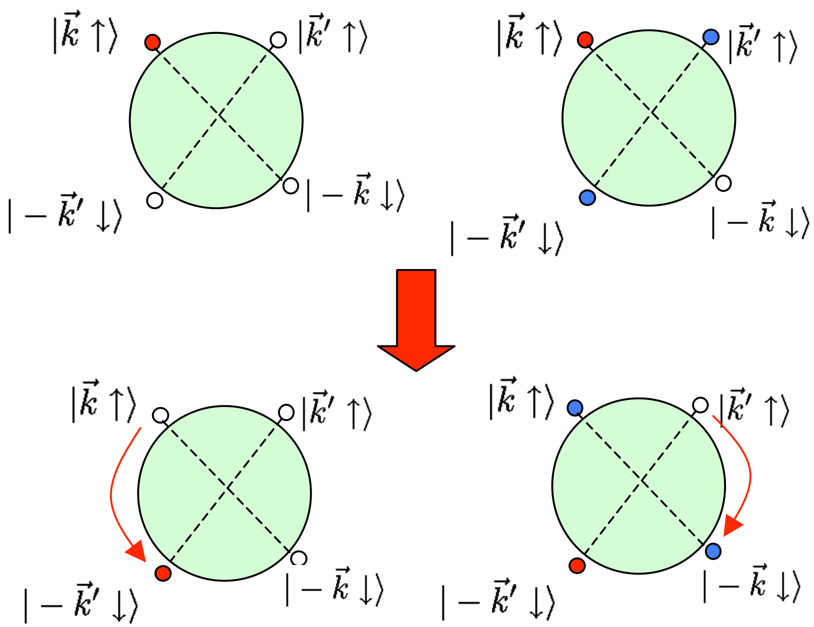
$$\mathcal{H}_{I\cdot S} = A\sum_{\vec{k},\vec{k}'}\sum_{s,s'}\vec{I}\cdot c^{\dagger}_{\vec{k}s}\vec{\sigma}_{ss'}c_{\vec{k}'s'}$$

I nuclear spin

spin flip rate: $\alpha_s = W\left(|\vec{k}\uparrow\rangle \rightarrow |-\vec{k}'\downarrow\rangle\right)$

$$|\vec{k}\uparrow\rangle\left(u_{\vec{k}'} + v_{\vec{k}'}|\vec{k}'\uparrow;-\vec{k}'\downarrow\rangle\right)$$

$$\left(u_{\vec{k}} + v_{\vec{k}}|\vec{k}\uparrow;-\vec{k}\downarrow\rangle\right)|-\vec{k}'\uparrow\rangle$$



Coherence Factor - transition probabilities

Nuclear magnetic resonance

$$\mathcal{H}_{I\cdot S} = A \sum_{\vec{k},\vec{k}'} \sum_{s,s'} \vec{I} \cdot c_{\vec{k}s} \vec{\sigma}_{ss'} c_{\vec{k}'s'}$$

I nuclear spin

spin flip rate: $$\alpha_s = W\left(|\vec{k} \uparrow\rangle \rightarrow | -\vec{k}' \downarrow\rangle \right)$$

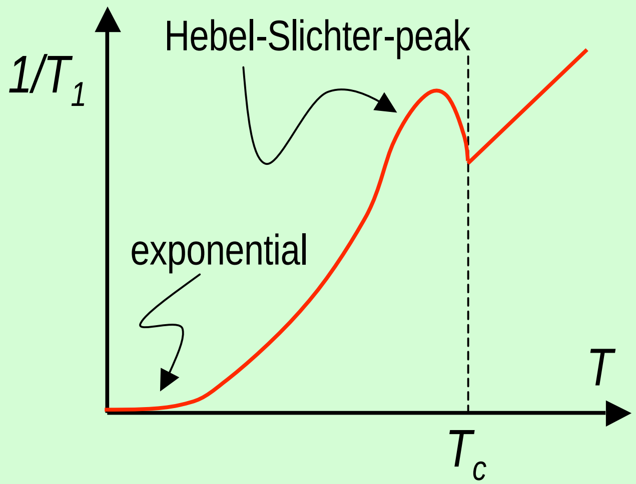
$$\alpha_s = \frac{2\pi}{\hbar} \sum_{\vec{k},\vec{k}'} C_+(\vec{k},\vec{k}') f(E_{\vec{k}}) \left(1 - f(E_{\vec{k}'})\right) \delta(E_{\vec{k}'} - E_{\vec{k}} - \omega)$$

Coherence factor: $$C_\pm(\vec{k},\vec{k}') = \begin{cases} \frac{1}{2}\left(1 + \frac{\xi_{\vec{k}}\xi_{\vec{k}'} \pm Re\Delta^2 \psi(\vec{k})\psi(\vec{k}')}{E_{\vec{k}}E_{\vec{k}'}}\right) & S=0 \\ \frac{1}{2}\left(1 + \frac{\xi_{\vec{k}}\xi_{\vec{k}'} \pm Re\Delta^2 \vec{d}(\vec{k})\cdot\vec{d}(\vec{k}')}{E_{\vec{k}}E_{\vec{k}'}}\right) & S=1 \end{cases}$$

Coherence Factor - transition probabilities

Conventional superconductor

$$\langle C_+(\vec{k},\vec{k}')\rangle_{\vec{k},\vec{k}'} = \frac{1}{2}\left(1+\frac{\Delta^2}{E^2}\right)$$

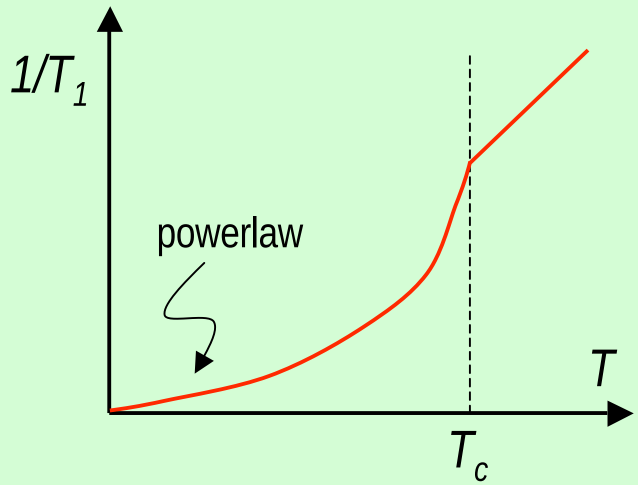


Enhancement due to

- density of states
- coherence factor

Unconventional superconductor

$$\langle C_+(\vec{k},\vec{k}')\rangle_{\vec{k},\vec{k}'} = \frac{1}{2}$$

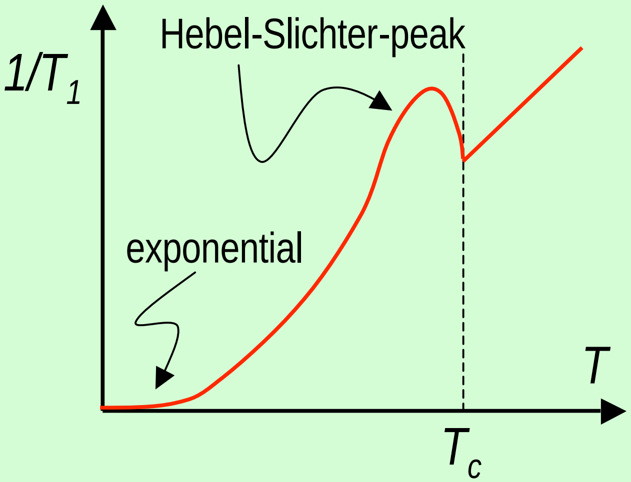


No enhancement

Coherence Factor - transition probabilities

Conventional superconductor

$$\langle C_+(\vec{k},\vec{k}')\rangle_{\vec{k},\vec{k}'} = \frac{1}{2}\left(1+\frac{\Delta^2}{E^2}\right)$$

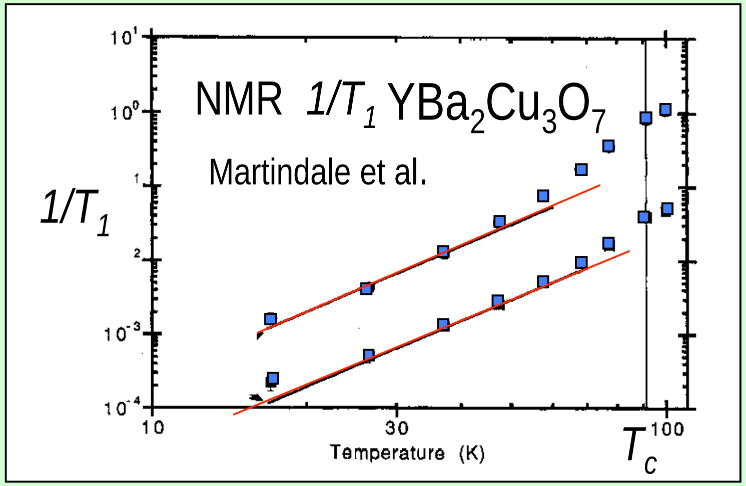


Enhancement due to

- density of states
- coherence factor

Unconventional superconductor

$$\langle C_+(\vec{k},\vec{k}')\rangle_{\vec{k},\vec{k}'} = \frac{1}{2}$$



No enhancement