Explicit thermalisation models I

Entanglement entropy and quantum field theory, International Journal of Quantum Information: Calabrese , Cardy, 2004

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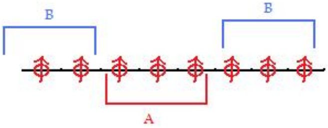
Context

- The concept of entanglement plays a crucial role in nowadays models of thermalisation. *(Popescu, S. et al.)*
- The von Neuman entropy $S_A = -Tr(\rho_A ln(\rho_A))$ can be used as a measure of the entanglement between a system A and its environment B.
- The main goal of this talk is the computation of S_A within 2dim conformal field theory.



$$T = 0 \Rightarrow \rho = |0\rangle \langle 0|$$

Example: Two spin degrees of freedom

- Pure state: $|\psi\rangle = cos\theta \, |1_A, 0_B\rangle + sin\theta \, |0_A, 1_B\rangle$,
- $Tr_B\rho =$
  $\langle 1_B| \otimes id_A \, |\psi\rangle \langle\psi| \, id_A \otimes |1_B\rangle + \langle 0_B| \otimes id_A \, |\psi\rangle \langle\psi| \, id_A \otimes |0_B\rangle$
- $\Rightarrow \rho_A = sin^2\theta \, |0_A\rangle \langle 0_A| + cos^2\theta \, |1_A\rangle \langle 1_A|$
- $\Rightarrow S_A = -Tr_A\rho_A ln(\rho_A) = -(cos^2\theta ln(cos^2\theta) + sin^2\theta ln(sin^2\theta))$
- $\Rightarrow max(S_A) = S_A(cos^2\theta = \frac{1}{2}) = -(ln(\frac{1}{2})) = ln(2)$
- The maximal entangled states are:
  $|\psi\rangle = \frac{1}{\sqrt{2}} (|1_A, 0_B\rangle \pm |0_A, 1_B\rangle)$

Plot

- Plot of $S_A$as a function of $cos^2(\theta)$:

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Plot[-(x*Log[x] + (1 - x)*Log[(1 - x)]), {x, 0, 1}]
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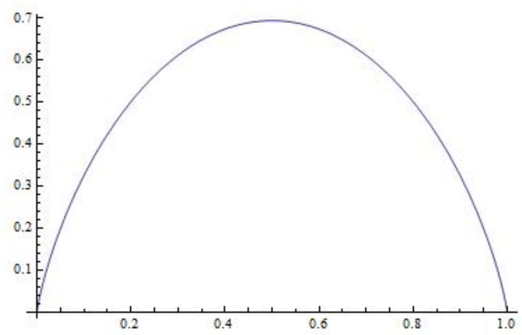


Outline

1. **Models in consideration: Critical quantum field theories in 1 spatial dimension**

1.a Transverse Ising model and quantum phase transitions (T=0)
1.b Scale invariance at the critical point

2. Field theoretical methods

2.a Conformal field theory in 2 dimensions
2.b Euclidean path integrals in quantum mechanics

3. **Explicit calculation of $S_A = -Tr_A \rho_A ln(\rho_A)$ for an 1 dimensional infinite system at T=0**

2.c Calculation of S_A, A being a finite interval on the x-axis
2.d Results for T=0 and T finite

4. **Conclusion**

Remarks

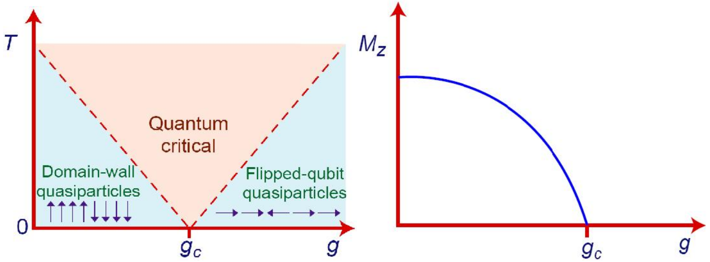
- We want to find the entanglement entropy between subsystem A and environment B for systems with many degrees of freedom where the total system is in a pure state $\rho = |\psi\rangle \langle\psi|$.
- In general this is not possible.
- For 1-dim lattice models at a *quantum critical point*, which can be described by a conformal field theory in 1+1 dimensions, explicit results have been obtained and this is the content of the talk.
- Whether these results have any importance for the issue of thermalisation is not considered here.

The Ising model

- Consider a spin degree of freedom at every point of a 1d lattice ($\sigma^z(x) = \pm 1$) with lattice constant a and a transverse tunable magnetic field that drives the phase transition at $T = 0$.
- This can be described by the Hamiltonian $H_I(g) = -g \sum_n \sigma_n^x - \sum_n \sigma_n^z \sigma_{n+1}^z$.
- At $g = 0$ we have $\langle \sigma_n^z \rangle = \pm 1$, at $g = \infty$ the transverse field dominates and $\langle \sigma_n^z \rangle = 0$. The continuous phase transition between these two regimes happens at $g = 1$.

Quantum Phase transitions

Pictures taken from Subir Sachdev: Quantum phase transitions, Yale 2004



Picture 1: Phase diagram for the 1d Ising model as function of g . In the non critical region the excitations of the system can be described by fermionic quasi-particles.

Picture 2: The ground state at $T = 0$: Dependence of the average magnetization as function of g. For $g < g_c$ the system is in a ***ferromagnetic state*** *described by a wave function similar to $|up\rangle = \otimes_x |1_z\rangle$ for $g > g_c$ in a* ***paramagnetic state*** *with $|right\rangle = \otimes_x |1_x\rangle$. Similar means that fluctuations don't break the phase.*

Generalisations

- In general we have some short range interaction $I(x-x')>0$ that vanishes for $|x-x'|$ larger than some multiple of a.
- Local observables $\phi^{lat}(x)$ are sums of products of nearby spins. For example the local spin $\sigma^z(x)$ itself or the energy density $\varepsilon(x)=\sum_{x'\in J} I(x-x')\sigma^z(x)\sigma^z(x')$.
- The correlation function $\left\langle\phi_1^{lat}(x)\phi_2^{lat}(x')\right\rangle$ falls off over the same distance scale as the interaction. Close to the QCP it is of the form $\langle\sigma^z(x)\sigma^z(x')\rangle \propto exp\left[-\frac{|x-x'|}{\xi}\right]$.

Scale Invariance

- The *correlation length* diverges at the QCP : $\xi \propto (g - g_c)^{-\nu}$. This implies that there is no length scale in the problem anymore; the theory becomes *scale invariant*.
- A Hamiltonian which is translational invariant for multiples of a, is replaced by one with arbitrary translational invariance.
- For the correlation function of scalar observables scale invariance means: $\langle \phi_1(bx)\phi_2(bx')\rangle = b^{-(h_1+h_2)} \langle \phi_1(x)\phi_2(x')\rangle$, where h_i are called scaling dimensions.
- Translational invariance then dictates: $\langle \phi_1(x)\phi_2(x')\rangle \propto |x - x'|^{-(h_1+h_2)}$.

The continuum limit

Towards the quantum field theory

- Consider the limit $a \to 0$.
- The lattice observables will become fields $\phi_j(x)$ of a continuous variable x.
- This means that the limit $\lim_{a\to 0}\left[a^{-(h_1+h_2)}\left\langle \phi_1^{lat}(x_1)\phi_2^{lat}(x_2)...\right\rangle\right]$ exists. It is denoted as $\langle\phi_1(x_1)\phi_2(x_2)...\rangle$.
- Together with translations and rotations, scale transformations form a group...

Remarks

- Conformal transformations leave angles invariant: $\tilde{x} \cdot \tilde{y} = \lambda^2 x \cdot y$, where the inner product is defined through the metric on $\mathbb{R}^m$: $x \cdot y = x^\mu \eta_{\mu\nu} y^\nu$.
- We will see that in two dimensions the symmetry algebra of an euclidean CFT is infinite dimensional and can be represented by analytic functions of a complex variable.
- This will significantly restrict the form of the correlation functions and their transformation properties, enabling us in the end to calculate S_A .

Conformal symmetry transformations

The conformal group for a 1+1dim euclidean theory

- Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $(x, y) \mapsto (\tilde{x}, \tilde{y})$ a conformal transformation, locally expressed as $d\tilde{x}^\mu = M^\mu_\nu dx^\upsilon,\ M^\mu_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}|_x$.
- The linear map M has to preserve angles: $M^\mu_\sigma \eta_{\mu\nu} M^\nu_\rho = \frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial \tilde{x}^\nu}{\partial x^\rho}\eta_{\mu\nu} = \lambda^2(x)\eta_{\sigma\rho}$. (For Poincaré transformations: $\lambda^2 = 1$)
- For $\eta_{\mu\nu} = \delta_{\mu\nu}$ this means that, the condition $d\tilde{x}^2 + d\tilde{y}^2 = \lambda^2(x, y)(dx^2 + dy^2)$ has to be fulfilled. This is equivalent to the Cauchy-Riemann equations: $\frac{\partial \tilde{x}}{\partial x} = \frac{\partial \tilde{y}}{\partial y}, \frac{\partial \tilde{x}}{\partial y} = -\frac{\partial \tilde{y}}{\partial x}$ or $\left(\frac{\partial \tilde{x}}{\partial x} = -\frac{\partial \tilde{y}}{\partial y}, \frac{\partial \tilde{x}}{\partial y} = \frac{\partial \tilde{y}}{\partial x}\right)$
- $\Rightarrow f = \tilde{x} + i\tilde{y}$ is an (anti)analytic function of $z = x + iy$.

Conformal symmetry transformations

Complex coordinates and infinite Lie Algebra

- Since f can be any analytic function, the Lie Algebra is infinite dimensional.
- Conformal field theories are quantum field theories for which conformal transformations leave the action S invariant.
- Let us now consider scalar field theory that can be described by a Lagrangian density $\mathcal{L}(\phi, \partial^{\mu}\phi, x)$, with the corresponding action $S[\phi] = \int dxdy\mathcal{L}$. The equations of motion follow from demanding $\delta S = 0$.

Conformal field theory

Relevant facts for the computation of the entanglement entropy

- Consider an infinitesimal transformation $g^\mu(x) = x^\mu + \epsilon\alpha^\mu(x)$. Conformal invariance is equivalent to $\delta S = 0$.
- The *stress tensor* $T_{\mu\nu}$ *is defined* by $\delta S = -\frac{1}{2\pi}\int T_{\mu\nu}\alpha^{\mu,\nu}dxdy$ and describes the response of S to a general infinitesimal transformation.
- Conformal symmetry implies that it is conserved, symmetric and traceless.
- In complex coordinates: $T_{z\bar{z}}$, $T_{\bar{z}z}$ vanish; $T(z) = T_{zz} = \frac{1}{2}(T_{xx} + T_{x\tau})$ and $\bar{T}(\bar{z}) = \frac{1}{2}(T_{xx} - T_{x\tau})$ are holomorphic and antiholomorphic.

Correlation functions in 2 dimensions

Primary fields and conformal weights

- In general fields do not transform trivially. In the simplest case the transformation is given by:

$$\phi(z,\bar{z}) \mapsto \left|f'(z)\right|^{2h} \phi(f(z), \overline{f(\bar{z})})$$

- In this case we call ϕ a *primary field.*
- It can be shown that this implies:

$$\langle \phi_1(z_1,\bar{z}_2), \phi_2(z_1,\bar{z}_2)\rangle = C\left|z_1 - z_2\right|^{-4h}$$

- We see that primary fields are the simplest example of continuum limits of lattice observables.

Euclidean path integrals in quantum mechanics

- To calculate $Tr\hat{\rho}$ we need to consider matrix elements of $\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z}$, β being the inverse temperature. For illustration we treat first one degree of freedom.
- $\frac{it}{\hbar} \mapsto \tau$, suggests that $e^{-\beta\hat{H}}$ describes "evolution" in imaginary time $0 \leq \tau \leq \beta$
- We can write $\hat{\rho}$ as a product of operators corresponding to arbitrarily small intervals $\varepsilon = \frac{\beta}{n}$,:

$$\left\langle q''\right| e^{-\beta\hat{H}} \left|q'\right\rangle = \int \prod_{k=1}^{n-1} dq_k \prod_{k=1}^{n} \left\langle q_k\right| e^{-(\tau_k,-\tau)\hat{H}} \left|q_{k-1}\right\rangle$$

- Where $\tau_k = k\varepsilon \; q_0 = q', q_n = q''$
- At each time step τ_k we insert the unity operator to "sum" over all possible ways of evolution

Euclidean path integrals in quantum mechanics

- Now we take the limit $n \to \infty$.
- Omitting some technical steps we arrive at:

$$\langle q''| \, e^{-\beta \hat{H}} \, |q'\rangle = \lim_{n\to\infty} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{n/2} \int \prod_{k=1}^{n-1} dq_k \, e^{[-S(q)]}$$

- Where S is just the euclidean action: $S = \int_0^\beta d\tau \frac{1}{2} m\dot{q}^2(\tau) + V(q(\tau))$
- Finally we write symbolically

$$\langle q''| \, e^{-\beta \hat{H}} \, |q'\rangle = \int_{q(0)=q'}^{q(\beta)=q''} [dq(\tau)] \exp\left[-S(q)\right]$$

Setup for the calculation

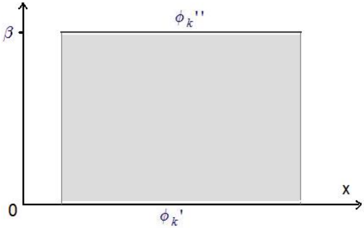
- We now consider a lattice quantum theory in one space (discrete variable x) and one continuous "time" dimension.
- $T = 0 \Rightarrow \hat{\rho} = |0\rangle\langle 0|$. which corresponds to $\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z}$, $\beta \to \infty$. Now the subsystem A consists of the points x in an interval (u, v) of length l.
- In order to quantify the entanglement between these systems we can use CFT methods to calculate S_A.
- Compute $Tr\rho_A^n$ using CFT for $n \in \mathbb{N}$, then treat n as a continuous variable and finally $S_A = \lim_{n \to 1} \frac{\partial}{\partial n} Tr\rho_A^n$.

The Path integral expression

- *For any CFT the density matrix elements are given by a path integral over some fundamental set of fields $\phi(x,\tau)$:*

$$\langle\phi(x,\beta)|\,\hat{\rho}\,|\phi(x,0)\rangle = Z^{-1}\int' [d\phi(x,\tau)]\, e^{-S[\phi]}$$

- *The rows and columns of $\hat{\rho}$ are labeled by the values of the fields at $\tau = 0, \beta$ and the path integral is over all histories (system configurations) consistent with these initial and final values.*



Integration structure 1

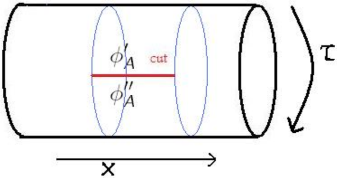
- The quantum partition function $Z(\beta) = Tr e^{-\beta \hat{H}}$ ensures that $Tr \hat{\rho} = 1$. It is found by setting $\phi(x, \beta) = \phi(x, 0)$ and integration $\int [d\phi(x, 0)]$. This has the effect of "sewing" together the edges of the integration domain along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β
- The reduced density matrix $\hat{\rho}_A = Tr_B \hat{\rho}$ is found by sewing together the points x outside of A and integration over the environment B.

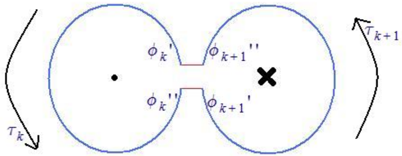
$$\langle \phi''_A | \hat{\rho}_A | \phi'_A \rangle = \int [d\phi_B(x, 0)] \langle \phi_A(x, \beta) | \otimes \langle \phi_B | \hat{\rho} | \phi_B \rangle \otimes | \phi_A(x, 0) \rangle$$

- This will leave an open cut in the cylinder along the line $\tau = 0$.

Integration structure 2

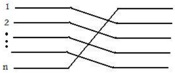
- Path integral representation of ρ_A and ρ_A^2 where $\langle\phi''_A|\,\hat{\rho^2}_A\,|\phi'_A\rangle = \int d\phi_A\,\langle\phi''_A|\,\hat{\rho}_A\,|\phi_A\rangle\,\langle\phi_A|\,\hat{\rho}_A\,|\phi'_A\rangle$:
- By making n copies of the cylinder and sewing them together cyclically along the cuts so that $\phi(x)''_k = \phi(x)'_{k+1}$ for all $x \in A$ we can compute the quantity $Tr\rho_A^n$, which is a starting point for the calculation of S_A





Final preparations

- The path integral on this n-sheeted structure ($\Re_n$), created by the cyclical sewing procedure, is:
  $Tr(\hat{\rho}_A^n) \equiv \frac{Z_n(A)}{Z(\beta)^n} = \int d\left[\phi_A^1\right] \int \prod_{k=1}^n d\phi_A^k \left\langle \phi_A^k \right| \hat{\rho}_A \left| \phi_A^{k+1} \right\rangle$, with
  $\phi_A^1 = \phi_A(x,0) = \phi_A^{n+1}$
- In the limit $T \to 0$, ($\beta \to \infty$) the n-sheeted integration domain can be regarded as n-times the complex w-plain sewn together, since the curvature of the cylinder also goes to zero.



- We won't calculate any path integral explicitly but will employ results of CFT to actually receive a value for $Tr(\hat{\rho}_A^n)$.

Calculation of S_A

- Write $\frac{Z_n}{Z^n}$ as $f(n) = Tr\hat{\rho}_A^n = \sum_j \lambda_j^n$, $\lambda_j \in [0,1)$ where the λ_j are the eigenvalues of $\hat{\rho}_A$
- $\Rightarrow f(n)$ converges and is $\in C^1$ for $Re(n) > 1$. If S_A exists:

$$\lim_{n\to 1}(\partial_n f(n)) = \lim_{n\to 1}(\sum_j e^{n\cdot ln(\lambda_j)}\cdot ln(\lambda_j)) = \sum_j \lambda_j \cdot ln(\lambda_j) = -S_A$$

- $\Rightarrow S_A = -\lim_{n\to 1}\left[\frac{\partial}{\partial n}\frac{Z_n}{Z^n}\right]$.
- We need to find a way to calculate the path integral expression $\frac{Z_n}{Z^n}$.

Remarks again

- It is possible to map $\Re_n$ in a conformal way to $\mathbb{C}$ where translational and rotational invariance ($\delta S = 0$) yields $\langle T(w) \rangle_{\mathbb{C}} = 0$.
- Two things will now be used and not derived: The explicit map and the transformation law for the stress tensor $T(w)$ under conformal transformation.

The conformal map

- We use the transformation law of CFT
  $T(w) = (z'')^2 T(z) + \frac{c}{12}\frac{z'''z' - \frac{3}{2}(z'')^2}{(z')^2}$ for $w \mapsto z(w) = (\frac{w-u}{w-v})^{\frac{1}{n}}$,
  which maps the n-sheeted w-surface $\Re_n$ to the z-plane $\mathbb{C}$.
- $w \mapsto \zeta = \frac{w-u}{w-v}$ maps (u, v) to $(0, -\infty)$; this is then combined with $\zeta \mapsto \zeta^{1/n}$. (Pictures on the o.p.)
- Calculating the three derivatives we get
  $\langle T(w)\rangle_{\Re_n} = \frac{c}{24}(1 - \frac{1}{n^2})\frac{(v-u)^2}{(w-u)^2(w-v)^2}$.
- c is called the central charge. For the Ising model $c = 1/2$, for the free boson $c = 1$.

Infinitesimal transformation and the stress tensor

- Change the length $l = |v - u|$ slightly by an infinitesimal transformation $g: \ x \to x + \delta l \theta(x - x_0)$, where $u \leq x_0 \leq v$.
- This leads to a discontinuity, giving rise to a non-vanishing modification of S according to
  $\delta S = -\frac{1}{2\pi} \int T_{\mu\nu} \alpha^{\mu,\nu} dx d\tau = -\frac{1 \delta l}{2\pi} \int_{-\infty}^{\infty} T_{xx}(x_0, \tau) d\tau$,
  $T_{xx} = T(w) + T(\bar{w})$,
- Inserted in the path integral expression for $Z_n(S) \to Z_n(S + \delta S)$ and expanded up to first order $exp(-S[\phi] - \delta S) \approx (1 - \delta S) exp(-S[\phi])$ resulting in a change δZ_n of the partition function Z_n .

Contour integration

- Interchanging the order of integration and inserting a factor n to consider the insertion on each of the n sheets, $w = x_0 + i\tau$:

$$\frac{\delta Z_n}{Z_n} = \delta ln(Z_n) = -\frac{n\delta l}{2\pi}\int_{-\infty}^{\infty} \langle T(w)\rangle_{\Re_n}\, d\tau$$

- Treating w as complex variable, it can be solved by a contour integration around v in $\langle T(w)\rangle_{\Re_n}$:
$\frac{\partial ln(Z_n)}{\partial l} = \frac{1}{Z_n}\frac{\partial Z_n}{\partial l} - \frac{(c/6)(n-1/n)}{l} \Rightarrow Z_n/Z^n \propto l^{-(c/6)(n-1/n)}$
- Finally: $Tr\rho_A^n = c_n\,(l)^{-(c/6)(n-1/n)}$

$$S_A = \frac{\partial}{\partial n} Tr\rho_A^n(n=1) = \frac{c}{3} ln(l) + ln(c_n)$$

Finite Temperature

- Take two primary fields $\phi_n(u)$ and $\phi_n(v)$ which have the same scaling dimension $h_n = \bar{h}_n = \frac{c}{24}(1 - \frac{1}{n^2})$
- From the CFT section we know that $\langle \phi_n(u)\phi_n(v)\rangle_{\mathbb{C}} = |l|^{-c/6(1-\frac{1}{n^2})}$. This implies that $Tr\rho_A^n = c_n \left(\frac{l}{a}\right)^{-(c/6)(n-1/n)}$ transforms as the correlation function of two primary fields: $w \mapsto \widetilde{w} = z(w)$

$$\langle \phi(w_1)\phi(w_2)\rangle = \left|z'(w_1)\right|^{2h} \left|z'(w_2)\right|^{2h} \langle \phi(z_1)\phi(z_2)\rangle$$

- The map $w \mapsto \widetilde{w} = (\beta/2\pi)ln(w)$ maps each sheet of $\Re_n$ into an infinite long cylinder of circumference β. The result for a thermal mixed state at $\beta^{-1} = T < \infty$ is then:

$$S_A = (c/3)ln(\frac{\beta}{\pi})sinh(\frac{\pi l}{\beta}) + c_1$$

- For $l \ll \beta$ this is the previous result, for $l \gg \beta$ S_A becomes extensive and equals the Gibbs entropy for an isolated system

Conclusion

- Close to a QCP (T=0) where the correlation length is much larger than the lattice spacing, 1d lattice models are believed to be described by a CFT in 1+1 dimensions.
- In this case the quantity $Tr\rho_A^n$ is represented by a path integral expression over some set of fundamental fields.
- By slightly changing the length of the subsystem A, $Tr\rho_A^n$ can be computed up to constant using the transformation property of the conformal stress tensor and complex analysis respectively.
- These method hay also been used to get more general results for similar situations: Finite system with boundary, several distinct intervals and for finite correlation length.