Arrow of Time and Entropy

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Pro-Seminar: Information in Statistical Mechanics

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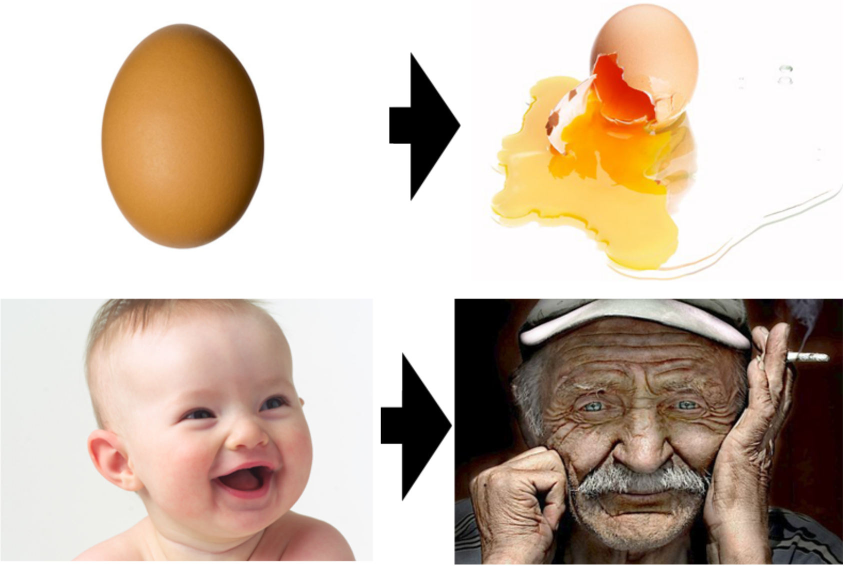
Time arrow

Time seems to have a preferred direction in the macroscopic world.

$\Rightarrow$ Many processes occur just in one direction i.e. the direction of (global) increasing entropy (2nd law of thermodynamics).

Time arrow

Take for example all the irreversible processes in our everyday life:



What is a time arrow?

What is a time arrow? Two different time arrows:

1. **The psychological time arrow:** Feeling of a directed flow in time caused by accumulating memories from a continuous change of perception.
2. **The thermodynamical time arrow:** An arrow on a macroscopic scale pointing in the direction of increasing entropy (2nd law of thermodynamics).
3. . . .

No microscopic time arrow

There is <u>no</u> "mircoscopic time arrow" i.e. an arrow on a microscopic scale pointing in a specific direction of time *does not exist* in the fundamental theories such as: Hamilton mechanics, QM, QFT, GR ...

(Exception: weak-interaction e.g. kaon decay).

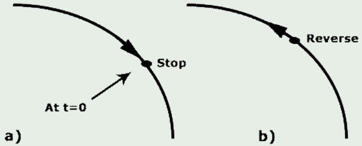
No microscopic time arrow (classical)

Time has <u>no</u> preferred direction in the microscopic world:

The fundamental laws of physics are time symmetric (actually CPT-invariant).

Classical: If $x(t)$ is a solution to $m\ddot{x} = -\nabla V(x)$ then also $x(-t)$.

Example: A classical particle in a central force field



(a) A classical trajectory which stops at $t = 0$ and (b) reverses its motion $p|_{t=0} \rightarrow -p|_{t=0}$. If you run the motion picture of trajectory (a) backward as in (b), you can't tell whether this is the correct sequence.

No microscopic time arrow (quantum mechanical)

Quantum mechanical: It's more difficult in QM. We need a antiunitary time-reversal operator T.

Example: The Schrödinger equation for a free particle

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\Delta\Psi \;\Rightarrow\; \Psi(x,t) = Ae^{i(\mathbf{kx}-\omega t)}$$

$$\Rightarrow\; T\Psi(x,t) = \Psi^*(x,-t) = A^* e^{-i(\mathbf{kx}+\omega t)}$$

Both wavefunctions are solutions of the Schrödinger equation.

The Loschmidt Paradox

Problem: How can one derive time-asymmetric laws from time-symmetric laws (Loschmidt Paradox).

Presented attempts to solve the paradox:

1. *Standart approach:* In a classical (statistical) framework by Boltzmann (1877).
2. *New approach:* In a quantum mechanical framework by Maccone (2008).

The statistical approach by Boltzmann

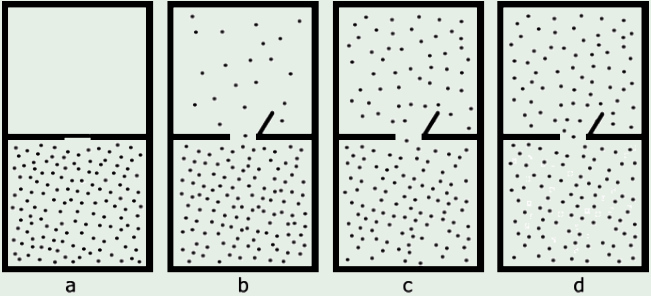
A qualitative description of the approach:

- Boltzmann states that entropy decreasing processes can occur (without doing any work), it is just tremendously improbable.
- The evolution of a microstate in phase space is such that the entropy is most likely to increase (or stay constant).

The statistical approach by Boltzmann

The box example

Consider an isolated system at four different times. The two halves of the box are initially seperated by a damper (a), which is opened at a later time:



What is the right time ordering?

$\Rightarrow$ The microscopic laws allow a-b-c-d and d-c-b-a!

The statistical approach by Boltzmann

Why do we only see a-b-c-d in nature, i.e. entropy increasing (or constant) evolutions?

Lets consider the phase space Γ of a system composed of N particles:

- $X(t) = (\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_N, \mathbf{p}_N) \in \Gamma$ denotes the microstate of the system in its $6N$-dimensional phase space.
- $M(X(t))$ denotes the macrostate of the system (e.g. temperature). There are many X corresponding to M (in fact a continuum).

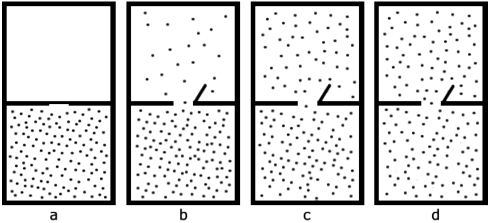
Boltzmann entropy

Boltzmann defined the entropy as ($k_B = 1$):

$$S_B = \log |\Gamma_M| \,,$$

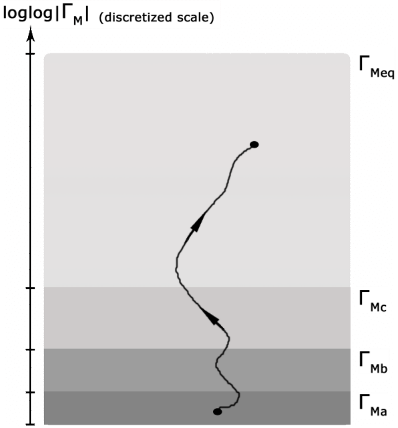
where $|\Gamma_M| = \int_{\Gamma_M} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}$ denotes the phase space volume corresponding to the macrostate $M = M(X(t))$.

The statistical approach by Boltzmann



- Macrostate M_i corresponds to the phase space Γ_{M_i} with $i = a, b, c, d$. The Γ_{M_i} are subspaces of the systems total phase space.
- The position of X in phase space does correspond to a specific macrostate $M(X(t))$.
- The phase space volume correspondig to the macrostate $M(X(t))$ can change as the system evolves.

The statistical approach by Boltzmann



Simplified diagramm of the phase space volume from the box example (don't take it too serious)

- The phase space Γ_E on the left is a constant-energy-subspace of the whole phase space (box example).
- The phase space Γ_{M_a} is very tiny in comparison to $\Gamma_{M_{eq}}$ (consider the double logarithmic scale).
- The system X is initially in Γ_{M_a}. We let it evolve due to Hamilton mechanics.
- The phase space volume $|\Gamma_{M(X(t))}|$ does depend on the region in which we find $X(t)$.
- The system will (with a random evolution) most likely explore the available phase space.
- $\Rightarrow$ $|\Gamma_{M(t)}|$ will increase until it reaches the maximum value $|\Gamma_{M_{eq}}|$ where it will most certainly stay. $|\Gamma_{M_{eq}}|/|\Gamma_E| \simeq 1$.

The statistical approach by Boltzmann

From this point of view, the 2nd law of thermodynamics is just a statistical principle of a system consisting many particles.

Lets have a look on how likely an increase of the entropy is:

Example: Enlargement of the phase space

1. Consider a qubic box with a length L containing 1 mol of gas (N_A particles) and the corresponding phase space volume $|\Gamma_L|$.
2. We then double the sides of the box (8 times bigger spatial volume) and denote the new available phase space by Γ_{2L} (no new energy).
3. If we now calculate the ration $|\Gamma_{2L}|/|\Gamma_L|$, we get a result of $2^{N_A} \approx 2^{\cdot 10^{23}}$.
4. The probability that the system will stay in the initial phase space is very roughly the inverse of this number.

$\Rightarrow$ The probability that the entropy decreases is ≈ 0.

The statistical approach by Boltzmann

Now, we have an explanation why entropy does increase for an arbitrary evolution.

But the **Loschmidt paradox isn't solved yet** since entropy does increase in t and $-t$ direction (for an arbitrary initial state, which has not max. entropy).

For instance figure (b) in the box example. If we didn't know that M_b is coming from M_a, we would (with the statistical arguments) conclude that the entropy does also increase in $-t$ direction.

$\Rightarrow$ **We need an assumption**.

The statistical approach by Boltzmann

Assumption: The system started in a lower entropy state i.e. the initial conditions of the system need to have low entropy.

- In our box example, the low entropy initial conditions were due to an experimentalist.
- In a general view, we need a **low entropy of the "early universe"** to explain the apparent thermodynamic arrow.

Problem: A low entropy early universe is not explained by any current theory.

$\Rightarrow$ The Loschmidt paradox is now a problem for cosmologists.

The quantum mechanical approach by Maccone

Boltzmann's solution to the paradox has an ad-hoc assumption (initial condition of the universe).

Now, **another approach to the problem** in a quantum mechanical framework by Lorenzo Maccone (2008).

Statement of the paper

Entropy increasing and decreasing transformations can occur (as time reversal dictates), but all entropy decreasing-transformations can't leave any trace (e.g. in a memory) of them having happend.

- This is indistinguishable from them not having happend.
- "The past exist only insofar as it is recorded in the present" i.e. the only physical evolution that can be studied are those where entropy has not decreased.
- The second law is then forcefully valid and it's reduced to a tautology.

The quantum mechanical approach by Maccone

The paper makes just one assumption.

Assumption:

Quantum mechanics is valid at all scales.

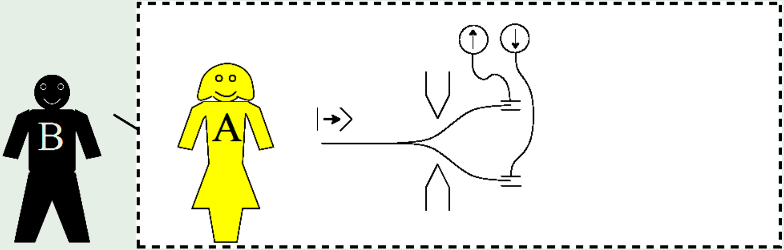
The quantum mechanical approach by Maccone

Correlations between systems:

- It is generally impossible to exclude that two systems might be correlated.
- A physical process might either increase or reduce this correlations.
- Each bit of memory is one bit of correlation.
- "Information is physical" $\Rightarrow$ Any record of an occurred event can be decorrelated from such event by appropriate physical interaction.
- As long as the memory is not erased, the correlations are not eliminated.

The quantum mechanical approach by Maccone: Example

Thought experiment: Bob and Alice (1)



- Alice is in an isolated lab which contains a Stern-Gerlach apparatus oriented along the z-axis. Bob (outside observer) gives Alice a pure spin-1/2 particle oriented along the x-axis (pure states have zero entropy):
  $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.
- Alice performs now a measurement on $|\rightarrow\rangle$ which will be in the maximally mixed state $(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)/2$ before the readout.

The quantum mechanical approach by Maccone: Example

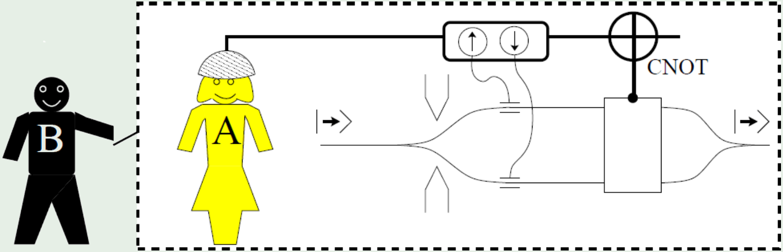
Thought experiment: Bob and Alice (2)

- The entropy of the spin system will increase by one bit because of the measurement.
- By looking at the result of the measurement, Alice transfers this one bit of entropy to her memory.
- The entropy of the laboratory system has therefore been increased by 1 bit (for an additional observer in the lab), whereas the entropy for Bob stayed the same.
- For Bob, the measurement is simply a quantum correlation (entanglement) between Alice and her measurement apparatus.

$$\frac{1}{\sqrt{2}}(\underbrace{|\uparrow\rangle}_{\text{spin}}\underbrace{|\text{Alice sees up}\rangle}_{\text{rest of laboratory}}+\underbrace{|\downarrow\rangle}_{\text{spin}}\underbrace{|\text{Alice sees down}\rangle}_{\text{rest of laboratory}})\,.$$

The quantum mechanical approach by Maccone: Example

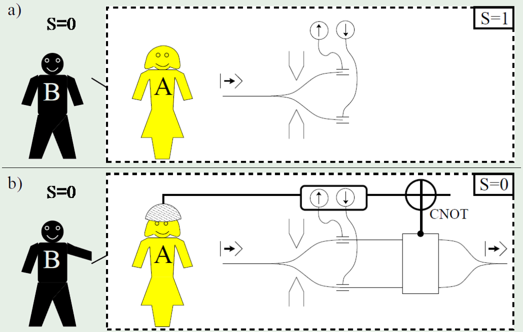
Thought experiment: Bob and Alice (3)



- For Bob, Alice's measurement is an evolution similar to a controlled-NOT unitary transformation: $U_{\mathrm{CNOT}}[(|0\rangle + |1\rangle) \otimes |0\rangle] = |00\rangle + |11\rangle$, where qubit b gets inverted if qubit a is 1. U_{CNOT} can easily be inverted.
- Bob can now perform such an inverted transformation. After this transformation, all records (brain cells, notepads . . .) of the measurement will have been decorrelated from the spin state.

The quantum mechanical approach by Maccone: Example

Thought experiment: Bob and Alice (4)



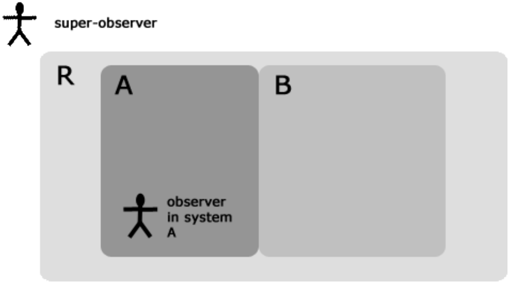
- Alice will remember having performed the measurement, but she must be unable to recall the result. Moreover, the spin has returned to the pure state $|\rightarrow\rangle$
- Entropy has certainly once been created from the viewpoint of Alice. Bob has then subsequently decreased the entropy by erasing the correlations.

The quantum mechanical approach by Maccone: Example

Thought experiment: Bob and Alice (5)

- **The second law has never been violated** from neither viewpoint, since Alice can't remember and Bob always had zero entropy.

The quantum mechanical approach by Maccone



Statements of the paper:

- Any interaction between A and B which decreases their entropy by a certain quantity, must also reduce their quantum mutual information by the same amount (unless the entropy is dumped into a reservoir R).
- An observer in system A will only be aware of entropy non-decreasing processes.
- Not even a super-observer would see any entropy decrease. Since he sees all correlations $\Rightarrow$ all processes are always zero entropy processes.

The quantum mechanical approach by Maccone

The assertion of the paper is summarized in Eq.1:

Claim:

$$\Delta S(\rho_A) + \Delta S(\rho_B) - \Delta S(\rho_R) - \Delta S(\rho_A : \rho_B) = 0 \qquad (1)$$

- **The density matrix** ρ_X describing the system X.
- **The von Neumann entropy**: $S(\rho_X) \equiv -\mathrm{Tr}(\rho_\mathrm{X} \log_2 \rho_\mathrm{X})$.
- **The quantum mutual information**:
  $S(\rho_A : \rho_B) \equiv S(\rho_A) + S(\rho_B) - S(\rho_{AB})$.
  With ρ_A and ρ_B reduced states of ρ_{AB}.
- $\Delta S(\rho_X) \equiv S_t(\rho_X) - S_0(\rho_X)$, where $S_0(\rho_X)$ denotes the inital entropy and $S_t(\rho_X)$ the entropy at some later time.
- $\Delta S(\rho_A : \rho_B) \equiv S_t(\rho_A : \rho_B) - S_0(\rho_A : \rho_B)$

The quantum mechanical approach by Maccone

What does Eq.1 mean?

Claim:

$$\Delta S(\rho_A) + \Delta S(\rho_B) - \Delta S(\rho_R) - \Delta S(\rho_A : \rho_B) = 0$$

- $\Delta S(\rho_X)$: Change of the entropy in system X.
- $\Delta S(\rho_A : \rho_B)$: Change of the quantum mutual information of system A and B.
- $\Rightarrow$ If we want to reduce the entropy of the system A and B, without increasing the entropy of a reservoir R, we need to reduce the quantum mutual information between A and B.

The quantum mechanical approach by Maccone

Does Eq.1 hold?

$$\Delta S(A) + \Delta S(B) - \Delta S(R) - \Delta S(A:B) = 0$$

By inserting the definitions we obtain:

$$-S_t(R) + S_0(R) + S_t(AB) - S_0(AB) = 0$$

- We choose the reservoir R such that ABR is a pure state and the evolution maintains the purity (we can always do that). R is known as a purification space.
- If we consider AB to be the reduction of the pure state ABR $(\mathrm{Tr_R(ABR) = AB})$

  $\Rightarrow S_0(AB) = S_0(R)$ and $S_t(AB) = S_t(R)$
  (follows from Schmidt decomposition).
- Equation (1) is therefore valid.

The quantum mechanical approach by Maccone

- A memory of an event is a physical system A having nonzero classical mutual information on a system C which bears the consequences of that event.
- In order to proof that **every entropy decreasing transformation entails a memory erasure in the classical sense** we need the property that the quantum mutual information is an upper bound to the classical mutual information:

$$S(A:C) \geq I(A:C)$$

- This inequallity has been proofed [Yuen and Ozawa].

The quantum mechanical approach by Maccone: Analysis

- *Is quantum mechanics really necessary to these arguments?*

  Yes, we used the property that the entropy of a joint system can be smaller than that of each of its subsystems.

- *By how much must any system be extended until we can take advantage of this quantum reduction of the global entropy?*

  Since the timescale on which a system can be considered as isolated is very small, the presented effects become relevant only at a scale that approaches the whole universe very rapidly.

Summary

- We analyzed the **arrow-of-time dilemma** and gave solutions from two different points of view.
- Arrow-of-time dilemma is actually an arrow-of-thermodynamics problem. Since the fundamental laws are time-symmetric, we were questioning how one could derive the time-asymmetric 2nd law (**Loschmidt paradox**).
- **Approach by Boltzmann:** Global entropy can decrease it is just tremendously improbable. $\Rightarrow$ Needs an ad-hoc assumption.
- **Approach by Maccone:** Every global entropy decreasing transformation must entail a memory erasure of this transformation having happened, which is indistinguishable from their not having happened at all.