Exercise sheet V

due 8.4.2008.

Problem 1 [*Uncertainty relation I*]: Recall from the lectures that the n-th eigenfunction of the particle confined to the one-dimensional box is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) , \qquad 0 \leq x \leq a \, . \tag{1}$$

(i) Show that the expectation values in the state ψ_n satisfies

$$\langle x \rangle_n = \frac{a}{2}, \qquad \langle (x - \langle x \rangle)^2 \rangle_n = \frac{a^2}{12} \left(1 - \frac{6}{n^2\pi^2}\right) \, . \tag{2}$$

(ii) Determine also the expectation values and the variance of p in the state ψ_n, and confirm the Heisenberg uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$. Here the variance of a variable A in the state Ψ is defined to be

$$\Delta A_\Psi = \sqrt{\langle A^2 \rangle_\Psi - \langle A \rangle_\Psi^2} \, . \tag{3}$$

Problem 2 [*Uncertainty relation II*]: A particle of mass m moves in one dimension subject to the potential $\frac{1}{2}kx^2$ $(k > 0)$. Express the expectation value of the energy E in terms of $\langle x \rangle$, $\langle p \rangle$, Δx and Δp. Hence, using the uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$, show that

$$\langle E \rangle \geq \tfrac{1}{2}\hbar \left(\frac{k}{m}\right)^{1/2} \, . \tag{4}$$

This implies that there is a nonzero lower bound for the energy.

Problem 3 [*Commutator Algebra*]: Let A,B,C be linear operators. Show the following commutator relations:

(i) $[A, BC] = [A, B]C + B[A, C]$ and $[AB, C] = [A, C]B + A[B, C]$.

(ii) Suppose that $[A, [A, B]] = 0 = [B, [A, B]]$. Show that

$$[A, B^n] = nB^{n-1}[A, B], \qquad [A^n, B] = nA^{n-1}[A, B] \, .$$

(iii) If A and B are as in part (ii), prove that $e^A e^B = e^{A+B+[A,B]/2}$.
Hint: Define $f(t) = e^{tA}e^{tB}e^{-t(A+B)}$ and show that $f' = t[A, B]f$. Integrate.