CHAPTER III: IRREDUCIBLE TENSOR REPRESENTATIONS OF GL(n), SL(n), U(n) and SU(n)

In this chapter we construct the irreducible tensor representations of the full linear group and certain of its subgroups by specializing the reciprocity established in Theorem 3 of Chapter II to a reciprocity between GL(n) and the symmetric group $\mathfrak{S}_n$.

Let V be an n-dimensional vector space over a field $\emptyset$ of characteristic 0. Let G = GL(V) be the group of all invertible linear transformations on V. For a fixed m, let $V^{\otimes m}$ be the m-fold tensor product of V with itself; $V^{\otimes m}$ consists of all tensors

$$u = \sum_i v_1^i \otimes v_2^i \otimes \dots \otimes v_m^i, \quad v_j^i \in V.$$

Let $\{e_1, \dots, e_n\}$ be a basis of V. The space $V^{\otimes m}$ has a basis consisting of the n^m tensors $e_{i_1} \otimes \dots \otimes e_{i_m}$, $1 \leqslant i_j \leqslant n$. The group G acts on $V^{\otimes m}$ in a natural way (tensor product representation). If $F \in G$, we let $F^{\otimes m}$ denote the linear transformation

$$u \longrightarrow F^{\otimes m} u = (F \otimes \dots \otimes F)\, u = \sum_i F v_1^i \otimes F v_2^i \otimes \dots \otimes F v_m^i. \qquad (1)$$

The mapping $F \longrightarrow F^{\otimes m}$ defines a representation of G in the tensor space $V^{\otimes m}$. We shall determine the irreducible G-submodules of $V^{\otimes m}$.

We attack this problem indirectly. Let us consider the symmetric group $\mathfrak{S}_m$ and the group algebra $A = \emptyset \mathfrak{S}_m$. For each $s \in \mathfrak{S}_m$ we can define the following action of s on the space $V^{\otimes m}$

$$s\,(v_1 \otimes \dots \otimes v_m) = v_{s^{-1}(1)} \otimes \dots \otimes v_{s^{-1}(m)} \qquad (2)$$

in other words, the result of applying s to the tensor $v_1 \otimes \dots \otimes v_m$ is to move the i^{th} factor v_i into the $s(i)^{th}$ place. One immediately verifies that (2) defines a representation of $\mathfrak{S}_m$ in $V^{\otimes m}$, and therefore $V^{\otimes m}$ becomes a left A-module. An element $a \in A$ viewed as a linear transformation on $V^{\otimes m}$, is called a <u>symmetry operator</u> on $V^{\otimes m}$. Our first observation is that the transformations $F^{\otimes m}$ all commute with the symmetry operators. For this it is sufficient to show

$$F^{\otimes m}(s(v_1 \otimes \dots \otimes v_m)) = s(F^{\otimes m}(v_1 \otimes \dots \otimes v_m))$$

and this is obvious from the definitions. Now $A = \phi\,\mathfrak{S}_m$ is semi-simple and hence we can apply the Theorem 3 of Chapter II to determine the irreducible C-submodules of $V^{\otimes m}$ where

$$C = \mathrm{Hom}_A(V^{\otimes m}, V^{\otimes m}) .$$

These modules all have the form $eV^{\otimes m}$, where e is a primitive idempotent in A, and for any primitive idempotent e in A, either $eV^{\otimes m} = 0$ or $eV^{\otimes m}$ is an irreducible C-submodule of $V^{\otimes m}$. The primitive idempotents have all been determined in Chapter I.

<u>Definition:</u>

Let $T : G \longrightarrow GL(M)$ be a representation of the group G in the vector space M over ϕ. The ϕ-subspace of $\mathrm{Hom}_\phi(M,M)$ spanned by all the $\{ T(x) , x \in G \}$ is called the <u>enveloping algebra</u> of T. It consists of all linear combinations

$$\alpha_1 T(x_1) + \dots + \alpha_r T(x_r) , \quad x_i \in G , \quad \alpha_i \in \phi.$$

Observe that the elements of the enveloping algebra need not be invertible.

The following lemma shows that the C-submodules of $V^{\otimes m}$ are identical with the G-submodules of $V^{\otimes m}$.

Lemma 1:

Let $C = \mathrm{Hom}_A (V^{\otimes m}, V^{\otimes m})$. Then C is the enveloping algebra of the set of transformations $F^{\otimes m}$, $F \in G$. Therefore the C-submodules of $V^{\otimes m}$ are identical with the G-submodules of $V^{\otimes m}$.

Proof:

We show first that it is sufficient to prove that any linear function φ on C which vanishes on all the $F^{\otimes m}$, $F \in G$ vanishes on all of C. Let $T \equiv \{ F^{\otimes m}, F \in G \}$, $\mathcal{E}' \equiv \{ \gamma \in GL(V^{\otimes m})$; $\varphi(T) = 0 \implies \varphi(\gamma) = 0 \}$ and let $\mathcal{E}$ denote the enveloping algebra of T. Clearly $T \subset \mathcal{E} \subset \mathcal{E}'$ and thus $T^{\perp} \supset \mathcal{E}^{\perp} \supset \mathcal{E}'^{\perp}$. But from the definition of $\mathcal{E}'$ we have at once $T^{\perp} \subset \mathcal{E}'^{\perp}$ and therefore $\mathcal{E}^{\perp} = \mathcal{E}'^{\perp}$, $\mathcal{E}^{\perp\perp} = \mathcal{E}'^{\perp\perp}$. For finite dimensional linear spaces $\mathcal{E}^{\perp\perp} = \mathcal{E}$, $\mathcal{E}'^{\perp\perp} = \mathcal{E}'$ and the above characterisation of the enveloping algebra is proven.

Let $\gamma \in C$; then γ is described by its coefficients $\gamma(j_1, \ldots, j_m; i_1, \ldots, i_m)$ in a basis, where

$$\gamma(e_{i_1} \otimes \ldots \otimes e_{i_m}) = \sum_{(j_1,\ldots j_m)} e_{j_1} \otimes \ldots \otimes e_{j_m} \gamma(j_1,\ldots,j_m; i_1, \ldots, i_m).$$

It is easy to verify that $\gamma \in C$ is equivalent to the conditions

$$\gamma(j_1,\ldots,j_m; i_1,\ldots,i_m) = \gamma(j_{s^{-1}(1)},\ldots,j_{s^{-1}(m)}; i_{s^{-1}(1)},\ldots i_{s^{-1}(m)}) \tag{3}$$

A linear functional on C assigns to each $\gamma \in C$ an element of $\emptyset$ given by $\gamma \to \sum \alpha(j_1,\ldots,j_m; i_1,\ldots, i_m) \, \gamma(j_1,\ldots j_m; i_1, \ldots, i_m)$ where the α's are fixed elements of $\emptyset$ assumed without loss of generality to fulfill the symmetry conditions (3). If we put $F e_i = \sum e_j \xi_{ji}$ then by assumption

$$\sum \alpha(j_1,\ldots,j_m; i_1, \ldots, i_m) \, \xi_{j_1 i_1} \ldots \xi_{j_m i_m} = 0 . \tag{4}$$

Let us rename the $\{\xi_{ji}\}$ calling them $\lambda_1, \dots \lambda_{n^2}$. Then (4) can be rewritten as a polynomial

$$P(\lambda_1, \dots, \lambda_{n^2}) = \sum \beta(k_1 \dots, k_{n^2}) \lambda_1^{k_1} \dots \lambda_{n^2}^{k_{n^2}} = 0 \quad (5)$$

where $k_1 + \dots + k_{n^2} = m$ and $\beta(k_1, \dots, k_{n^2})$ is $m!/k_1! \dots k_{n^2}!$ times any one of the coefficients $\alpha(j_1, \dots, j_m; i_1, \dots, i_m)$ of $\xi_{j_1 i_1} \dots \xi_{j_m i_m}$ in which k_1 of the ξ_{ji} are equal to λ_1 etc. The relation (5) holds for all λ's in ϕ for which a second polynomial relation $Q(\lambda_1, \dots, \lambda_{n^2}) \neq 0$ holds, namely, the relation which expresses the fact that the determinant of F is different from zero. It follows, that in the polynomial ring $\phi[X_1, \dots, X_{n^2}]$ we have $P(X_1, \dots, X_{n^2})\, Q(X_1, \dots, X_{n^2}) = 0$. Since $Q(X_1, \dots, X_{n^2}) \neq 0$, we have ($\phi[X_1, \dots, X_{n^2}]$ is an integral domain see e.g. Van der Waerden Algebra I) $P(X_1, \dots, X_{n^2}) = 0$. Therefore all the coefficients $\beta(k_1, \dots, k_{n^2}) = 0$, and we have shown that all $\alpha(j_1, \dots j_m; i_1, \dots, i_m) = 0$. This completes the proof of the Lemma.

This Lemma, together with Theorem 3 of Chapter II proves the following result:

Theorem 1

Let $G = GL(V)$ be the general linear group on a vector space V over a field of characteristic zero, and let $V^{\otimes m}$ be the space of m-fold tensors over V. Then $V^{\otimes m}$ is a completely reducible G-module, and the irreducible G-submodules are obtained as follows. Let e be a primitive idempotent in the group algebra ϕS_m; then $eV^{\otimes m}$ is either zero or an irreducible G-submodule of $V^{\otimes m}$. All irreducible G-submodules of $V^{\otimes m}$ are obtained in this way. Moreover, two irreducible G-modules $eV^{\otimes m}$ and $e'V^{\otimes m}$ are G-isomorphic if and only if $e \phi S_m$ and $e'\phi S_m$ are isomorphic right ϕS_m-ideals.

Let now ϕ be the complex field $\mathbb{C}$. We show that an irreducible G-module of Theorem 1 remains irreducible under the action of the subgroups $SL(n, \mathbb{C})$, $U(n)$ and $SU(n)$. This follows from $GL(n, \mathbb{C}) \supset SL(n, \mathbb{C}) \supset SU(n)$, $GL(n, \mathbb{C}) \supset U(n) \supset SU(n)$ and the following

Lemma 2:

With the notations of Lemma 1, the algebra C is the enveloping algebra of the set of transformation $U^{\otimes m}$, $U \in SU(n)$.

Proof:

We use the same notations as in the proof of Lemma 1. In (4) the restriction $\det(\xi_{ij}) = 1$ can be dropped (if $\det(\xi_{ij}) = \Delta$ then $\Delta^{-1/n}(\xi_{ij}) \in SU(n)$ and since the equation (4) is homogeneous in the ξ_{ij} we can drop the factors $\Delta^{-1/n}$); hence we assume that $U = (\xi_{ji})$ in (4) is unitary. If we change U by an infinitesimal amount $U \longrightarrow U + dU$, $dU = (d\xi_{ji})$ we obtain from (5)

$$\frac{\partial P}{\partial \xi_{ij}} \, d\xi_{ij} = 0 \qquad (6)$$

From $\qquad (U + dU)^* (U + dU) = \mathbb{1}$,

we get $\qquad (U^*dU)^* + U^* dU = 0. \qquad (7)$

If we put $\qquad dU = i\,U \cdot \delta U$, $\quad \delta U = \delta\xi_{ij}$, $\qquad (8)$

equation (7) shows that $\delta U^* = \delta U$ i.e. δU is an arbitrary hermitian matrix. We rewrite (6) as

$$\frac{\partial p}{\partial \xi_{ij}} \xi_{ik} \delta \xi_{kj} = 0$$

Since an arbitrary matrix $\delta\beta_{ij}$ can be written as a sum of a hermitian and i times another hermitian matrix we conclude $\frac{\partial p}{\partial \xi_{ij}} \xi_{ik} = 0$. The matrix (ξ_{ik}) is non-singular, which implies $\frac{\partial p}{\partial \xi_{ij}} = 0$. The left hand side is a homogeneous polynomial of degree m - 1. Hence we can repeat the argument to conclude (by m-fold differentiation) that the coefficients β in (5) are equal to zero. This proves the Lemma.

Remark:

It can happen that inequivalent G-submodules $eV^{\otimes m}$ and $e'V^{\otimes m}$ become equivalent if we consider them as modules for the subgroups SL(n,C), U(n) or SU(n). This will be studied later.

In Chapter I we have determined the primitive idempotents e of ϕS_m for which $\phi S_m e$ is a minimal left ideal. In Theorem 1 we need the primitive idempotents for which $e \phi S_m$ is a minimal right ideal. In order to find these we look at the mapping

$$a = \sum_s \alpha(s)\ s \longrightarrow \hat{a} = \sum_s \alpha(s^{-1})\ s = \sum_s \alpha(s)\ s^{-1}$$

of ϕS_m onto ϕS_m. Obviously $\hat{\hat{a}} = a$ and $\widehat{ab} = \hat{b}\hat{a}$. In other words $a \longrightarrow \hat{a}$ is an involutive anti-isomorphism. Left ideals are mapped onto right ideals and inversely. If L is a minimal left ideal, then $\hat{L}$ is a minimal right ideal (show that L and $\hat{L}$ are in the same two-sided ideal). If e is a primitive idempotent then the same is true for $\hat{e}$ and vice versa.

Now let $\hat{e}$ be a primitive idempotent such that $\phi S_m e$ is a minimal left ideal. According to Theorem 14 of Chapter I it has the form

$$\hat{e} = \sum_{\substack{p \in R(D) \\ q \in C(D)}} \varepsilon_q\ pq$$

and hence

$$e = \hat{\hat{e}} = \sum \varepsilon_q \, q^{-1} \, p^{-1} = \sum \varepsilon_q \, q \, p \, , \quad (\varepsilon_q = \varepsilon_{q^{-1}}). \qquad (9)$$

This solves our problem.

Next we want to characterize the primitive idempotents for which $eV^{\otimes m} = 0$. For a fixed Young diagram D, a general element of $e(D)\, V^{\otimes m}$ is of the form

$$F' = e_{(D)}\, F = \sum_{i_1 \ldots i_m} \Big(\sum_{\substack{p \in R(D) \\ q \in C(D)}} \varepsilon_q \, qp \Big) \, F^{i_1 \ldots i_m} \, e_{i_1} \otimes \cdots \otimes e_{i_m}$$

where $F \in V^{\otimes m}$. It is convenient to arrange the indices $i_1 \ldots i_m$ of $F^{i_1 \ldots i_m}$ in an index table of the same form as D with each number k replaced by i_k. If $D = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ we find for example

$$F' = e_{(D)}\, F = \sum_{i_1 \ldots i_3} \Big(\sum_{p,q} \varepsilon_q \, qp \Big) \, F^{\substack{i_1 i_2 \\ i_3}} \, e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$$

$$= \sum_{i_1 \ldots i_3} \Big(F^{\substack{i_1 i_2 \\ i_3}} + F^{\substack{i_2 i_1 \\ i_3}} - F^{\substack{i_3 i_2 \\ i_1}} - F^{\substack{i_2 i_3 \\ i_1}} \Big) \, e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$$

$$\equiv \sum_{i_1 i_2 i_3} F'^{\, i_1 i_2 i_3} \, e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$$

<u>Proposition:</u>

The idempotents e with $eV^{\otimes m} = 0$ are precisely the ones belonging to Young diagrams with more than n rows ($n = \dim V$).

Proof:

First we note that by (I,(44)) every tensor F' in $eV^{\otimes m}$ is antisymmetric in the columns: $q\, e\, F = \mathcal{E}_q\, e\, F, \quad q \in C(D)$. Let now the number r of rows in D be bigger than n = dim V, and let $F' = (F'^{i_1 \ldots i_m})$ be a tensor in $eV^{\otimes m}$ with components $F'^{i_1 \ldots i_m}$. Let 1',...,r' be the r numbers in the first column of D. Since $r > n$ at least two of the $i_{1'}, \ldots, i_{r'}$ in $F'^{i_1 \ldots i_m}$ must be equal. If t denotes the transposition of these two numbers, then according to the above remark $t\, F' = - F'$. On the other hand t does not affect $F'^{i_1 \ldots i_m}$, hence $F'^{i_1 \ldots i_m} = 0$. This holds for every component implying $F' = 0$.

For $r \leqslant n$ we consider the special tensor

$$F_o = \underbrace{(e_1 \otimes \ldots \otimes e_1)}_{f_1} \otimes \underbrace{(e_2 \otimes \ldots \otimes e_2)}_{f_2} \otimes \ldots$$

where $[f_1, \ldots, f_r]$ denotes the length's of the rows of the table. Since diagrams belonging to the same table give isomorphic submodules it is sufficient to consider the following special diagram D

D :

1	2	. . .	f_1
f_1+1		...	f_1+f_2
etc.			

$P = \sum\limits_{p \in R(D)} p$ applied to F_o gives a number different from zero times F_o. $Q = \sum\limits_{q \in C(D)} \mathcal{E}_q\, q$ applied to F_o gives a sum of different terms, and since the $e_{i_1} \otimes \ldots \otimes e_{i_m}$ are linearly independent, $e(D)\, F_o \neq 0$ proving our proposition.

CHAPTER IV: IRREDUCIBLE CHARACTERS FOR U(n) AND S_m :

1. Irreducible characters for U(n)

In this section we compute the irreducible characters for U(n). This is possible without knowing the representations.

Let M be an irreducible U(n)-module over $\mathbb{C}$. We consider the subgroup $D \subset U(n)$ consisting of all diagonal transformations

$$A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}, \quad |\varepsilon_j| = 1, \quad \varepsilon_j = e^{i\alpha_j} .$$

This group is isomorphic to $U(1) \times U(1) \times \ldots \times U(1)$ (n times). Each element of U(n) is conjugate to an element from D. A permutation of the ε_j in A leads also to a conjugate element. Let χ be the character of M. Since χ is a class function, it is sufficient to know χ on D. $\chi(\varepsilon_1, \ldots, \varepsilon_m)$ is a symmetric function on $U(1) \times \ldots \times U(1)$. M considered as a $U(1) \times \ldots \times U(1)$ module is completely reducible with one-dimensional irreducible components. Let v_k, $k = 1, \ldots, \dim M$, be a basis of M, which adapts this decomposition. Then $A\, v_k = f_k(\alpha_1, \ldots, \alpha_n)\, v_k$, $A \in D$. The representation property implies

$$f_k(\alpha_1, \ldots, \alpha_n)\, f_k(\beta_1, \ldots, \beta_n) = f_k(\alpha_1+\beta_1, \ldots, \alpha_n+\beta_n)$$

Clearly $|f_k| = 1$, hence f_k is a primitive character of $U(1) \times \ldots \times U(1)$. We assume that the representation of U(n) is continuous. Then f_k is a continuous character. Now we use the following

<u>Lemma 1:</u>

Let $A_1, \ldots, A_n$ be (topological) abelian groups, then every primitive character of $A = A_1 \times \ldots \times A_n$ is of the form

$$(x_1, \ldots, x_n) \longrightarrow \chi_1(x_1) \ldots \chi_n(x_n), \quad x_i \in A_i$$

where the χ_i are primitive characters of the A_i.

Proof:

If χ_i is a primitive character of A_i, $i = 1, \ldots n$, then obviously $\chi = \chi_1 \ldots \chi_n$ is a primitive character of A. Conversely, let χ be a primitive character of A. If e_i is the identity element of A_i then

$$\chi(x_1, x_2, \ldots, x_n) = \chi(x_1, e_2, \ldots, e_n)\, \chi(e_1, x_2, e_3, \ldots, e_n) \ldots$$
$$\ldots \chi(e_1, \ldots, e_{n-1}, x_n)$$

If we put $\chi_j(x_j) = \chi(e_1, \ldots, e_{j-1}, x_j, e_{j+1}, \ldots, e_n)$ then χ_j is a primitive character of A_j and $\chi = \chi_1 \ldots \chi_n$. This proves the Lemma.

The continuous characters of U(1) are

$$\chi(\alpha) = e^{ih\alpha}, \quad h \in \text{integers}.$$

Hence $f_k(\alpha_1, \ldots, \alpha_n)$ is of the form

$$f_k(\alpha_1, \ldots, \alpha_n) = e^{i \sum h_m^{(k)} \alpha_m} = \varepsilon_1^{h_1^{(k)}} \ldots \varepsilon_n^{h_n^{(k)}} \equiv f_k(\varepsilon_1, \ldots, \varepsilon_n).$$

From

$$\chi(\varepsilon_1, \ldots, \varepsilon_n) = \sum f_k(\varepsilon_1, \ldots, \varepsilon_n)$$

we conclude that $\chi(\varepsilon_1, \ldots, \varepsilon_n)$ is a symmetric polynomial in the ε_j with positiv integer coefficients.

Next we need the Haar measure for U(n). Let f be a class function on U(n) and $d\mu$ the Haar measure of U(n); then (for a proof see H.Weyl: "The Theory of Groups and Quantum Mechanics", Dover publications, page 386):

$$\int_{U(n)} f\, d\mu = c \int d^n\alpha\, |\Delta|^2 f(\alpha_1, \ldots, \alpha_n) \qquad (1)$$

where

$$\Delta = \prod_{i<k}(\varepsilon_i - \varepsilon_k) = \begin{vmatrix} \varepsilon_1^{n-1} & \varepsilon_1^{n-2} & \cdots & \varepsilon_1 & 1 \\ \varepsilon_2^{n-1} & \varepsilon_2^{n-2} & \cdots & \varepsilon_2 & 1 \\ & & \vdots & & \\ \varepsilon_n^{n-1} & & \cdots & \varepsilon_n & 1 \end{vmatrix} \equiv |\varepsilon^{n-1}, \ldots, \varepsilon, 1| \quad (2)$$

We assume that the Haar measure is normalized, hence c is fixed by the equation

$$c \int |\Delta|^2 \, d\alpha_1 \ldots d\alpha_n = 1.$$

The expansion of the determinant Δ gives

$$\Delta = \sum_{\pi \in S_n} \varepsilon_\pi \, \pi \, e^{i[(n-1)\alpha_1 + (n-2)\alpha_2 + \ldots + \alpha_{n-1} + 0 \cdot \alpha_n]}$$

where π permutes $\alpha_1, \ldots, \alpha_n$. Now let $p_{h_1 \ldots h_n}$ be an antisymmetric polynomial of the form

$$p_{h_1 \ldots h_n} = \sum \varepsilon_\pi \, \pi \, e^{i \sum h_k \alpha_k}, \quad h_1 > h_2 > \ldots > h_n, h_i \in \text{ integers.}$$

Then

$$\int d^n \alpha \, \bar{p}_{h_1 \ldots h_n}(\alpha) \, p_{h'_1 \ldots h'_n}(\alpha) = \begin{cases} n! \, (2\pi)^n & \text{for } h_k = h'_k, k=1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Especially we obtain $c = [n! \, (2\pi)^n]^{-1}$. Since χ is irreducible, we must have

$$[n! \, (2\pi)^n]^{-1} \int d^n \alpha \, \overline{\Delta \cdot \chi} \, \Delta \cdot \chi = 1, \quad \text{defined in (2).} \quad (4)$$

$$\text{Let} \quad \Gamma = \Delta \cdot \chi \quad (5)$$

Γ is an antisymmetric polynomial in the ε_j with integer coefficients, hence of the form

$$\Gamma = \sum_{(h)} \alpha_{h_1 \dots h_n} p_{h_1 \dots h_n} \; ; \; \alpha_{h_1 \dots h_n} \in \text{integers.}$$

From (3) (4) we obtain

$$\sum_{(h)} \alpha^2_{h_1 \dots h_n} = 1$$

hence for some integers $h_1 > h_2 > \dots > h_n$

$$\Gamma = \pm \; p_{h_1 \dots h_n} \qquad (6)$$

We now exclude the (-) sign in (6). For this, we consider the coefficient of the "highest" term on both sides of (6) in the sence of following ordering:

$$\varepsilon_1^{k_1} \dots \varepsilon_n^{k_n} > \varepsilon_1^{k_1'} \dots \varepsilon_n^{k_n'} \text{ if } (k_1, \dots, k_n) > (k_1', \dots k_n')$$

and where $(k_1, \dots, k_n) > (k_1', \dots, k_n')$ if, by reading from the left, the first k_i different from k'_i is bigger than k'_i.

The right hand side in (6) has highest term $\pm \; \varepsilon_1^{h_1} \dots \varepsilon_n^{h_n}$. The highest term coming from Δ in the left hand side of (6) is $\varepsilon_1^{n-1} \varepsilon_2^{n-2} \dots \varepsilon_n^{o}$, and χ has only positive integer coefficients. This shows that the (-) sign in (6) is excluded.

We arrive at the result that a primitive character has necessarily the form

$$\chi_{h_1 \dots h_n}(\varepsilon_1, \dots, \varepsilon_n) = \Delta^{-1} p_{h_1 \dots h_n}(\varepsilon_1, \dots, \varepsilon_n)$$

$$= \frac{|\,\varepsilon^{h_1}, \dots, \varepsilon^{h_n}\,|}{|\,\varepsilon^{n-1}, \dots, \varepsilon, 1\,|} \; ; \quad h_1 > h_2 > \dots > h_n, \; h_i \in \text{integers} . \qquad (7)$$

Applications of the Character Formula:

We have shown that the functions of the form (7) are orthonormal with respect to the Haar measure (see equation (3)). From the Peter-Weyl theory one knows that the primitive characters of a compact group from a complete orthonormal set for the class functions on the group. This fact implies that all functions of the form (7) are primitive characters of U(n). We will obtain this result below without using the Peter-Weyl theory. First we compute the character of a representation of U(n) corresponding to the Young table $D = [f_1, \dots, f_n]$. We know that this character has necessarily the form (7).

It is clear that $e(D)V^{\otimes f}$, $f = f_1 + \dots + f_n$ is spanned by vectors $e(D)\, e_{i_1} \otimes \dots \otimes e_{i_f}$ which contain at most f_i identical factors e_i, otherwise the application of $Q \equiv \sum_{q \in C(D)} \varepsilon_q\, q$ gives zero. On the other hand we have seen that $e(D)\, x_0 \neq 0$ for

$$x_0 = \underbrace{(e_1 \otimes \dots \otimes e_1)}_{f_1} \otimes \underbrace{(e_2 \otimes \dots \otimes e_2)}_{f_2} \otimes \dots$$

Clearly

$$A^{\otimes f}\, e(D)\, x_0 = \varepsilon_1^{f_1} \dots \varepsilon_n^{f_n}\, e(D)\, x_0 \;, \quad A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}.$$

Hence the highest term in the character χ of $[f_1,\dots,f_n]$ is $\varepsilon_1^{f_1} \dots \varepsilon_n^{f_n}$. This imples that the highest term of $\Delta\chi$ is

$\varepsilon_1^{f_1+n-1}\, \varepsilon_2^{f_2+n-2} \dots \varepsilon_n^{f_n}$ and thus

$$h_1 = f_1 + n - 1 \;, \quad h_2 = f_2 + n - 2 \;, \quad h_n = f_n \;. \tag{8}$$

Completeness:

In (7) the h_i are not restricted to have $f_n \geqslant 0$ (the f_i are now always given by (8). If we, however, multiply the tensor representation $[f_1,\dots,f_n]$ by $(\det A)^{-k}$, $A \in U(n)$, k positive integer, the representation so obtained is of course still irreducible and has a character of the from (7) where the h'_i are given by $h'_i = h_i - k$. We see that <u>we obtain in this way all continuous irreducible representations of U(n).</u>

The same argument shows that if $A \longrightarrow [f_1, \dots, f_n]$ is a tensor representation of U(n) then

$A \longrightarrow (\det A)^{f_n} [f_1 - f_n, f_2 - f_n, \dots, 0]$ gives the same representation. Hence the tensor representations $[f_1, \dots f_{n-1}]$ are the complete system of continuous representations of SU(n). Two different Young tables with $f_n = 0$ give inequivalent representations for SU(n). This can easily be seen from the character formula (7) (note that $\varepsilon_1, \dots, \varepsilon_{n-1}$ are independent variables).

Dimension Formula:

The dimension of the representation belonging to the character $\chi_{h_1 \dots h_n}$ is equal to $\chi_{h_1 \dots h_n}(\mathbb{1})$. If we take formula (7) then $\chi(\mathbb{1})$ is of the form o/o. For this reason we adopt the following limiting procedure. We put $\alpha_1 = (n-1)\,\alpha$, $\alpha_2 = (n-2)$, $\dots$, $\alpha_n = o\,\alpha$ and let $\alpha \longrightarrow 0$. For $\alpha \longrightarrow 0$

$$e^{h_1 \alpha} - e^{h_2 \alpha} \simeq \alpha\,(h_1 - h_2)\ , \text{ hence for } \alpha \longrightarrow 0$$

$$|\,\varepsilon^{h_1}, \dots, \varepsilon^{h_n}\,| \simeq \prod_{i<k} (h_i - h_k)\,\alpha$$

$$|\,\varepsilon^{n-1}, \dots, \varepsilon, 1\,| \simeq \prod_{i<k} (k - i)\,\alpha$$

and the ratio of the difference products in (7) becomes

$$N_{[h_1, \dots, h_n]} = \frac{\prod_{i<k} (h_i - h_k)}{\prod_{i<k} (k-i)} = \frac{\Delta(h_1, \dots h_n)}{\Delta(n-1, \dots 1, 0)} \qquad (9)$$

For SU(3) we get for the representations $[h_1, h_2, 0]$

$$N_{[h_1,h_2,0]} = \frac{(h_1-h_2)(h_1-0)(h_2-0)}{(2-1)(2-0)(1-0)} = \frac{1}{2}(h_1-h_2)\, h_1 h_2 \qquad (10)$$

The complete system of irreducible representations of SU(2) is given by $[f,0]$. This shows that all spaces of irreducible representations are already spanned by symmetric tensors.

Let us return for a moment to the full linear group $GL(n,\mathbb{C})$. The tensor representation belonging to $[f_1,\dots,f_n] = [1,\dots,1]$ is obviously the representation $A \longrightarrow (\det A)$. The representation $A \longrightarrow (\det A)^k$, k positive integer, is the k-fold tensor product of the representation $A \longrightarrow \det A$ and hence must belong to the tensor representations. To which Young table does it belong ? Since polynomial degree is k, we must have $f_1 + \dots + f_n = k \cdot n$. From the dimension formula (9), which obviously holds also for the tensor representations of $GL(n,\mathbb{C})$, it is clear, that the only one dimensional representation with this property is $f_1 = f_2 = \dots = f_n = k$.

Furthermore the representation $A \longrightarrow (\det A)^k\ [f_1,\dots,f_n]$ is irreducible and belongs to the Young table $[f_1+k, f_2+k, \dots, f_n+k]$ since this is true for the unitary subgroup U(n). Hence for $SL(n,\mathbb{C})$ we can restrict ourselves to Young tables with $f_n = 0$. Two different Young tables with $f_n = 0$ give inequivalent irreducible representations for $SL(n,\mathbb{C})$ since this is the case for the subgroup SU(n).

We add a further remark. Sometimes it is useful (especially for SU(3)) to consider the following representations. Let $\overset{*}{V}$ be an n-dimensional vector space with the following action of U(n)

$$A \in U(n):\ \xi_i \longrightarrow \bar{a}_{ik}\, \xi_k\ ,\ (a_{ik}) = A\ ,$$

and consider the U(n)-module $\overset{*}{V}{}^{\otimes f}$. It is obvious from our earlier result that the set $\bar{A}^{\otimes f}$, $A \in U(n)$, generates the centralizer $C = \mathrm{Hom}_{\mathbb{C} S_f}(\overset{*}{V}{}^{\otimes f}, V^{*\otimes f})$. Hence the U(n)-submodules are identical with the C-submodules and all irreducible U(n)-submodules are of the form

$$e\,\overset{*}{V}{}^{\otimes f}\,, \tag{11}$$

e a primitive idempotent. If e belongs to the Young table $[f_1,\ldots, f_n]$ we denote the representation (11) by $[f_1,\ldots,f_n]^*$. To which character does this representation belong ? Using the same arguments as above, it is clear that the tensor

$$\overset{*}{x}_o = \underbrace{(\overset{*}{e}_1 \otimes \ldots \otimes \overset{*}{e}_1)}_{f_n} \otimes \underbrace{(\overset{*}{e}_2 \otimes \ldots \otimes \overset{*}{e}_2)}_{f_{n-1}} \otimes \ldots$$

where $\overset{*}{e}_i$ is a basis of $\overset{*}{V}$ picks up the highest term, namely

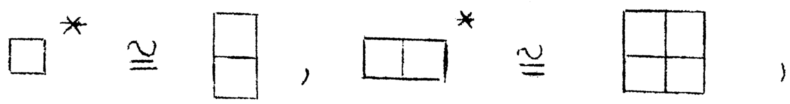
$$\bar{\varepsilon}_1^{\,f_n}\;\bar{\varepsilon}_2^{\,f_{n-1}}\ldots\bar{\varepsilon}_n^{\,f_1} = \varepsilon_1^{-f_n}\,\varepsilon_2^{-f_{n-1}}\ldots\varepsilon_n^{-f_1}$$

From this it is obvious that the representation $[f_1,\ldots,f_n]^*$ is isomorphic to the representation

$$A \longrightarrow (\det A)^{-f_1}\,[f_1 - f_n\,,\; f_1 - f_{n-1}\,,\;\ldots\;,\;0]$$

If we restrict ourselves to SU(n), we conclude that the representation $[f_1,\ldots,f_{n-1}]^*$ is equivalent to $[f_1,\; f_1-f_{n-1},\ldots,f_1-f_2]$. (Draw a picture of this relation).

<u>Examples for SU(3):</u>



etc.

Without proof, we give now the construction of all irreducible (always finite-dimensional) representations of $SL(n, \mathbb{C})$. Let V be an n-dimensional vector space with vectors ξ_α and consider the "fundamental" representation of $SL(n, \mathbb{C})$:

$$A \in SL(n, \mathbb{C}) : \xi_\alpha \longrightarrow a_{\alpha\beta} \xi_\beta , \quad A = (a_{\alpha\beta}).$$

Beside V consider a second n-dimensional vector space $\dot{V}$ with vectors $\eta_{\dot\alpha}$ and the following action of $SL(n, \mathbb{C})$:

$$A \in SL(n, \mathbb{C}) : \eta_{\dot\alpha} \longrightarrow \bar{a}_{\dot\alpha\dot\beta} \eta_{\dot\beta} , \quad \bar{A} = (\bar{a}_{\dot\alpha\dot\beta}).$$

Then the following theorem is true: (i.e. Boerner page 159)

<u>Theorem 1:</u>

The $SL(n, \mathbb{C})$ submodules

$$e\, V^{\otimes m} \otimes e'\, \dot{V}^{\otimes m'} \tag{12}$$

where e and e' are primitive idempotents of $\mathbb{C}S_m$ and $\mathbb{C}S_{m'}$ belonging to Young tables $[f_1, \ldots, f_{n-1}]$ and $[f'_1, \ldots, f'_{n-1}]$ are irreducible. One gets in this way all irreducible continuous $SL(n, \mathbb{C})$ representations. Moreover, two modules (12) are isomorphic if and only if their corresponding pair of Young tables (with $f_n = f'_n = 0$) are the same.

<u>Remark:</u>

If we restrict the representation (12) to $SU(n)$ then it is the tensor product representation

$$[f_1, \ldots, f_{n-1}] \otimes [f'_1, \ldots, f'_{n-1}]^*$$

$$= [f_1, \ldots, f_{n-1}] \otimes [f'_1,\ f'_1 - f'_{n-1}, \ldots,\ f'_1 - f'_2].$$

2. Relation between the Characters of U(n) and $\mathfrak{S}_m$

The correspondence existing between the representations of $\mathfrak{S}_m$ and U(n), suggests, that we can find an expression for the characters of $\mathfrak{S}_m$ in terms of those of U(n). In the following we shall exhibit such a relation and give a formula for the dimension of an irreducible $\mathfrak{S}_m$ - representation.

Let V be a n-dimensional vector space over $\mathbb{C}$ and let $M = V^{\otimes m}$. If we define the action of U(n) rsp. $\mathfrak{S}_m$ on M as in III.(1) rsp. III.(2), then M becomes a U(n) × $\mathfrak{S}_m$ - module. Let $\pi \in \mathfrak{S}_m$, $A \in U(n)$. We now calculate $Tr(\pi A)$, πA considered as an operator in M, in two different ways.

a) We begin with a decomposition of M into irreducible U(n) × $\mathfrak{S}_m$-modules

$$M = \bigoplus_{(r)} M^{(r)} \quad , \quad r = [r_1, \ldots, r_k] \tag{13}$$

where the sum runs over all Young tables. As shown in the Appendix to chapter II, $M^{(r)}$ is isomorphic to a tensorproduct $M_1^{(r)} \otimes M_2^{(r)}$ where $M_i^{(r)}$ are irreducible U(n), $\mathfrak{S}_m$-modules respectively. If we denote the corresponding characters of $\mathfrak{S}_m$ with $\chi^{(r)}$ and of U(n) with $X^{(r)}$ we get

$$Tr\ (\pi A) = \sum_{(r)} \chi^{(r)}_{(\pi)}\ X^{(r)}_{(A)}\ . \tag{14}$$

b) Let $\{ e_i\ , i = 1, \ldots, n \}$ be a basis for V such that $Ae_i = \varepsilon_i e_i$. The character is a class function and it is therefore enough to consider

$$A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}$$. We find at once

$$\pi A\, e_{i_1} \otimes \dots \otimes e_{i_m} = \varepsilon_{i_1} \dots \varepsilon_{i_m}\; e_{i_{\pi^{-1}(1)}} \otimes \dots \otimes e_{i_{\pi^{-1}(m)}} \tag{15}$$

The Matrix πA with respect to the basis $e_{i_1} \otimes \dots \otimes e_{i_m}$ has diagonal elements if and only if

$$i_{\pi^{-1}(j)} = i_j \quad \text{for all} \quad j. \tag{16}$$

We now determine the subspace of $V^{\otimes m}$ with the property that $i_{\pi^{-1}(j)} = i_j$. Let $(\gamma_1, \gamma_2, \dots)$ be the cycle decomposition of π. γ_1 permutes certain factors in $e_{i_1} \otimes \dots \otimes e_{i_m}$ i.e. the positions $\alpha_1, \dots, \alpha_{\gamma_1}$, and similarly for the other cycles. Only if $i_{\alpha_1} = i_{\alpha_2} = \dots = i_{\alpha_{\gamma_1}}$, can (16) be satisfied. Let M_{γ_1} be the subspace generated by the elements

$$\left\{ e_p(\alpha_1) \otimes e_p(\alpha_2) \otimes \dots \otimes e_p(\alpha_{\gamma_1}) \;;\; p = 1,\dots,n \right\} .$$

In this subspace (γ_1) acts "diagonal". If we repeat this construction with the other cycles and note that πA acts "diagonal" exactly in $M_{\gamma_1} \otimes M_{\gamma_2} \otimes \dots$ we finally get

$$\mathrm{Tr}(\pi A) = \prod_j \mathrm{Tr}\,(A)\Big|_{M_{\gamma_j}} = \prod_j \Big(\sum_{k=1}^{n} \varepsilon_k^{\gamma_j}\Big) = \prod_j \mathrm{Tr}\,(A^{\gamma_j}) \equiv \prod_j \sigma_j \tag{17}$$

A comparison of the two results for $\mathrm{Tr}(\pi A)$ leads to

$$\sum_{(r)} \chi^{(r)}_{(\pi)} \; \chi^{(r)}(\varepsilon_1, \ldots, \varepsilon_n) = \prod_j \sigma_j \qquad (18)$$

Note that the right hand side of (18) is a polynomial in the $\varepsilon_1 \ldots \varepsilon_n$. We have thus found a relation between the characters of S_m and U(n). With the help of the orthogonality relations for the characters $\chi^{(r)}$ we could in principle extract an expression for the characters $\chi^{(r)}$ of S_m. We shall not do this but derive instead a

<u>Dimension formula for the representations of</u> S_m:

The character $\chi^{(r)}(\varepsilon_1 \ldots \varepsilon_n)$ of U(n) is given by (7) and (8), so that

$$\sum_{(r)} \chi^{(r)}_{(\pi)} \; p_{h_1 \ldots h_n} = \Delta \cdot \prod_j \sigma_j \qquad (19)$$

and $\chi^{(r)}_{(\pi)}$ is equal to the coefficient of $\varepsilon_1^{h_1} \ldots \varepsilon_n^{h_n}$ in $\Delta \cdot \prod_j \sigma_j$. In general

$$\text{dimension } g = \chi^{(r)}_{(\mathbb{1})} \qquad (20)$$

If we set $\pi = \mathbb{1}$, then $\gamma_1 = \gamma_2 = \ldots = \gamma_m = 1$ and

$$\prod_j \sigma_j = \prod_j \left(\sum_{k=1}^{n} \varepsilon_k\right) = \left(\sum_{k=1}^{n} \varepsilon_k\right)^m.$$

The right hand side in (19) then becomes

$$\left(\sum_{k=1}^{n} \varepsilon_k\right)^m \cdot \sum_{\pi} (-1)^{\pi} \; \pi \; (\varepsilon_1^{n-1} \; \varepsilon_2^{n-2} \ldots \varepsilon_n^{o}) \qquad (21)$$

and we have to find the coefficient of $\varepsilon_1^{h_1} \ldots \varepsilon_n^{h_n}$ in this expression. Now

$$\left(\sum_{k=1}^{n} \varepsilon_k\right)^m = \sum_{k_1 \ldots k_n} \frac{m!}{k_1! \ldots k_n!} \; \varepsilon_1^{k_1} \ldots \varepsilon_n^{k_n}$$

and

$$\pi\,(\varepsilon_1^{n-1}\ \varepsilon_2^{n-2}\ \ldots\ \varepsilon_n^{o}) = \varepsilon_1^{f_1}\ \ldots\ \varepsilon_n^{f_n}$$

so that to get the desired term

$$k_i = h_i - f_i \tag{22}$$

This yields with $\pi\,(n-1, n-2, \ldots, o) = (f_1, \ldots, f_n)$

$$g = m! \sum_{\pi} (-1)^{\pi} \frac{1}{(h_1-f_1)!\ \ldots\ (h_n-f_n)!}$$

$$= m! \cdot \begin{vmatrix} \frac{1}{[h_1-(n-1)]!} & \frac{1}{[h_1-(n-2)]!} & \cdots & \frac{1}{h_1!} \\ \frac{1}{[h_2-(n-1)]!} & \cdots & & \vdots \\ \vdots & & & \\ \frac{1}{[h_n-(n-1)]!} & \cdots & & \frac{1}{h_n!} \end{vmatrix}$$

$$= \frac{m!}{h_1!\ldots h_n!} \begin{vmatrix} h_1(h_1-1)\ldots(h_1-(n-2)); & \ldots; h_1(h_1-1); & h_1; & 1 \\ & & h_2; & \\ \vdots & & \vdots & \\ & ; & h_n; & 1 \end{vmatrix}$$

The elements in a fixed row are always polynomials with degree descending from left to right. It is thus possible (by addition of multiples of columns) to arrive at the form

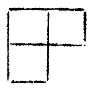
$$g = \frac{m!}{h_1! \dots h_n!} \begin{vmatrix} h_1^{n-1} & \dots\dots & h_1^2 & h_1 & 1 \\ & & & & \\ h_n^{n-1} & \dots\dots & h_n^2 & h_n & 1 \end{vmatrix}$$

which is again a Van der Monde determinant and the final result for the dimension g reads

$$g_{[r_1,r_2,\dots,r_n]} = \frac{m! \cdot \prod\limits_{i<k} (h_i - h_k)}{h_1! \dots\dots h_n!} \qquad (23)$$

g does not depend on the parameter n. It is in general enough to chose for n the number of rows in the Young table of the representation. But $n \geqslant 2$ is a necessary condition for the validity of the foregoing calculations.

<u>Example:</u> S_3 ; $r = [r_1, r_2] = [2, 1]$

r: 

Chose $n = 2$. Then

$$h_1 = r_1 + (n-1) = 3$$
$$h_2 = r_2 + (n-2) = 1$$

and g becomes

$$g_{[2,1]} = \frac{3!\ 2}{3!\ 1!} = 2 \quad .$$

CHAPTER V PHYSICAL APPLICATIONS

In this chapter we illustrate the usefulness of the group theoretical results developed in the previous chapters for the solution of a variety of physical problems. We start with a discussion of the (L,S)-terms of a complicated atom.

1. (L,S) - terms for an Atom

Let us consider an atom with f electrons. We would like to classify the eigenstates of the Hamiltonian

$$H = \frac{1}{2m} \sum_{i=1}^{f} \vec{p}_i^{\,2} - \sum_{i=1}^{f} \frac{Ze^2}{r_i} + \sum_{i<k} \frac{e^2}{r_{ik}} \qquad (1)$$

that is, we consider the case where the spin-orbit coupling is small and neglect it in a first approximation. The Hilbert space of f electrons is the antisymmetric part of

$$\mathcal{L}^2(\mathbb{R}^{3f}) \otimes (\mathbb{C}^2)^{\otimes f}$$

and the symmetry group is $O(3) \times \mathcal{S}_f$. The subspace belonging to an energy E is the antisymmetric part of

$$M \otimes (\mathbb{C}^2)^{\otimes f}, \quad M \subset \mathcal{L}^2(\mathbb{R}^{3f}) \qquad (2)$$

where M is an irreducible $O(3) \times \mathcal{S}_f$ - module if the energy is not accidentally degenerate.

Proposition:

M has a definite parity.

Proof:

The parity P: $\vec{x} \longrightarrow -\vec{x}$ commutes with all rotations and the elements $\{\mathbb{1}, P\}$ form an abelian, cyclic, invariant subgroup of O(3); $O(3) \cong SO(3) \times \{\mathbb{1}, P\}$. Then according to the reasoning on p.59 we can reduce M into a direct sum of blocks such that the rows in each block transform irreducibly under $SO(3) \times \mathcal{S}_f$ whereas the columns transform irreducibly under $\{ \mathbb{1}, P \}$. M is irreducible and therefore only one block (consisting of one row) can occur.

We conclude that M is also an irreducible $SO(3) \times \mathcal{S}_f$ - module. Hence we can introduce a basis in M (see Appendix to Chapter II)

$$M = M(\Delta, L): \quad \begin{matrix} \varphi^L_{-L,1} & \varphi^L_{-(L-1),1} & \cdots & \varphi^L_{L,1} \\ \vdots & & & \\ \varphi^L_{-L,r} & & \cdots\cdots & \varphi^L_{L,r} \end{matrix} \qquad (3)$$

such that the rows transform all in the same way according to the representation $D^{(L)}$ of $SO(3)$ and such that the columns transform irreducibly and in the same way according to a representation Δ of $\mathcal{S}_f$.

In focussing our attention to the module $(\mathbb{C}^2)^{\otimes f}$, we first remark that we can choose the parity operator equal to $\mathbb{1}$ as long as we consider only one sort of particles (According to Schur's Lemma, the parity operator in $\mathbb{C}^2$ is $\pm \mathbb{1}$. An overall phase of the parity operator is, however, arbitrary). The reduction of $(\mathbb{C}^2)^{\otimes f}$ with respect to $SU(2) \times \mathcal{S}_f$ gives

$$(\mathbb{C}^2)^{\otimes f} = \bigoplus_{\Delta', S} N(\Delta', S) \qquad (4)$$

In $N(\Delta', S)$ we introduce a basis similar to the one in (3):

$$
N(\Delta', S):\quad
\begin{array}{ccccc}
U^{S}_{-S,1} & \cdots\cdots & U^{S}_{S,1} & \longrightarrow & D^{(S)} \\
\vdots & & \vdots & & \\
\vdots & & \vdots & & \\
U^{S}_{-S,t} & \cdots\cdots & U^{S}_{S,t} & \longrightarrow & D^{(S)} \\
\downarrow & & \downarrow & & \\
\Delta' & & \Delta' & &
\end{array}
\qquad (5)
$$

<u>Remarks:</u>

1.) The Young table corresponding to Δ' cannot have more than two rows since the rows in (5) span a representation of SU(2).

2.) Δ' is uniquely determined by S. By construction, we have collected in a block <u>all</u> isomorphic SU(2) - modules and <u>all</u> isomorphic S_f - modules.

Now we form

$$M(\Delta, L) \otimes (\mathbb{C}^2)^{\otimes f} = \bigoplus_{S} M(\Delta, L) \otimes N(\Delta', S)$$

and look for the antisymmetric part. In the Appendix to this chapter we show that the antisymmetric representation is contained in $\Delta \otimes \Delta'$ if and only if and then exactly once if Δ' belongs to the Young table which is obtained from the one corresponding to Δ by interchanging rows and columns. We shall call this representation the associated representation of Δ and denote it by $\tilde{\Delta}$.

The considerations so far show that an energy eigenvalue E has a definite parity, a definite L and a definite S (if we ignore accidental degeneracy). The symmetry character of the space function is uniquely determined by the spin S. Its Young table cannot have more than two columns. This is the effect of the Pauli-principle.

Clearly the elements

$$\underline{\Phi}^{(L,S)}_{m_L,m_S} \equiv \frac{1}{\sqrt{r}} \sum_{k=1}^{r} \varphi^{L}_{m_L,k} \otimes U^{S}_{m_S,k} \qquad (6)$$

$r = \dim \Delta\ , \quad -L \leqslant m_L \leqslant L\ , \quad -S \leqslant m_S \leqslant S$

transform under $\pi \in S_f$ (see Appendix) according to (using the unitarity of Δ)

$$\underline{\Phi}^{(L,S)}_{m_L,m_S} \xrightarrow{\ \pi\ } \frac{1}{\sqrt{r}} \sum_{k} \sum_{k'\,k''} \Delta_{kk'}(\pi)\ \varphi^{L}_{m_L,k'} \otimes \varepsilon_\pi\, \overline{\Delta}_{kk''}(\pi)\ U^{S}_{m_S,k''}$$

$$= \varepsilon_\pi\ \underline{\Phi}^{(L,S)}_{m_L,m_S}$$

so that (6) gives a basis of the antisymmetric part in $M(\Delta,L) \otimes N(\tilde{\Delta},S)$.

In order to obtain the manifold of (L,S) - terms of a complicated atom one starts with a Hamiltonian

$$H_o = \frac{1}{2\,m} \sum_{i=1}^{f} \vec{p}_i^{\,2} + \sum_{i} V\,(r_i)$$

where V(r) is a screened Coulomb-potential which takes into account the attraction of the nucleus and the averaged repulsion of the electrons (central field approximation). By switching on the full interaction continuously, the manifold of (L,S) - terms is not changed. In section 6 of the following chapter we show that separate shells can be treated independently (the Pauli principle is not "effective" between different shells). Thus, there remains the task of determining the (L,S) - terms of f electrons in a given shell. We now apply our group theoretical tools to solve this problem. (This problem can be solved with more elementary methods; the amount of labour increases rapidly, however, if the number of equivalent electrons increases; see e.g. Van der Waerden: Die Gruppentheoretischen Methoden in der Quantenmechanik.)

Let us consider a shell with basis φ_m^ℓ. Let V be the space spanned by the φ_m^ℓ , $-\ell \leq m \leq \ell$. In order to determine the (L,S)-terms of f electrons in this shell, we have to reduce $V^{\otimes f}$ with respect to SO(3) x $\mathcal{S}_f$, whereby only representations of $\mathcal{S}_f$ are allowed which belong to Young tables with not more than two columns (Pauli principle). Since SO(3) commutes with $\mathcal{S}_f$, the group SO(3) operates by a subgroup of the centralizer

$$C = \mathrm{Hom}_{\mathbb{C}\mathcal{S}_f} (V^{\otimes f}, V^{\otimes f}) .$$

From III. Lemma 2, we know that C is generated by SU(2 ℓ + 1). We imagine that we have decomposed $V^{\otimes f}$ into blocks of irreducible ($\mathcal{S}_f$, C) - modules. These modules belong to a Young table R. The spin is given by the associated table $\tilde{R}$ and the possible L values are given by reducing the irreducible tensor representation of SU(2ℓ + 1) with respect to SO(3). The embedding SO(3) $\subset$ SU(2ℓ + 1) is such that the basis vectors φ_m^ℓ transform under SO(3) according to

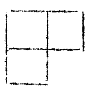
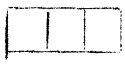
$$\varphi_m^\ell \xrightarrow{SO(3)} \varphi'^{\ell}_m = \sum_{m'} D^\ell_{mm'} \varphi^\ell_{m'} . \qquad (7)$$

Note that the Matrix $(D^\ell_{mm'})$ is an element of SU(2ℓ + 1).

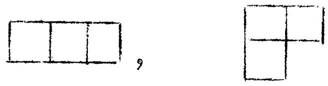
Before discussing the branching SU(2ℓ + 1) $\supset$ SO(3) more systematically, we work out a simple

Example: 3 p electrons.

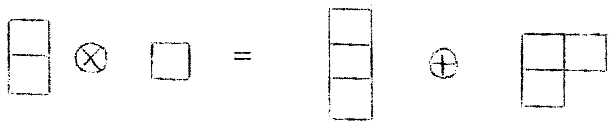
For p electrons ℓ = 1. The only allowed tables R are

R:  ,  (not ).

The corresponding $\tilde{R}$ are

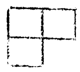
$\tilde{R}$: 

whose dimensions as irreducible SU(2) representations are (by IV.(9)) 4 and 2 respectively, leading to spin values $S = {}^3/2$ and $S = {}^1/2$. Now we have to compute the branching $SU(3) \supset SO(3)$ for the tables R. Since  has dimension 1 it contains only L = 0. The table  has only L = 1 since the dimension of the corresponding SU(3) representation is 3.  contains obviously only L = 1. Now (see Appendix to this chapter)

 (8)

Restricting (8) to the subgroup $SO(3) \subset SU(3)$, we obtain for the left hand side

$$D^{(1)} \otimes D^{(1)} = \underset{L=2}{D^{(2)}} \oplus \underset{L=1}{D^{(1)}} \oplus \underset{L=0}{D^{(0)}}$$

which shows that the L-values of  are L = 2,1.

With the notation

^{2S+1}X :

L	0	1	2	3	4	..
X	S	P	D	F	G	..

we find the following (L,S) - terms of three equivalent p electrons:

$$^4S\,, \quad ^2P\,, \quad ^2D\,.$$

Decomposition of Representations of SU(n) into Representations of SO(3):

We describe first a general method for the reduction $SU(2\ell+1) \supset SO(3)$.

Let $X(\varepsilon_{-\ell}, \ldots, \varepsilon_{+\ell})$ be the character of an irreducible representation of $SU(2\ell+1)$. If we restrict this representation to the subgroup SO(3) we obtain a representation with character (see (7)) $Y(\varepsilon) \equiv X(\varepsilon^{-\ell}, \varepsilon^{-(\ell-1)}, \ldots, \varepsilon^{\ell})$. Here $\varepsilon = e^{i\varphi}$; φ = rotation angle around z-axis.

Let $\chi^{(L)}(\varepsilon)$ be a primitive character of SO(3). It has the form

$$\chi^{(L)}(\varepsilon) = \sum_{m=-L}^{L} \varepsilon^{m} = \frac{\varepsilon^{L+1} - \varepsilon^{-L}}{\varepsilon - 1} . \tag{9}$$

The expansion of $Y(\varepsilon)$ into primitive characters is given by

$$Y(\varepsilon) = \sum_{L} a_L \chi^{(L)}(\varepsilon) \tag{10}$$

where a_L is the multiplicity of L in $Y(\varepsilon)$ (see next chapter for some general results on characters).

If we expand $Y(\varepsilon)$ as

$$Y(\varepsilon) = \sum_{n} b_n \varepsilon^{n}$$

and compare with (10) then

$$a_L = b_L - b_{L+1} \quad (= b_{-L} - b_{-(L+1)}) \tag{11}$$

We apply this method first to the representation $\square\square$.
Evaluation of IV.(7) for $\square\square$ shows that

$$X(\varepsilon_1, \ldots, \varepsilon_{2\ell+1}) = \sum_{i \leqslant j} \varepsilon_i \varepsilon_j \qquad (12)$$

and hence

$$Y(\varepsilon) = \sum_{-\ell \leqslant m_1 \leqslant m_2 \leqslant \ell} \varepsilon^{m_1} \varepsilon^{m_2} . \qquad (13)$$

From this, one easily finds the L-content:

$$\square\square \; : \; L = 2\ell, 2\ell - 2, \ldots, 0 \qquad (14).$$

For the antisymmetric representation $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ the character is
$X = \sum_{i<j} \varepsilon_i \varepsilon_j$ and from this, one finds the L-content:

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \; : \; L = 2\ell - 1, 2\ell - 3, \ldots, 1. \qquad (15)$$

More generally, the character X for the representation belonging to

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline r \\ \hline\end{array} \text{ is } X(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{i_1 < i_2 < \ldots < i_r} \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_r} \qquad (16)$$

and the highest L value for SO(3) becomes

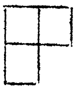
$$L_{max} = \ell + (\ell - 1) + \ldots + (\ell - r + 1) = r\ell - \frac{r(r-1)}{2} \qquad (17)$$

with multiplicity 1. General formulas are complicated, but in simple examples the L-content can easily be computed. The dimension of the representation (16) is dim  $= X(1) = \binom{2\ell+1}{r}$ (18)

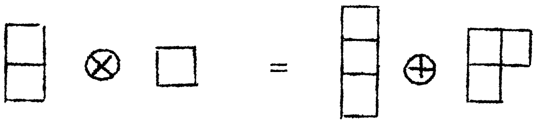
This is sometimes useful.

<u>Example:</u> Three d electrons

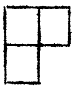
The Pauli principle allows only tables with at most two columns

R_1 :  , R_2 : 

The irreducible representations of SU(2) corresponding to $\tilde{R}_1$ and $\tilde{R}_2$ are characterized by $S = {}^3/2$ and $S = {}^1/2$ resp.. We have to study the branching $SU(5) \supset SO(3)$. The highest value of L for R_1 is $L_{max} = 3$. Since the dimension is 10 and L = 0 obviously does not occur ($a_o = b_o - b_1 = o$) the only other value is L = 1. In the Appendix we show that

 (19)

dim  = 10 and with (15) L = 3,1. dim  = 5 and L = 2. Reduction on the left hand side of (19) gives (if we write L instead of $D^{(L)}$) $5 \oplus 4 \oplus (3)^2 \oplus (2)^2 \oplus (1)^2$ and we obtain for

 : $L = 5, 4, 3, (2)^2$.

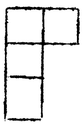
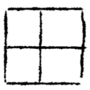
Hence the (L,S) - terms of three d electrons are

^2P, $(^2D)^2$, ^2F, ^2G, ^2H, ^4P, ^4F .

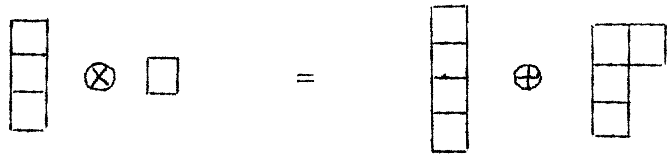
To show how fast the method works, we consider also the following

<u>Example:</u> Four d electrons

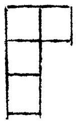
The allowed Young tables are

R_1 :  , R_2 :  , R_3 : 

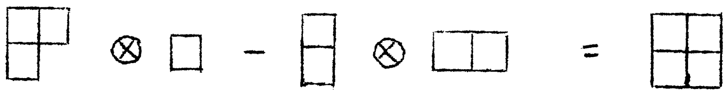
so that the spin is found to be S = 2, S = 1, S = 0 resp.. We first consider R_1. Since for SU(5)  is the contragredient representation of  the L content is L = 2. From



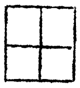
and the already known L-content of  we get for

 : $L = 5, 4, (3)^2, 2, (1)^2$.

Using



we obtain for

 : $L = 6, (4)^2, 3, (2)^2, (0)^2$

and hence the (L,S) - terms of four d electrons are

$({}^1S)^2$, $({}^1D)^2$, ${}^1F$, $({}^1G)^2$, ${}^1I$, $({}^3P)^2$, ${}^3D$, $({}^3F)^2$, ${}^3G$, ${}^3H$, ${}^5D$.

Exercise: Determine the terms for five d electrons.

2. Nuclear Spectra in (L,S) - coupling.

Supermultiplets

If the nuclear forces do not depend strongly on the spins, we have essentially the same problem as before - the only difference is that the spin-space has to be replaced by the spin $\otimes$ isospin-space $\mathbb{C}^4$. The Hilbert space of f nucleons is the antisymmetric part of

$$\mathcal{L}^2(\mathbb{R}^{3f}) \otimes (\mathbb{C}^4)^{\otimes f} \qquad (20)$$

The energy of a state will depend critically on the orbital wave function. Since the nuclear forces are primarily attractive, the energy will be lowered if the symmetry of the orbital wave function is increased. Thus, we may expect that the state whose orbital function has the highest symmetry will have the lowest energy. Since the energy depends only on the orbital function, while the multiplicity depends on the charge-spin function, each energy level will be a supermultiplet. Let us first look at the isospin-spin content of a supermultiplet. We begin with $\square$ (self-representation of SU(4)). If the elements $\{\alpha_i \,;\, i = 1, 2\}$ form a basis of $\mathbb{C}^2$, so do the $\{\alpha_i \otimes \alpha_j \,;\, i,j = 1,2\}$ of $\mathbb{C}^4$. $SU(2) \times SU(2)$ is canonically embedded in SU(4) according to

$$SU(2) \times SU(2) \ni (U_1, U_2) : \alpha_i \otimes \alpha_j \longrightarrow U_1\alpha_i \otimes U_2\alpha_j$$

and we obtain

$$\square \;:\; (2 I + 1 \,,\, 2 S + 1) = (2,2)$$

From $\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$ and $\dim \begin{array}{c}\square\\[-4pt]\square\end{array} = 6$ (see(18))

it follows $\dim \square\square = 10$. Reduction of the left hand side gives

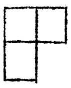
$$(2,2) \otimes (2,2) = (3,3) \oplus (1,1) \oplus (3,1) \oplus (1,3)$$

and by looking at the dimensions we get the result

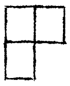
 : $(3,3) \oplus (1,1)$

 : $(3,1) \oplus (1,3)$

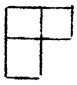
By IV.(11)ff the representation  has the same $SU(2) \times SU(2)$ content as  , that is $(2,2)$. From

 $\otimes$  $=$  $\oplus$ 

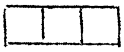
we obtain

 : $(2,2) \oplus (2,4) \oplus (4,2)$.

Using this result and

 $\otimes$  $=$  $\oplus$ 

we find

 : $(2,2) \oplus (4,4)$.

This ladder process can be continued. The computation of the L values in a shell model proceeds in the same way as for the atomic shell model. Here we get in general more terms since more than two columns are allowed in the Young tables.

3. Quark Model

In the quark model the mesons are considered as bound states of a quark and an antiquark, whereas the baryons are considered to be bound states of three quarks. Furthermore, one assumes that the q-q forces are independent of the spin and unitary spin. Formally the difference between the quark model and the super-multiplet model for nuclei which we considered before is that we have to replace $\mathbb{C}^4$ by $\mathbb{C}^6$.

Let us consider the possible three quark states in the spin-unitary spin space $\mathbb{C}^6$. We have to reduce $\square \otimes \square \otimes \square$ into irreducible SU(6) representations and determine their SU(3) $\times$ SU(2) content. The method is the same as before.

For the self-representation $\square$ of SU(6) we have at once

$$\square \; : \; (\dim SU(3), \; \dim SU(2)) = (3,2)$$

From

$$\square \otimes \square = \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\!\square$$

we conclude with (18) that dim $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ = 15 and dim $\square\!\square$ = 21.

Then

$$(\square, S = 1/2) \otimes (\square, S = 1/2) = (\square \otimes \square \; , 1/2 \otimes 1/2) =$$

$$= (\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\!\square \; , 1 \oplus (0)) = (\square^* \oplus \square\!\square \; , 1 \oplus (0))$$ where $\square$ etc always denotes a SU(3) representation. Thus we obtain

$$(3,2) \otimes (3,2) = (3^* \oplus 6 \; , 3 \oplus 1)$$

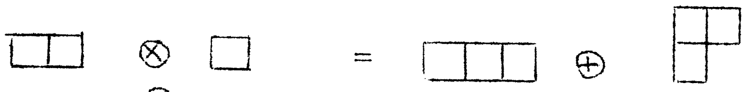
$$= (3^*,3) \oplus (3^*,1) \oplus (6,3) \oplus (6,1)$$

and

$$\square\!\square \; : \; (6,3) \oplus (3^*,1)$$

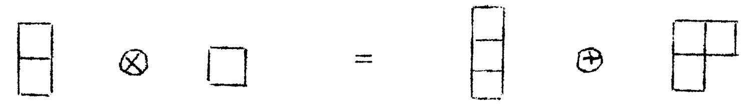
$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \; : \; (6,1) \oplus (3^*,3) .$$

Next we consider



(see IV.(9))

$$21 \otimes 6 \qquad 56 \oplus 70$$



$$15 \otimes 6 = 20 \oplus 70$$

On the other hand

$$[(6,3) \oplus (3^*,1)] \otimes (3,2)$$

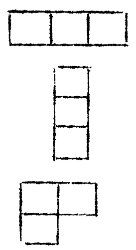
$$= (10,4) \oplus (10,2) \oplus (8,4) \oplus 2(8,2) \oplus (1,2)$$

and

$$[(6,1) \oplus (3^*,3)] \otimes (3,2)$$

$$= (10,2) \oplus (8,4) \oplus 2(8,2) \oplus (1,4) \oplus (1,2)$$

We find, by looking at the dimensions, that the only possibility is



$$: \quad 56 = (10,4) \oplus (8,2)$$

$$: \quad 20 = (8,2) \oplus (1,4) \qquad (21)$$

$$: \quad 70 = (10,2) \oplus (8,4) \oplus (8,2) \oplus (1,2)$$

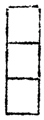
We also note the result

$$6 \otimes 6 \otimes 6 = 56 \oplus 2(70) \oplus 20 \; .$$

For a more systematic solution of this reduction which makes use of the characters of SU(n) see Macfarlane J.M.P., 1965.

Particle Assignment:

By the same reason as in the last paragraph one would assume that the lowest bound state of three quarks has a totally symmetric space wave function, i.e. all quarks are in relative s-states. If the quarks satisfy the Fermi-Dirac statistics, the spin-unitary spin wave function should then be totally antisymmetric, i.e.



$$: \quad 20 = (8,2) \oplus (1,4)$$

Since L = 0 in this case, we have a $1/2^+$ octet and a $3/2^+$ singlet. This solution was proposed by Sakita, but had to be abandoned soon.

An attractive idea is to consider the representation 56 with L = 0 for the lowest baryon states. In this case we get a $1/2^+$ octet and a $3/2^+$ decuplet [(21)]:

$1/2^+$ octet: $n, p;\ \Sigma^+, \Sigma^0, \Sigma^-;\ \Lambda^0;\ \Xi^-, \Xi^0$

$3/2^+$ decuplet: $\underset{1}{\Omega^-};\ \underset{2}{\Xi^*};\ \underset{3}{Y_1^*};\ \underset{4}{\Delta}$

The space function in this case must be antisymmetric and we have to assume that the particles in p-states will have a larger binding energy than in s-states. How is this possible ?

APPENDIX TO CHAPTER V

In this appendix we derive a few results already used in Chapter V.

1. Associated Representation

Let G be a finite group of order h and let M and M' be two irreducible G-modules with character χ and χ' respectively. Consider the decomposition

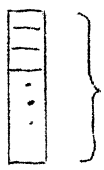
$$M \otimes M' = \bigoplus m_i N_i$$

into irreducible modules N_i with characters χ_i and multiplicities m_i. The multiplicities m_i are given by

$$m_i = \frac{1}{h} \sum_{g \in G} \overline{\chi}_i (g) \chi(g) \chi'(g) \qquad (22)$$

The antisymmetric representation of S_f is defined by $S_f \ni \pi \longrightarrow \varepsilon_\pi$. It is one dimensional, with character $\chi^{(-)} = \varepsilon_\pi$, generating idempotent $e^{(-)} = \frac{1}{\sqrt{f!}} \sum_{\pi \in S_f} \varepsilon_\pi \pi$ and Young table

$R^{(-)}$:



f .

Now we specialize G = S_f. The multiplicity $m^{(-)}$ of the antisymmetric representation is then

$$m^{(-)} = \frac{1}{f!} \sum_{\pi \in S_f} \varepsilon_\pi \chi(\pi) \chi'(\pi). \qquad (23)$$

<u>Definition:</u>

The <u>associated</u> representation of a given representation Δ of $\mathfrak{S}_f$ is defined to be ε_π times the complex conjugate matrix Δ:

$$\mathfrak{S}_f \ni \pi \longrightarrow \varepsilon_\pi \overline{\Delta}(\pi).$$

Formula (22) shows that

$$m^{(-)} = \begin{cases} 1 & \text{if } \chi' \text{ is the associated character of } \chi \\ 0 & \text{otherwise} \end{cases}$$

We would like to determine the Young table of the associated representation. Let the original representation be isomorphic to the minimal left ideal $L = Ae$, $A = \mathbb{C}\,\mathfrak{S}_f$ and $e = \sum e(s)\, s$ a primitive idempotent corresponding to the Young table D. First we compute the character corresponding to e. Consider the projection

$$x \longrightarrow g\,x\,e \quad ; \quad g \in \mathfrak{S}_f \;,\; x \in A \qquad (24)$$

of A on L. Within L this is just the left-multiplication by g. On choosing a coordinate system in A in such a way that the first m = dim L vectors span the subspace L, the last (f!-m) rows of the matrix corresponding to (24) consist only of zeros; hence the trace of the projection (24) of the total space A is equal to the character $\chi(g)$ belonging to e. We rewrite (24) as

$$x = \sum_s x(s)\, s \longrightarrow \sum_{s,t} x(s)\; e(t)\; g\; s\; t$$

$$= \sum_{s,r} x(s)\; e\,(s^{-1}g^{-1}r)\; r.$$

In components the transformation is thus

$$x(s) \longrightarrow y(r) = \sum_t e(t^{-1}g^{-1}r)\; x(t)$$

and the character $\chi(g)$ is therefore

$$\chi(g) = \sum_{s} e\,(s^{-1}g^{-1}s) \qquad . \tag{25}$$

The element e has the form

$$e = \sum_{\substack{p \in R(D) \\ q \in C(D)}} \varepsilon_q\, p\, q$$

and from this we see that $\chi(g)$ is real.

Now we consider the idempotent which belongs to the diagram $\tilde{D}$ arising from D by interchanging rows and columns

$$\tilde{e} = \sum \varepsilon_p\, q\, p = \sum \varepsilon_p\, q^{-1}\, p^{-1} = \sum \varepsilon_p\, (pq)^{-1} \quad .$$

Hence if we put $\tilde{e} = \sum \tilde{e}(s)\, s$ we obtain

$$\tilde{e}(s) = \varepsilon_s\, e(s^{-1})\ , \quad \varepsilon_s = \varepsilon_p\, \varepsilon_q \quad .$$

The character $\tilde{\chi}$ belonging to $\tilde{D}$ is thus

$$\tilde{\chi}(g) = \sum_{t} \tilde{e}\,(t^{-1}g^{-1}t) = \sum_{t} \varepsilon_{(t^{-1}g^{-1}t)}\ e\,(t^{-1}g\,t)$$

$$= \sum_{t} \varepsilon_g\, e\,(t^{-1}g\,t) = \varepsilon_g\, \chi\,(g^{-1}) \quad .$$

The elements g and g^{-1} are conjugate in $\mathfrak{S}_f$ and since χ is a class-function we finally obtain

$$\tilde{\chi}(g) = \varepsilon_g\, \chi(g) = \varepsilon_g\, \overline{\chi}(g) \ .$$

This proves that $\tilde{\chi}(g)$ is the character of the associated representation and hence this representation belongs to the Young table $\tilde{D}$.

2. Reduction of the Tensor Product of SU(n) - modules

We give now without proof a recipe for the reduction of a tensor product of two irreducible SU(n) - representations into a direct sum of irreducible representations. The student interested in the proof should read chapter VI and then see e.g. Robinson "representation theory of the symmetric group".

We begin with

Definition 6:

A lattice permutation of $a^n b^m c^p \ldots$ is a sequence of the n a's, m b's, p c's, ... like aaabbac ... such that to the left of any point in the sequence there are not less a's than b's, and not less b's than c's, and so on.

Now for the recipe: To find the irreducible representations in the tensor product, draw the Young table for one of the factors that enter in the product. In the table of the other factor, assign the same symbol, say a, to all boxes in the first row, the same b to all boxes in the second row, etc. Now attach boxes labeled by the symbol a to the first table in all possible ways subject to the rule that no two a's appear in the same column and that the resultant graph is still a Young table; repeat this process with the b's, etc. One further restriction: If, after all symbols have been added to the table, we read the added symbols from the right to the left in the first row, then in the second row, etc., they must form a lattice permutation of the a's, b's, ...

Example for SU(3):

We rewrite this as

```
.  .        a  a
       ⊗
.           b
```

and enlarge the first pattern by adjoining the a's

```
.  .  a  a       .  .  a       .  .  a       .  .
.                .  a          .             .  a
                               a             a

    (i)             (j)           (k)          (l)
```

From (i) we get

```
.  .  a  a       .  .  a  a
.  b             .
                 b
```

(note that

```
.  .  a  a  b
.
```

is not a lattice permutation).

From (j) we obtain

```
.  .  a          .  .  a
.  a  b          .  a
                 b
```

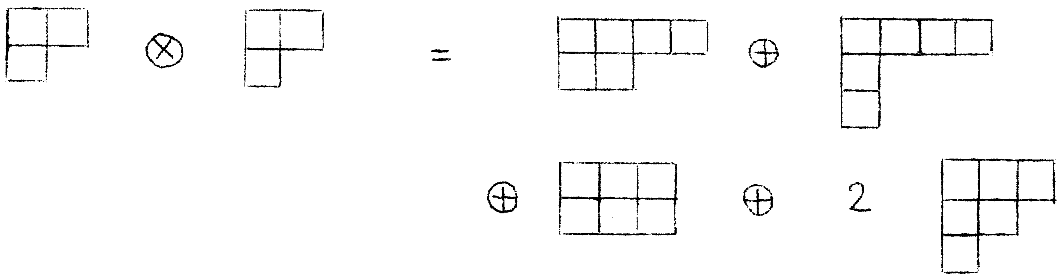
From (k)

```
.  .  a          .  .  a
.  b             .
a                a              = 0   for SU(3)
                 b
```

From (1)

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & a \\ a & b \end{array} \qquad \begin{array}{cc} \cdot & \cdot \\ \cdot & a \\ a & \\ b & \end{array} \qquad = 0 \quad \text{for } SU(3)$$

Hence we get the SU(3) - result:



$\oplus$ unit representation.

CHAPTER VI INDUCED REPRESENTATIONS AND DECOMPOSITION OF THE TENSOR PRODUCT OF U(n) - MODULES.

1. Tensor Product of Modules

The concept of tensor product we used in III and V is not general enough for the following. Let us begin with a few definitions.

Definition 1:

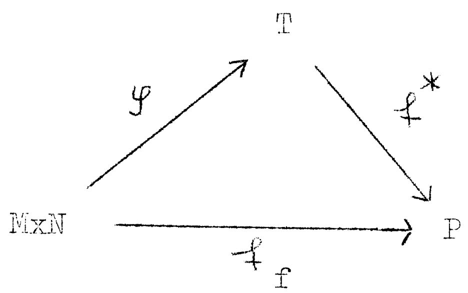
Let M be a right module and N a left module over an arbitrary ring R with an identity element. Let P be an abelian group, written additively. A balanced map f of the Cartesian product set M x N into P assigns to each pair $(m,n) \in M \times N$ an element $f(m,n) \in P$, so that

$$f(m_1+m_2,n) = f(m_1, n) + f(m_2,n)$$
$$f(m,n_1+n_2) = f(m,n_1) + f(m,n_2)$$
$$f(m,r\,n) = f(m\,r,n)$$

for all $r \in R$, $m_i \in M$, $n_i \in N$.

Definition 2:

Let $f: M \times N \longrightarrow P$ and $\varphi: M \times N \longrightarrow T$ be balanced maps of M x N into the additive abelian groups P and T resp.. We say that f can be factored through T if there exists a homomorphism $f^*: T \longrightarrow P$ such that $f(m,n) = f^*[\varphi(m,n)]$ for all $(m,n) \in M \times N$. In other words, f can be factored through T if there exists a homomorphism $f^* : T \longrightarrow P$ such that the diagram



is commutative.

We now state without proof

Theorem 1

Let M and N be right and left R-modules, resp.. There exists an abelian group T and a balanced map $t: M \times N \longrightarrow T$ such that

(1) The elements $t(m,n)$ generate T, and in fact every element of T is a sum $\sum t(m_i, n_i)$, $m_i \in M$, $n_i \in N$.

(2) Every balanced map of $M \times N$ into an arbitrary abelian group P can be factored through T.

For a proof of this theorem see the already mentioned book of Curtis, Reiner.

Definition 3:

The group T constructed in Theorem 1 is called the tensor product of M and N, and will be denoted by $M \otimes_R N$.

The next result shows that the tensor product is uniquely determined up to isomorphism by the properties (1) and (2) of Theorem 1.

Corollary 1

Let $(M \otimes_R' N, t')$ be another pair consisting of an abelian group $M \otimes_R' N$ and a balanced map $t': M \times N \longrightarrow M \otimes_R' N$ such that (1) and (2) of Theorem 1 hold. Then there exists a group isomorphism λ of $M \otimes_R N$ onto $M \otimes_R' N$ such that for all $(m,n) \in M \times N$ we have $\lambda[t(m,n)] = t'(m,n)$.

Proof:

Applying Theorem 1, there exist homomorphisms $\lambda : M \otimes_R N \longrightarrow M \otimes_R' N$ and $\mu : M \otimes_R' N \longrightarrow M \otimes_R N$ such that $\lambda[t(m,n)] = t'(m,n)$ and $\mu[t'(m,n)] = t(m,n)$. Because the elements $\{ t(m,n) \}$ and $\{ t'(m,n) \}$ generate the groups $M \otimes_R N$ and $M \otimes'_R N$ resp., it follows that $\mu\lambda$ and $\lambda\mu$ are the identity mappings on $M \otimes_R N$ and $M \otimes'_R N$ resp.. Therefore both λ and μ are isomorphisms onto, and the corollary is proved.

We have defined the tensor product $M \otimes_R N$ and have constructed a balanced map $M \times N \longrightarrow M \otimes_R N$. Henceforth we shall write $m \otimes n$ for the image of (m,n) under this map. We see that

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \qquad (1)$$

$$m\, r \otimes n = m \otimes r\, n$$

for all $m_i \in M$, $n_i \in N$, $r \in R$.

As we are ultimately interested in representations, we now investigate the circumstances under which $M \otimes_R N$ is a module over some ring. We say that an abelian group M is an <u>(S,R) - bimodule</u> over the rings R and S if M is a left S-module and a right R-module, and if $(s\, m)\, r = s\, (m\, r)$ for all $s \in S$, $r \in R$, $m \in M$. E.g., any left module M over a commutative ring R is an (R,R) - bimodule if we define $m\, r = r\, m$, $r \in R$, $m \in M$.

<u>Proposition:</u>

If M is an (S,R) - bimodule and N a left R-module, then $M \otimes_R N$ is a left S-module.

<u>Proof:</u>

Let s be a fixed element of S. Then the mapping $(m,n) \longrightarrow sm \otimes n$ of $M \times N$ into $M \otimes_R N$ is a balanced map, and, by Theorem 1, there exists an endomorphism ψ_s of $M \otimes_R N$ such that $\psi_s (m \otimes n) = sm \otimes n$. We can now define, for each $s \in S$,

$$s(\sum m_i \otimes n_i) = \psi_s (\sum m_i \otimes n_i) = \sum sm_i \otimes n_i ,$$

and conclude that $M \otimes_R N$ is a left S-module with respect to this operation.

We shall need the following theorems.

<u>Theorem 2</u>

Let M be a right R-module such that $M = M_1 \oplus M_2$ where M_1 and M_2 are submodules, and let N be a left R-module. Then

$$M \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$$

i.e. $M \otimes_R N$ is the external direct sum of the $M_i \otimes_R N$, $i = 1,2$.

<u>Proof:</u>

The direct sum decomposition $M = M_1 \oplus M_2$ implies the existence of $\pi_1, \pi_2 \in \mathrm{Hom}_R(M,M)$ such that

$$\mathbb{1} = \pi_1 + \pi_2 , \quad \pi_i \pi_j = \pi_j \pi_i = \delta_{ij} \pi_i$$
$$\pi_1 M = M_1 , \quad \pi_2 M = M_2 \qquad (2)$$

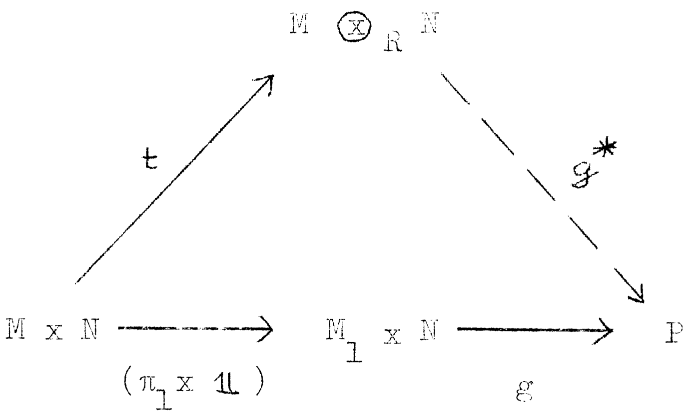
Set $\theta_i = \pi_i \otimes \mathbb{1}$, $i = 1,2$; then each θ_i is an endomorphism of $M \otimes_R N$, and, from (2), we have

$$\mathbb{1} = \theta_1 + \theta_2 , \ \theta_i \theta_j = \theta_j \theta_i = \delta_{ij} \theta_i$$

Setting $T_i = \theta_i (M \otimes_R N)$, we conclude that $M \otimes_R N = T_1 \oplus T_2$. In order to complete the proof, it is sufficient to show, for example, that $T_1 \cong M_1 \otimes_R N$. We shall prove this by showing that T_1 has the characteristic properties of a tensor product as given in Theorem 1. We must find a balanced map $\varphi: M_1 \times N \longrightarrow T_1$ for which the images $\{ \varphi(m_1,n): m_1 \in M_1, n \in N \}$ generate T_1 and such that every balanced map $g: M_1 \times N \longrightarrow P$ can be factored through T_1 by means of φ. Let us write

$$t : M \times N \longrightarrow M \otimes_R N$$

for the mapping determined in Theorem 1. Since $M_1 \times N \subset M \times N$, we may set $\varphi = t\big|_{M_1 \times N}$, so that $\varphi(m_1, n) = t(m_1, n)$ for all $m_1 \in M_1$, $n \in N$. It is clear from $M_1 = \pi_1 M$ and the definition of T_1 that the image of $M_1 \times N$ under φ does indeed generate T_1. Now let $g: M_1 \times N \longrightarrow T$ be a balanced map, where P is any additive abelian group. The bottom line of the following diagram gives a balanced map $M \times N \longrightarrow P$, and so there exists a homomorphism $g^* : M \otimes_R N \longrightarrow P$ making the diagram commutative.



Let us set $g_1 \equiv g^*\big|_{T_1}$, so that g_1 is a homomorphism of T_1 into P. To complete the proof, we need only verify that $g_1 \varphi = g$ on $M_1 \times N$. But for $m_1 \in M_1$, $n \in N$, we have

$$g_1 \varphi (m_1, n) = g_1 t (m_1, n) = g^* t (m_1, n)$$
$$= g(\pi_1 \times 1\!\!1) (m_1, n) = g(\pi_1 m_1, n) = g(m_1, n).$$

This establishes the result.

We remark that if $M = M_1 \oplus M_2$, all these being (S,R)-bimodules, and if N is a left R-module, then the isomorphism

$$M \otimes_R N \cong M_1 \otimes_R N \oplus M_2 \otimes_R N$$

obtained in Theorem 2 is an isomorphism of left S-modules.

Now let N be a left R-module; since R is an (R,R)-bimodule, the tensor product $R \otimes_R N$ is a left R-module. The following result is basic:

Theorem 3

$R \otimes_R N \cong N$ as left R-modules.

Proof:

The map $(r,n) \longrightarrow rn$ is a balanced map of $R \times N$ into N, and so by Theorem 1 there exists a homomorphism $\varphi : R \otimes_R N \longrightarrow N$ such that $\varphi(r \otimes n) = rn$. On the other hand, we may define a homomorphism $\psi : N \longrightarrow R \otimes_R N$ by $\psi(n) = 1 \otimes n$, $n \in N$. Clearly $\varphi\psi$ = identity map on N; furthermore, $\psi\varphi(r \otimes n) = \psi(rn) = 1 \otimes rn = r \otimes n$, so $\psi\varphi$ acts as the identity map on $R \otimes_R N$. This implies that φ is an isomorphism of $R \otimes_R N$ onto N, and it is easily seen to be an R-isomorphism of left R-modules. Thus the theorem is proved.

We add a few remarks on tensor products of vector spaces.

Let M and N be finite-dimensional left vector spaces over a field $\mathbb{K}$. As we have seen earlier, M is a $(\mathbb{K}, \mathbb{K})$ - bimodule, and $M \otimes_{\mathbb{K}} N$ becomes a $\mathbb{K}$ - space if we define

$$\xi (\sum u_i \times v_i) = \sum \xi u_i \times v_i$$

where $\xi \in \mathbb{K}$, $u_i \in M$, $v_i \in N$. Now M is isomorphic to an external direct sum of r copies of $\mathbb{K}$ where r = dim M, and so by repeated use of Theorems 2 and 3, we have

$$M \otimes_{\mathbb{K}} N \cong N \oplus \ldots \oplus N \quad \text{(r copies)}$$

as left $\mathbb{K}$ - modules. This proves that

$$\dim M \otimes_{\mathbb{K}} N = \dim M \cdot \dim N.$$

Suppose now that

$$M = \mathbb{K} m_1 \oplus \ldots \oplus \mathbb{K} m_r \qquad r = \dim M$$
$$N = \mathbb{K} n_1 \oplus \ldots \oplus \mathbb{K} n_s \qquad s = \dim N.$$

Then using the distributivity (1) we see that every element of $M \otimes_{\mathbb{K}} N$ is expressible as

$$\sum_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} \alpha_{ij} (m_i \otimes n_j) , \quad \alpha_{ij} \in \mathbb{K} .$$

Since dim $M \otimes_{\mathbb{K}} N$ is rs, this proves that the elements $\{ m_i \otimes n_j : 1 \leqslant i \leqslant r , 1 \leqslant j \leqslant s \}$ form a basis of $M \otimes_{\mathbb{K}} N$.

2. Induced Representations:

Let H be a subgroup of a finite group G, and let $\mathbb{K}$ be an arbitrary field. All modules will be assumed to be finite-dimensional $\mathbb{K}$-spaces. Since $\mathbb{K}H$ is a subalgebra of $\mathbb{K}G$, every $\mathbb{K}G$-module L is also a $\mathbb{K}H$-module which we shall denote by L_H. Thus L_H has the same underlying vector space as L, but the domain of left operators is $\mathbb{K}H$ instead of $\mathbb{K}G$. Of course, a matrix representation T_H of H afforded by L_H is obtained from a matrix representation T of G afforded by L by setting $T_H = T\big|_H$.

Our objective here is to describe a construction which associates with each $\mathbb{K}H$-module M an "induced" $\mathbb{K}G$-module M^G.

Definition 4:

Let H be a subgroup of G, and let M be a left $\mathbb{K}H$-module. Then $\mathbb{K}G$ is a ($\mathbb{K}G$, $\mathbb{K}H$) - bimodule, and we may form the left $\mathbb{K}G$ -module

$$M^G \equiv \mathbb{K}G \otimes_{\mathbb{K}H} M$$

which is said to be induced from M. The representation of G afforded by M^G is called an induced representation.

Let us start with a left coset decomposition

$$G = g_1H \cup g_2H \cup \dots \cup g_tH, \qquad t = \dim G \text{ over } H$$

where $g_1 = 1$. Every element of G is expressible as a product g_ih, $1 \leqslant i \leqslant t$, $h \in H$, with uniquely determined g_i and h, and so every element of $\mathbb{K}G$ is uniquely expressible as $\sum g_ib_i$, $b_i \in \mathbb{K}H$. Thus we have

$$\mathbb{K}G = g_1\mathbb{K}H \oplus \dots \oplus g_t\mathbb{K}H$$

so that $\mathbb{K}G$ is a free right $\mathbb{K}H$-module with basis $\{g_1,\dots,g_t\}$.

Using Theorem 2, we obtain

$$M^G = g_1\mathbb{K}H \otimes_{\mathbb{K}H} M \oplus \dots \oplus g_t\mathbb{K}H \otimes_{\mathbb{K}H} M$$

which we may rewrite as

$$M^G = g_1 \otimes M \oplus \dots \oplus g_t \otimes M \qquad (3)$$

by virtue of the formula

$$g_i b \otimes m = g_i \otimes bm , \quad b \in \mathbb{K} H, \quad m \in M.$$

In (3), we have a decomposition of M^G into $\mathbb{K}$-subspaces which are, in general, neither left $\mathbb{K}G$-modules nor $\mathbb{K}H$-submodules of M^G. Note that, since $g_i \mathbb{K}H \cong \mathbb{K}H$ as right $\mathbb{K}H$-modules, with the isomorphism being given by $g_i b \longrightarrow b$, $b \in \mathbb{K}H$, it follows that

$$g_i \mathbb{K}H \otimes_{\mathbb{K}H} M \cong \mathbb{K}H \otimes_{\mathbb{K}H} M \cong M$$

by Theorem 3. Thus $g_i b \otimes m \longrightarrow bm$ gives an isomorphism

$$g_i \mathbb{K}H \otimes_{\mathbb{K}H} M \cong M$$

which is easily seen to be a $\mathbb{K}$-isomorphism. Therefore

$$\dim M^G = (\dim G \text{ over } H) \cdot \dim M$$

and we conclude further that every element of M^G is expressible as $\sum g_i \otimes u_i$ with uniquely determined $u_1, \dots, u_t$ in M. It follows that if $\{ m_1, \dots, m_r \}$ is a $\mathbb{K}$-basis for M, then the elements

$$\{ g_i \otimes m_j : 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant r \} \qquad (4)$$

form a $\mathbb{K}$-basis for M^G.

We are now going to determine a matrix representation afforded by M^G once we know a matrix representation afforded by M.

Suppose that, relative to its $\mathbb{K}$-basis $\{ m_1, \dots, m_r \}$, M affords the matrix representation T, so that

$$h\, m_i = \sum \alpha_{ji}(h)\, m_j , \quad T(h) = (\alpha_{ij}(h))$$

for $h \in H$. Relative to the $\mathbb{K}$-basis (4) of M^G, let us compute the matrix representation U afforded by M^G. To do this, we must express

$g(g_i \otimes m_j)$ as $\mathbb{K}$-linear combinations of the basis elements. We may write $gg_i = g_k h$ for some $h \in H$ and for some k, $1 \leqslant k \leqslant t$; then

$$g(g_i \otimes m_j) = gg_i \otimes m_j = g_k \otimes hm_j$$
$$= \sum_{s=1}^{r} \alpha_{sj}(h)\, g_k \otimes m_s \,.$$

Now we have $h = g_k^{-1} g\, g_i$. If we extend the domain of definition of α_{sj} from H to G by setting $\alpha_{sj}(x) = 0$, $x \in G$, $x \notin H$, then we may rewrite our formula as

$$g(g_i \otimes m_j) = \sum_{s=1}^{r} \sum_{k=1}^{t} \alpha_{sj}(g_k^{-1} g g_i) \cdot g_k \otimes m_s \,.$$

If we arrange the basis elements (4) in the order

$g_1 \otimes m_1, \ldots, g_1 \otimes m_r, g_2 \otimes m_1, \ldots, g_2 \otimes m_r, \ldots,$
$g_t \otimes m_1, \ldots, g_t \otimes m_r$

then the preceding equation implies that for each $g \in G$

$$U(g) = \left[\begin{array}{c|c|c} & (i,1)\ldots(i,r) & \\ \hline & T(g_j^{-1} g g_i) & \\ \hline & & \end{array}\right] \begin{array}{c} (j,1) \\ \vdots \\ (j,r) \end{array}$$

where T is extended to all G by setting $T(x) = 0$ for $x \in G$, $x \notin H$. Thus $U(g)$ is partitioned into a $t \times t$ array of $r \times r$ blocks, and the block in the j th block row and i th block column is $T(g_j^{-1} g\, g_i)$. Specifically we have

$$U(g) = \begin{bmatrix} T(g_1^{-1} g g_1) & \ldots & T(g_1^{-1} g g_t) \\ \vdots & & \vdots \\ T(g_t^{-1} g g_1) & \ldots & T(g_t^{-1} g g_t) \end{bmatrix} \qquad (5)$$

<u>Definition 5:</u>

If μ is the character afforded by M and μ^G that of M^G, we call μ^G an <u>induced character</u> and say that μ^G is <u>induced</u> from μ.

If the $\mathbb{K}$ H-module M affords a matrix representation T, we extend as before the domain of definition of T to G by letting T vanish outside H:

$$\dot{T}(y) \equiv \begin{cases} T(y) , & y \in H \\ 0 , & y \notin H \end{cases}$$

Then relative to a suitable basis, M^G affords a matrix representation T^G given by

$$T^G(x) = (\dot{T}(g_j^{-1} x g_i)) \quad 1 \leq i, j \leq n, \quad x \in G .$$

The induced character μ^G obtained from this satisfies:

$$\mu^G(x) = \sum_{i=1}^{n} \dot{\mu}(g_i^{-1} x g_i) , \quad x \in G \tag{6}$$

where $\dot{\mu}$ coincides with μ on H and vanishes outside H. Since $\dot{\mu}(h^{-1} y h) = \dot{\mu}(y)$, $h \in H$, $y \in G$ we deduce that

$$\mu^G(x) = \frac{1}{h_H} \sum_{t \in G} \dot{\mu}(t^{-1} x t) \tag{7}$$

h_H = order of H, $x \in G$.

A map ω: G $\longrightarrow$ $\mathbb{K}$ is called a class function if $\omega(t^{-1}g\, t) = \omega(g)$; $g,t \in G$. Since class functions can be added and multiplied, it is clear that the set cf(G) of all class functions on G forms an algebra over $\mathbb{K}$ of dimension equal to s, the number of conjugate classes in G. Formula ( ) then provides a method for obtaining from each $\mu \in$ cf(H) an induced class function $\mu^G \in$ cf(G).

3. Frobenius Reciprocity Theorem

We remind the reader that, since the character of a module determines the module up to isomorphism, it is clear that each result on characters will imply the corresponding result for modules, and vice versa.

Let ω, η be two class functions on G. Define an inner product (ω, η) by

$$(\omega, \eta) = \frac{1}{h_G} \sum_{x \in G} \omega(x) \overline{\eta(x)}$$

$$h_G = \text{order of } G$$

Then the orthogonality relations for characters imply $(\chi^{(i)}, \chi^{(j)}) = \delta_{ij}$ if $\chi^{(i)}$ and $\chi^{(j)}$ are members of the full set of irreducible characters of G. In particular, we have the

Proposition.

Let $\chi^{(i)}$ be the character of the irreducible $\mathbb{K}$G-module Z_i, and let μ be the character of an arbitrary $\mathbb{K}$G-module M. Then the inner product $(\mu, \chi^{(i)})$ is equal to the number of composition factors of M which are isomorphic to Z_i. We shall call $(\mu, \chi^{(i)})$ the multiplicity with which M contains Z_i.

Let μ be the character of a $\mathbb{K}$G-module. Then $\mu | H$ denotes the restriction of μ to the subgroup $H \subset G$. We now prove

Theorem 4 (Frobenius Reciprocity Theorem):

Let H be a subgroup of G, and let $\chi \in cf(G)$, $\psi \in cf(H)$. Then

$$(\psi, \chi|H) = (\psi^G, \chi) . \qquad (8)$$

In particular, if χ and ψ are characters of irreducible $\mathbb{K}G$ - and $\mathbb{K}H$ - modules, resp., the multiplicity of ψ in $\chi|H$ is the same as the multiplicity of χ in ψ^G.

Proof:

Define $\dot{\psi}$ to coincide with ψ on H and vanish outside H. Then

$$(\psi^G, \chi) = \frac{1}{h_G} \sum_{g \in G} \psi^G(g) \overline{\chi(g)}$$

$$= \frac{1}{h_G} \cdot \frac{1}{h_H} \sum_{x,g \in G} \dot{\psi}(x^{-1}gx) \overline{\chi(x^{-1}gx)}$$

For fixed $x \in G$, as g ranges over all elements of G, so does $x^{-1}gx$. Thus

$$(\psi^G, \chi) = \frac{1}{h_G} \cdot \frac{1}{h_H} \sum_{x \in G} \sum_{y \in G} \dot{\psi}(y) \overline{\chi(y)}$$

$$= \frac{1}{h_H} \sum_{y \in G} \dot{\psi}(y) \overline{\chi(y)}$$

$$= \frac{1}{h_H} \sum_{y \in G} \psi(y) \overline{(\chi|H)(y)}$$

since $\dot{\psi}$ vanishes outside H. This proves the theorem.

4. Analysis of the Tensor Product of U(n) - Modules

Let Λ_i be two irreducible representations of U(n) corresponding to Young tables D_i with f_i boxes and let Δ_i be the corresponding representations of S_{f_i}. Furthermore, let

$$\Lambda_1 \otimes \Lambda_2 = \bigoplus m_\Lambda \Lambda \qquad (9)$$

be the decomposition of the tensorproduct representation $\Lambda_1 \otimes \Lambda_2$ into irreducible representations Λ with multiplicities m_Λ.

The group $S_{f_1} \times S_{f_2}$ can naturally be embedded into $S_f, f = f_1+f_2$. The representations Λ in (9) correspond to Young tables D_Λ with f boxes. We also consider the representation Δ_Λ of S_f belonging to D_Λ.

In this section we shall prove the following

Theorem 5:

With the notations introduced above, the multiplicity m_Λ is equal to the multiplicity of the representation (Δ_1, Δ_2) of $S_{f_1} \times S_{f_2}$ in $\Delta_\Lambda\big|_{S_{f_1} \times S_{f_2}}$, and also equal to the multiplicity of Δ_Λ in the induced representation $(\Delta_1, \Delta_2)^{S_f}$ of S_f.

To prove this, we begin with a few preparations.

We require first a useful characterization of induced modules. In this result, H denotes a subgroup of a finite group G.

Lemma 1:

Let M be a $\mathbb{K}G$ - module such that for some $\mathbb{K}H$ - submodule L, M is a direct sum

$$M = \bigoplus_{i=1}^{m} g_i L \quad ,$$

where the $\{g_i\}$ form a set of representatives of the left cosets of H in G ($G = g_1 H \cup \ . \ . \cup g_m H$). Then $M \cong L^G$ as $\mathbb{K}G$ - modules.

Proof:

Using (3), we verify at once that $\sum g_i \otimes h_i \longrightarrow \sum g_i h_i$ is a $\mathbb{K}G$ - isomorphism of L^G onto M, and the Lemma is proved.

Corollary 2.

Let L be a left ideal in $\mathbb{K}H \subset \mathbb{K}G$ and consider the $\mathbb{K}G$ - left ideal M generated by L. Then $M \cong L^G$ as $\mathbb{K}G$ - modules.

Proof:

Since $\mathbb{K}G$ is free over G, we have

$$M = \mathbb{K}G\,L = (\bigoplus g_i\ \mathbb{K}H)\,L = \bigoplus g_i\,L$$

With Lemma 1 the Corollary is thus proved.

Now we consider again the situation of Corollary 2, but assume that L is a minimal left ideal. L has the form $\mathbb{K}H\,e'$, where e' is a primitive idempotent of $\mathbb{K}H$. Clearly $M = \mathbb{K}G\,e'$. In order to find the irreducible $\mathbb{K}G$ - submodules of M, we have to decompose e' into a sum of orthogonal primitive idempotents of $\mathbb{K}G$:

$$e' = \sum_{i=1}^{m} e_i + \ . \ . \ . \ . \ . \ . \qquad (10)$$

Here we have collected equivalent idempotents (which generate isomorphic left ideals in $\mathbb{K}G$) into partial sums. The number of irreducible modules isomorphic to $\mathbb{K}G\, e_i$ contained in $M \cong L^G$ is equal to m. According to the reciprocity theorem of Frobenius m is also the number of $\mathbb{K}H$ - submodules in $\mathbb{K}G\, e_i$ which are isomorphic to L.

We now apply these results for the particular case $H = S_{f_1} \times S_{f_2}$, $G = S_f$, $f_1 + f_2 = f$. It is easy to see that

$\mathbb{K}(S_{f_1} \times S_{f_2}) \cong \mathbb{K}S_{f_1} \otimes \mathbb{K}S_{f_2}$. In this case L and M have the form

$$L = (e_1 \otimes e_2)\, \mathbb{K}(S_{f_1} \times S_{f_2}),$$
$$M \cong L^{\mathbb{K}S_f} = (e_1 \otimes e_2)\, \mathbb{K}S_f\,, \qquad (11)$$

where e_i are primitive idempotents of $\mathbb{K}S_{f_i}$.

Now let V be a vector space of dimension n and consider the U(n) - modules $e_i\, V^{\otimes f_i}$, i = 1,2, and their tensor product

$$e_1\, V^{\otimes f_1} \otimes e_2\, V^{\otimes f_2} = (e_1 \otimes e_2)\, V^{\otimes f} \qquad (12)$$

The decomposition of $e_1 \otimes e_2$ into a sum of primitive orthogonal idempotents of $\mathbb{K}S_f$ goes parallel with the reduction of the U(n) - module (12) into irreducible U(n) - submodules.

This, and the underlined conclusions after formula (10), prove Theorem 5.

5. Coupling of inequivalent groups of electrons

Let us consider two groups of f_i, $i = 1,2$, electrons in shells (n_i, ℓ_i). Let V_i be the space spanned by the $(2\ell_i + 1)$ one particle states of each shell and let M_i be an S_{f_i} - submodule of $V_i^{\otimes f_i}$. Furthermore, let M be the S_f - submodule of $(V_1 \oplus V_2)^{\otimes f}$, $f = f_1+f_2$, generated by $L = M_1 \otimes M_2$. This is the subspace of f electrons with f_i electrons in M_i, $i = 1,2$. We claim that $M \cong L^{S_f}$. To prove this, let

$$S_f = g_1 (S_{f_1} \times S_{f_2}) \cup \ldots \cup g_m (S_{f_1} \times S_{f_2})$$

be a decomposition of S_f into $S_{f_1} \times S_{f_2}$ - left cosets. Now

$$M = \mathbb{K} S_f L = (\bigoplus_{i=1}^{m} g_i (S_{f_1} \times S_{f_2})) L = \sum g_i L \qquad (13)$$

This sum is direct. For this it is sufficient to show that

$$N = \sum_{i=1}^{m} g_i (V_1^{\otimes f_1} \otimes V_2^{\otimes f_2}) \qquad (14)$$

is direct. Now N is the direct sum of spaces

$$V_{i_1} \otimes \ldots \otimes V_{i_f}$$

where f_i indices i_k are equal to i, $i = 1,2$. Hence the dimension of N is equal to

$$\frac{(f_1+f_2)!}{f_1!\, f_2!} \dim (V_1^{\otimes f_1} \otimes V_2^{\otimes f_2}) = (G:H) \dim (V_1^{\otimes f_1} \otimes V_2^{\otimes f_2})$$

This proves that the sum (14) and hence the sum (13) is direct. Lemma 1 implies that <u>the coupled space of two inequivalent groups of electrons belonging to representations Δ_i of S_{f_i}, $i = 1,2$, carries the induced representation $(\Delta_1, \Delta_2)^{S_f}$ of S_f.</u> According to the Theorem of Frobenius, <u>the irreducible</u>

representation Δ of S_f appears as many times in this coupled space as $\Delta|_{S_{f_1} \times S_{f_2}}$ contains (Δ_1, Δ_2) of $S_{f_1} \times S_{f_2}$. The Pauli principle restricts the Δ's to those corresponding to Young tables with not more than two columns.

We now apply these results and Theorem 5 to show that the Pauli principle is ineffective between two different shells (this is one of Hund's rules).

With the notations used in this section, let (L_i, S_i) be two (L, S) terms of f_i electrons in the two shells considered. In configuration space these terms belong to representations (D^{L_i}, Δ_i) of $SO(3) \times S_{f_i}$. Now we couple these two groups of electrons. As we have seen, in configuration space the space of the coupled system carries the representation

$$(D^{L_1} \otimes D^{L_2}, (\Delta_1, \Delta_2)^{S_f}) = \bigoplus_{L, \Delta} (D^L, \Delta)$$

In this sum $|L_1 - L_2| \leqslant L \leqslant L_1 + L_2$.

Because of the Pauli principle, we have only to consider those Δ belonging to Young tables with not more than two columns. Now the multiplicity m_Δ of such a Δ is equal to the multiplicity of (Δ_1, Δ_2) in $\Delta|_{S_{f_1} \times S_{f_2}}$. It is easy to show that m_Δ is then also equal to the multiplicity of $(\tilde{\Delta}_1, \tilde{\Delta}_2)$ in $\tilde{\Delta}|_{S_{f_1} \times S_{f_2}}$. Theorem 5 finally proves that

$$m_\Delta = \begin{cases} 1 & \text{for } |S_1 - S_2| \leqslant S \leqslant S_1 + S_2 \\ & \qquad (S\text{: Spin belonging to } \tilde{\Delta}) \\ 0 & \text{otherwise} \end{cases}$$

.

The (L,S) terms of the combined system are thus $|L_1 - L_2| \leqslant L \leqslant L_1 + L_2$, $|S_1 - S_2| \leqslant S \leqslant S_1 + S_2$. In this sense, the Pauli principle is ineffective between the two different shells.

* * *

L I T E R A T U R E :

Curtis, Reiner	Representation Theory of Finite Groups and Associative Algebras
Van der Waerden	Die Gruppentheoretischen Methoden in der Quantenmechanik
H. Weyl	The Theory of Groups and Quantum Mechanics
Robinson	representation theory of the symmetric group