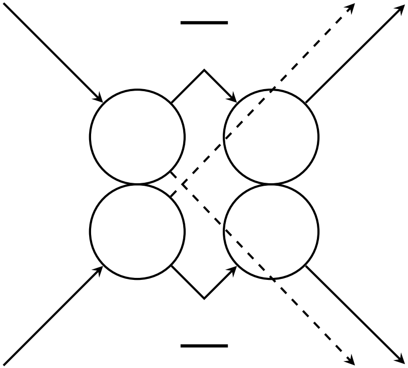
Reversible Computation

Martin Paesold

March 30, 2009
Proseminar FS 09
ETHZ

Supervisor: Dr. Andrei Lebedev

Billiard ball computer



Outline

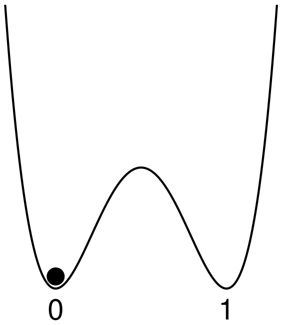
1. Logical and physical irreversibility
   - Landauer's erasure principle
   - "Computations have to be irreversible"
2. Turing Machines
   - Definition
   - Example
3. Bennett's reversible automaton
   - From quintuples to quadruples
   - Construction of the machine
   - Discussion and Implications

Outline

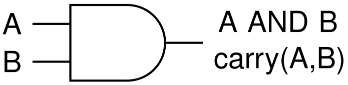
- conservative bistable-well
- operation RESTORE TO ONE dissipates energy
- computing depends on information erasure

Landauer's Erasure Principle

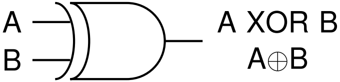
$$\Delta S/\text{erased bit} = -k_B \log 2$$



Logical Gates

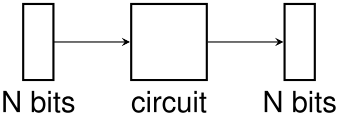


A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1



A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

Landauer's arguments for logical irreversibility

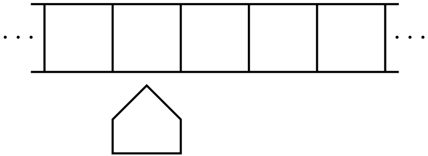


1. Assertion, that real computers depend on logical irreversibility. Every machine copying logical organization of real computers relies on logical irreversibility.
2. Computer that relies on logical gates taking one or two inputs and is reversible can only be built using XOR and ¬XOR. These form no complete set.
3. If we would write down every computational step, we would have to *erase* this history in the end.

Outline

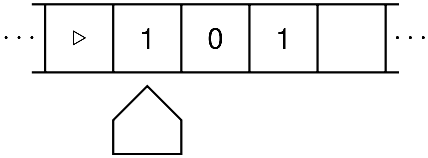
Standard Turing Machine

- Turing Machine consists of
  - unbounded tape,
  - read/ write head,
  - state control.
- Additionally there are
  - finite set of internal states $\{A_k\}$
  - alphabet $\Sigma = \{\triangleright, 0, 1, b\}$



Standard Turing Machine

- Turing Machine consists of
  - unbounded tape,
  - read/ write head,
  - state control.
- Additionally there are
  - finite set of internal states $\{A_k\}$
  - alphabet $\Sigma = \{\triangleright, 0, 1, b\}$



State Transitions

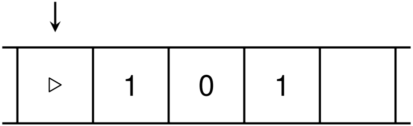
Definition

Quintuple describes a computational step and has the form

$$\underbrace{A_j, T}_{\text{domain}} \rightarrow \underbrace{T', \sigma, A_k}_{\text{range}}$$

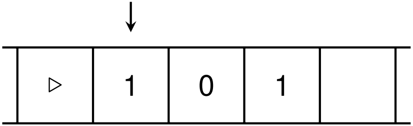
where $\sigma \in \{-, 0, +\}$.

- finite set of quintuples defines work of Turing Machine
- Turing Machines are *deterministic*: no two quintuples have the same domain.



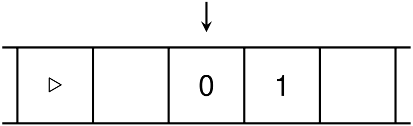
$f(x) = 1$

$$
\begin{aligned}
A_1, \triangleright &\rightarrow \triangleright, +, A_2 \\
A_2, 1 &\rightarrow b, +, A_2 \\
A_2, 0 &\rightarrow b, +, A_2 \\
A_2, b &\rightarrow b, -, A_2 \\
A_2, \triangleright &\rightarrow \triangleright, +, A_3 \\
A_3, b &\rightarrow 1, 0, A_f
\end{aligned}
$$



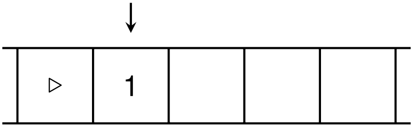
$f(x) = 1$

$$
\begin{aligned}
A_1, \triangleright &\rightarrow \triangleright, +, A_2 \\
A_2, 1 &\rightarrow b, +, A_2 \\
A_2, 0 &\rightarrow b, +, A_2 \\
A_2, b &\rightarrow b, -, A_2 \\
A_2, \triangleright &\rightarrow \triangleright, +, A_3 \\
A_3, b &\rightarrow 1, 0, A_f
\end{aligned}
$$



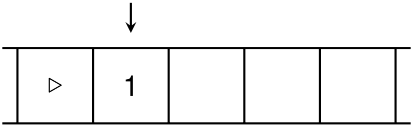
$f(x) = 1$

$$
\begin{aligned}
A_1, \triangleright &\longrightarrow \triangleright, +, A_2 \\
A_2, 1 &\longrightarrow b, +, A_2 \\
A_2, 0 &\longrightarrow b, +, A_2 \\
A_2, b &\longrightarrow b, -, A_2 \\
A_2, \triangleright &\longrightarrow \triangleright, +, A_3 \\
A_3, b &\longrightarrow 1, 0, A_f
\end{aligned}
$$



$f(x) = 1$

$$
\begin{array}{rcl}
A_1, \triangleright & \rightarrow & \triangleright, +, A_2 \\
A_2, 1 & \rightarrow & b, +, A_2 \\
A_2, 0 & \rightarrow & b, +, A_2 \\
A_2, b & \rightarrow & b, -, A_2 \\
A_2, \triangleright & \rightarrow & \triangleright, +, A_3 \\
A_3, b & \rightarrow & 1, 0, A_f
\end{array}
$$



$f(x) = 1$

$$\begin{aligned}
A_1, \triangleright &\rightarrow \triangleright, +, A_2 \\
A_2, 1 &\rightarrow b, +, A_2 \\
A_2, 0 &\rightarrow b, +, A_2 \\
A_2, b &\rightarrow b, -, A_2 \\
A_2, \triangleright &\rightarrow \triangleright, +, A_3 \\
A_3, b &\rightarrow 1, 0, A_f
\end{aligned}$$

Outline

- The inverse of read, write, shift quintuple has the form shift, read, write.
- Transitions and their inverse should have the same form.

Split quintuples into quadruples

$$A_j, T \to T', \sigma, A_k \sim \begin{cases} A_j, T & \to & T', A' \\ A', / & \to & \sigma, A_k \end{cases}$$

- $/$ indicates that the tape is not read.

- The inverse of read, write, shift quintuple has the form shift, read, write.
- Transitions and their inverse should have the same form.

Split quintuples into quadruples

$$A_j, T \rightarrow T', \sigma, A_k \;\sim\; \begin{cases} A_j, T & \rightarrow \quad T', A' \\ A', / & \rightarrow \quad \sigma, A_k \end{cases}$$

- $/$ indicates that the tape is not read.

Definition

A *reversible, deterministic n-tape Turing machine* is a finite set of quadruples, no two of which overlap either in domain or range.

Theorem

For every standard one-tape Turing machine ***S****, there exist a three-tape reversible, deterministic Turing machine* ***R*** *that has the same functionality as* ***S****.*

Definition

A *reversible, deterministic n-tape Turing machine* is a finite set of quadruples, no two of which overlap either in domain or range.

Theorem

*For every standard one-tape Turing machine **S**, there exist a three-tape reversible, deterministic Turing machine **R** that has the same functionality as **S**.*

First stage: computation

Input		

$$1)\quad \begin{cases} A_1[b/b] & \rightarrow & [b+b]A'_1 \\ A'_1[/b/] & \rightarrow & [+1\,0]A_2 \end{cases}$$

$$\vdots$$

$$m)\quad \begin{cases} A_j[T/b] & \rightarrow & [T'+b]A'_m \\ A'_m[/b/] & \rightarrow & [\sigma\, m\, 0]A_k \end{cases}$$

$$\vdots$$

$$N)\quad \begin{cases} A_{f-1}[b/b] & \rightarrow & [b+b]A'_N \\ A'_N[/b/] & \rightarrow & [0\,N\,0]A_f \end{cases}$$

Output	History	

Second stage: copy output

Output	History	

$$A_f[b\,N\,b] \rightarrow [b\,N\,b]B'_1$$

$$B'_1[/\,/\,/] \rightarrow [+0+]B_1$$

$$(x \neq b) \quad \{\ B_1[x\,N\,b] \rightarrow [x\,N\,x]B'_1\ \}$$

$$B_1[b\,N\,b] \rightarrow [b\,N\,b]B'_2$$

$$B'_2[/\,/\,/] \rightarrow [-0-]B_2$$

$$(x \neq b) \quad \{\ B_2[x\,N\,x] \rightarrow [x\,N\,x]B'_2\ \}$$

$$B_2[b\,N\,b] \rightarrow [b\,N\,b]C_f$$

Output	History	Output

Third stage: retrace

Output History _Output

N) $\begin{cases} C_f[/N/] \rightarrow [0\,b\,0]C'_N \\ C'_N[b/b] \rightarrow [b-b]C_{f-1} \end{cases}$

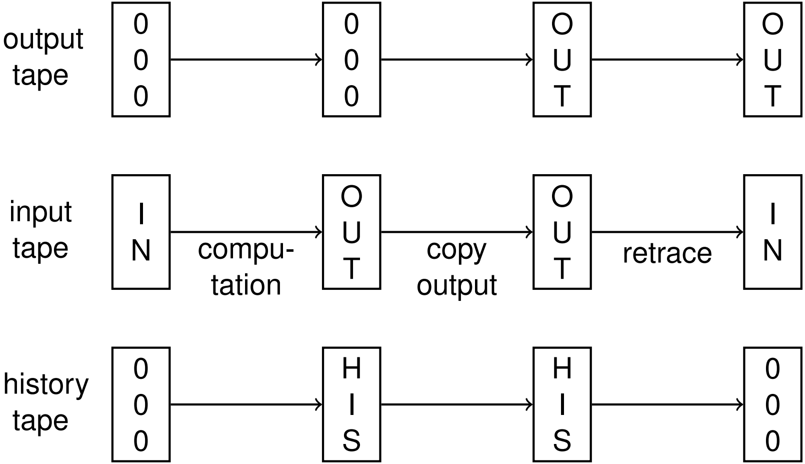
⋮

m) $\begin{cases} C_k[/m/] \rightarrow [-\sigma\,b\,0]C'_m \\ C'_m[T'/b] \rightarrow [b-b]C_j \end{cases}$

⋮

1) $\begin{cases} C_2[/1/] \rightarrow [-b\,0]C'_1 \\ C'_1[b/b] \rightarrow [b-b]C_1 \end{cases}$

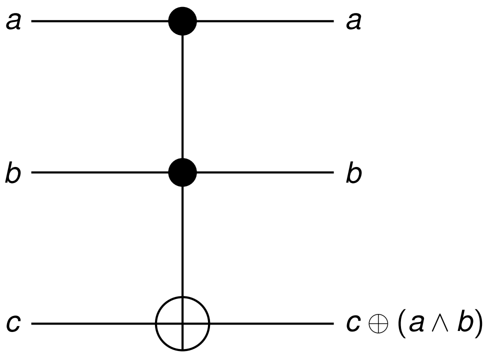
_Input _ _Output


output tape
0 0 0
0 0 0
O U T
O U T
input tape
I N
compu-tation
O U T
copy output
O U T
retrace
I N
history tape
0 0 0
H I S
H I S
0 0 0

- reversible machine requires
  - $2f + 2N + 4$ internal states
  - $4N + 2z + 3$ quadruples

  with N number of quintuples, f internal states, z number of letters of the alphabet of irreversible machine.
- We need great temporary storage for the history. - Segmenting of the computation helps.
- We gained better understanding of the physics of a computer.
- Good starting point to explore quantum computers.

Toffoli gate



Underwater Hockey and Rugby

- **Tuesday**
  Hallenbad City
  20:00 - 21:45
- **Thursday**
  Hallenbad
  Oerlikon
  20:00 - 21:45