Exercise sheet VII

due 22.4.2008.

Problem 1 [*Angular momentum*]: The angular momentum operator is defined by

$$\vec{L} = \vec{r} \wedge \vec{p} \ , \tag{1}$$

thus being a vector operator with components

$$L_i = \varepsilon_{ijk} r_j p_k \ , \tag{2}$$

with the convention that there is a sum over double indices and where we have used the totally antisymmetric tensor ε_{ijk}. ($\varepsilon_{123} = +1$, and ε_{ijk} is the sign of the permutation $(1\ 2\ 3) \to (i\ j\ k)$. One easily checks that $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.)

(i) Using $[r_i, p_j] = i\hbar\delta_{ij}$, derive the commutation relations

$$[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k \ , \tag{3}$$

and show that

$$[L_3, \vec{L}^2] = 0 \ , \tag{4}$$

where $\vec{L}^2 = \sum_i L_i L_i$.

(ii) Evaluate $[L_3, L_1L_2 + L_2L_1]$ and deduce that in an eigenstate $|l, m\rangle$ of both $\vec{L}^2$ and L_3 with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$, respectively, the expectation values of L_1^2 and L_2^2 are given by

$$\langle l, m|L_1^2|l, m\rangle = \langle l, m|L_2^2|l, m\rangle = \frac{1}{2}\hbar^2[l(l+1) - m^2] \ . \tag{5}$$

Hint: If ψ is an eigenstate of the self-adjoint operator $\mathbf{A}$, show that, for any operator $\mathbf{B}$,

$$\langle \psi | [\mathbf{A}, \mathbf{B}] \psi \rangle = 0 \ .$$

Problem 2 [$SO(4)$]: Construct the Lie algebra of $SO(4)$,

$$so(4) = \{\dot{\gamma}(t)|_{t=0} \ : \gamma(t) \text{ differentiable path in } SO(4),\ \gamma(0) = id\} \ .$$

First find a basis for the vector space $so(4)$, and then determine the commutators of these basis vetors. Finally, prove that $so(4)$ is equivalent, as a Lie algebra, to $su(2) \oplus su(2)$.

Hint: Find an obvious subset of generators that satify the $su(2)$ commutation relations. Calculate the commutators with the other generators, and construct two commuting sets of $su(2)$ generators.

Problem 3 [*Pauli matrices*]: The Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ define a basis for the Lie algebra of $su(2)$. They are explicitly given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(i) Show that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \ . \tag{6}$$

(ii) We define the exponential of these Lie algebra elements by

$$U(\omega\vec{n}) = \exp\left(-i\frac{\omega}{2}\vec{n}\cdot\vec{\sigma}\right) \ . \tag{7}$$

Show that

$$U(\omega\vec{n}) = \cos(\omega/2)\mathbf{1}_2 - i\sin(\omega/2)(\vec{n}\cdot\vec{\sigma}) \ . \tag{8}$$

Verify that $U(\omega\vec{n})$ is an element of the group $SU(2)$.

(iii) Regarded as an element of $SO(3)$, show that (7) describes the rotation by ω around the axis $\vec{n}$.

Hint: Use the isomorphism $SU(2)/\{\pm\mathbf{1}\} \simeq SO(3)$ and calculate $\tilde{x}' = U\tilde{x}U^\dagger$.