Asymptotic freedom and the beta-function

ϕ^4, 2d σ-model, QCD

David Oehri

Tutors: Dr. Ph. de Forcrand
Michael Fromm

20. April 2009

Introduction

- Renormalization is one of the most important concepts of quantum field theories.
- We start with considering the renormalization of ϕ^4 theory and encounter the following concepts:
  - running coupling $\lambda(p)$
  - beta-function $\beta(\lambda)$
  - asymptotic freedom
- We then see the nonlinear σ-model as an example of an asymptotically free theory.
- In the end, we consider the consequences of renormalization and asymptotic freedom on QCD.

Formalism and definitions

- A free scalar field ϕ describing the free propagation of particles is described by the Klein-Gordon Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

- In this case, the generating functional of n-point functions is

$$Z_0[J] = \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\right],$$

where $J(z)$ is the source of the field $\phi(z)$ and Δ_F is the Feynman propagator, obeying

$$(\Box + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x) \rightarrow \Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon}.$$

- We can expand the generating functional:

$$
\begin{aligned}
Z_0[J] = 1 &+ \left(-\frac{i}{2}\right) \int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y \\
&+ \frac{1}{2!}\left(-\frac{i}{2}\right)^2 \left[\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\right]^2 \\
&+ \frac{1}{3!}\left(-\frac{i}{2}\right)^3 \left[\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\right]^3 + \cdots
\end{aligned}
$$

- This can be represented diagrammatically using the rules:

$$
\mathrm{x}\;\text{———}\;\mathrm{y} \quad = i\Delta_F(x-y)
$$

$$
\times \quad = iJ(z)
$$

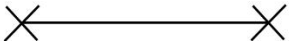
- Rules:

$$= i\Delta_F(x-y)$$

$$= iJ(z)$$

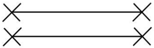
- Diagrammatic representation:

$$Z_0[J] = 1 + \left(\frac{1}{2}\right)\int$$



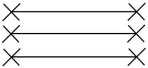
$$d^4x_1 d^4y_1$$

$$+ \frac{1}{2!}\left(\frac{1}{2}\right)^2 \int$$



$$d^4x_1 d^4y_1 d^4x_2 d^4y_2$$

$$+ \frac{1}{3!}\left(\frac{1}{2}\right)^3 \int$$



$$d^4x_1 d^4y_1 d^4x_2 d^4y_2 d^4x_3 d^4y_3$$

$$+ \cdots ,$$

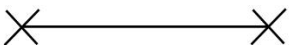
where x_i and y_i label the external points (sources).

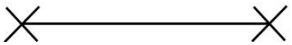
- n-point functions are defined as

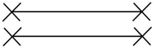
$$\tau(x_1, \ldots, x_n) := \frac{1}{i^n} \frac{\delta^n Z_0[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0}.$$

- Let us see how the functional derivative acts in our graphical picture:

$$\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\right) = \int i\Delta_F(x_1 - z) iJ(z)\mathrm{d}^4z$$

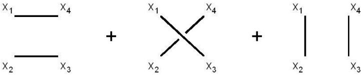
$\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{2}\int\right.$  $\left.\mathrm{d}^4x\mathrm{d}^4y\right) = \int x_1$  d^4z

$$Z_0[J] = 1 + \left(\frac{1}{2}\right)\int$$

$$d^4x_1 d^4y_1$$

$$+ \frac{1}{2!}\left(\frac{1}{2}\right)^2\int$$

$$d^4x_1 d^4y_1 d^4x_2 d^4y_2$$

$$+ \cdots$$

- $\tau(x,y) = \mathbf{x}$ —— $\mathbf{y}$ $= i\Delta_F(x-y)$
- $\tau(x_1,x_2,x_3,x_4) =$



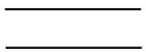
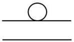
$$
\begin{aligned}
&= (i\Delta_F(x_1-x_4))(i\Delta_F(x_2-x_3)) \\
&\quad + (i\Delta_F(x_1-x_3))(i\Delta_F(x_2-x_4)) \\
&\quad + (i\Delta_F(x_1-x_2))(i\Delta_F(x_3-x_4)) \\
&= \tau(x_1,x_4)\tau(x_2,x_3) + \tau(x_1,x_3)\tau(x_2,x_4) \\
&\quad + \tau(x_1,x_2)\tau(x_3,x_4)
\end{aligned}
$$

- So far, we have considered a free scalar field ϕ described by the Klein-Gordon Lagrangian.
- If we have an additional interaction described by a Lagrangian $\mathcal{L}_{int}(\phi)$, the generating functional is:

$$Z[J] = N \exp\left(i \int \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) \mathrm{d}^4 x \right) Z_0[J],$$

  where N is a normalization constant.
- We will use this generating functional in the next section for the ϕ^4 interaction.

- If we consider the 4-point function of the ϕ^4 theory to first order,

$\tau(x_1, x_2, x_3, x_4) = 3\Big[$  $\Big] + 3(-i\lambda)\Big[$  $\Big] + (-i\lambda)\Big[$  $\Big],$

we see that there are two types of diagrams:

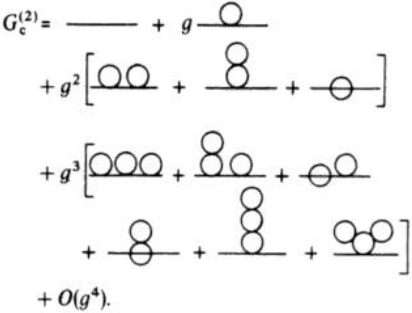
  - *Connected diagrams* $\longrightarrow$ all external points connected to each other.
  - *Disconnected diagrams* $\longrightarrow$ not all external points connected to each other.

- Without going into any details, we state:
  - There is a generating functional W, which generates only the connected part of the n-point functions, $\phi(x_1, \dots, x_n)$
  - Example:

$$\begin{aligned} i\phi(x_1, x_2, x_3, x_4) =& \tau(x_1, x_2, x_3, x_4) - \tau(x_1, x_2)\tau(x_3, x_4) \\ & - \tau(x_1, x_3)\tau(x_2, x_4) - \tau(x_1, x_4)\tau(x_2, x_3) \end{aligned}$$

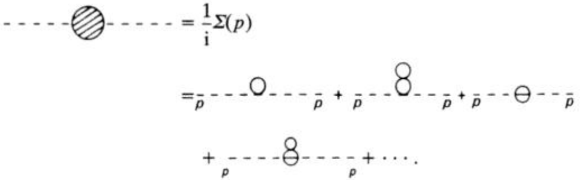
- It is convenient to talk about Green's functions which are directly related to n-point functions:
  - $G^{(n)}(x_1, \dots, x_n) := \tau(x_1, \dots, x_n)$
  - $G_c^{(n)}(x_1, \dots, x_n) := i\phi(x_1, \dots, x_n)$

- Let us introduce one further classification:
- Example: connected Green's function of ϕ^4 theory

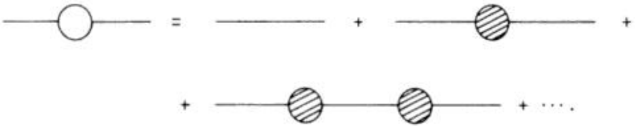


- *1-particle reducible* graphs can be divided into two subgraphs by cutting one internal line.
- *1-particle irreducible* (1PI) graphs cannot be divided into two subgraphs by cutting one internal line.

- Let us use this classification to define the *self-energy* part as the sum of all 1 PI graphs:



- Using this self-energy part $\Sigma(p)$ and the bare propagator (free 2-point function) G_0, we can write the full 2-point function as a graphical expansion:



- These expansion can be written as

$$\begin{aligned}G_c^{(2)}(p) =& G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) \\ &+ G_0(p)\frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \cdots \\ =& G_0\left(1 + \frac{\Sigma}{i}G_0 + \frac{\Sigma}{i}G_0\frac{\Sigma}{i}G_0 + \cdots\right) \\ =& G_0\left(1 - \frac{\Sigma}{i}G_0\right)^{-1} \\ =& \left[G_0^{-1}(p) - \frac{\Sigma(p)}{i}\right]^{-1} = \frac{i}{p^2 - m^2 - \Sigma(p)}.\end{aligned}$$

ϕ^4 theory - General properties

- ϕ^4 theory is a simple scalar field theory, where we investigate the concept of renormalization.
- Name of the theory comes from the interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4.$$

  - λ is a coupling constant, which should be positive,
  - factor 4! is due to symmetry reasons,
  - ϕ^4 leads to a interaction which involves four times the field.
- The whole Lagrangian is

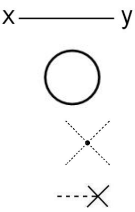
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

- The normalized generating functional is in general given as

$$Z[J] = \frac{\exp[i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})\mathrm{d}z]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y]}{\{\exp[i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})\mathrm{d}z]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y]\}\Big|_{J=0}},$$

  which is normalized to obey $Z[J=0]=1$.

- The Feynman rules of ϕ^4 theory are:



$\rightarrow i\Delta_F(x-y)$,

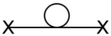
$\rightarrow i\Delta_F(0) = i\Delta_F(x-x)$,

$\rightarrow -i\lambda$ and integration over z

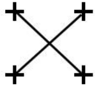
$\rightarrow iJ(x)$

- The generating functional can be calculated to any desired order.
- Making use of the Feynman rules, we write the generating functional to order g as

$$Z[J] = \Bigg[1 + \frac{(-i\lambda)}{4!}\int\Bigg(6$$



$$+$$

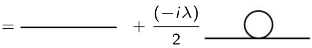


$$\Bigg)\mathrm{d}z\Bigg]\exp\Bigg(-\frac{i}{2}\int J\Delta_F J\Bigg).$$

- Let us again calculate the 2-point and 4-point functions:

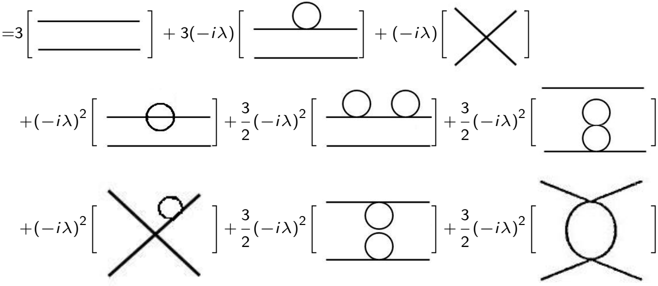
$$\tau(x_1, x_2) := -\left.\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0}$$

$$= i\Delta_F(x_1 - x_2) + \frac{(-i\lambda)}{2} i\Delta_F(0) \int \mathrm{d}z\, i\Delta_F(z - x_1) i\Delta_F(z - x_2) + \mathcal{O}(\lambda^2)$$



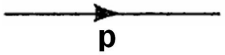
$+ \mathcal{O}(\lambda^2),$

$$\tau(x_1, x_2, x_3, x_4) := \left.\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}$$

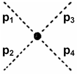


$+ \mathcal{O}(\lambda^3).$

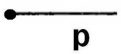
- As it is useful to work in momentum space, one can derive Feynman rules in momentum space:

for each propagator 

$= \dfrac{i}{p^2 - m^2 + i\epsilon}$ and $\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$

for each vertex 

$= -i\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3 + p_4)$

for each external point 

$= e^{-ipx}$

  - in the end only integration over independent momentas.

Symmetry properties

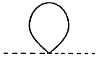
- Lagrangian has Z_2 symmetry: $\phi \to -\phi$
- Consequence: all n-point functions for odd n vanish.
- Generalization from one scalar field to a set of N real scalar fields:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 - \frac{1}{2}m^2(\phi^i)^2 - \frac{\lambda}{4!}[(\phi^i)^2]^2.$$

- This Lagrangian has a further symmetry: $O(N)$ symmetry.

n-point functions - Primitive divergences

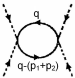
- Diverging contribution to 2-point function



$$= \lambda \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2}$$

is quadratically diverging.

- Diverging contribution to 4-point function



$$= \lambda^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\delta^{(4)}(q_1 + q_2 - p_1 - p_2)}{(q_1^2 - m^2)(q_2^2 - m^2)}$$

$$= \lambda^2 \int \frac{d^4 q}{(2\pi)^8} \frac{1}{(q^2 - m^2)((p_1 + p_2 - q)^2 - m^2)}$$

is logarithmically diverging.

n-point functions - Loop-Expansion

- It is important to note that we are performing perturbation theory. However, we are not performing perturbation series in λ but in the number of loops. Furthermore, we are then only considering the lowest order terms in λ as the coupling is assumed to be small.
- The reason to perform a expansion in the number of loop is that this expansion is equivalent to an expansion in $\hbar$, a diagram with L loop is of order $\hbar^{L-1}$. Thus, an expansion in loops is an expansion around the classical theory.

n-point functions - Superficial degree of divergence

- We now analyze the superficial degree of divergence of a particular diagram.
- Superficial means that this degree of divergence does not take internal divergences into account.
- The superficial degree of divergences D is given by

$$D = d - \left(\frac{d}{2} - 1\right)E + n\Big(d - 4\Big),$$

  for a diagram with n vertices and E external lines in d space-time dimensions.
- For $d = 4$, this simplifies to $D = 4 - E$, such that we have:
  - 2-point function has $E = 2$ and $D = 2$,
  - 4-point function has $E = 4$ and thus $D = 0$.

n-point functions - Dimensional analysis

- $[S] = [\int \mathrm{d}^d x \, \mathcal{L}] = 1$(in units with $\hbar = 1$) $\rightarrow [\mathcal{L}] = L^{-d} = M^d$
- $[\partial^\mu \phi \partial_\mu \phi] = L^{-d}$, $[\partial_\mu] = L^{-1} \rightarrow [\phi] = L^{1-d/2}$
- Considering the interaction term $\lambda\phi^4$ and supposing $[\lambda] = L^{-\delta} = M^\delta$, we find that $\delta = 4 - d$.
  $\rightarrow$ coupling constant λ in 4 dimensions is dimensionless.
- This is important $\rightarrow$ superficial degree of divergence:

$$D = d - \left(\frac{d}{2} - 1\right)E + n\left(d - 4\right) = d - \left(\frac{d}{2} - 1\right)E - n\delta.$$

- For a negative mass dimension, $\delta < 0$, the degree of divergence increases with increasing n.

Renormalization

- We have seen that we have diverging 2-point and 4-point diagrams.
- We will now see how we can renormalize our theory to give finite results for all measurable quantities:
  - mass m
  - coupling constant λ
  - field-strength ϕ
- So far, we have always considered bare quantities, which we will indicate in the following with a subscript B (m_B and λ_B).

- Let us start by considering the complete propagator

$$G^{(2)}(p) = \frac{i}{p^2 - m_B^2 - \Sigma(p)}.$$

- The pole of this propagator not at m_B anymore but at $\tilde{m}$ defined by

$$\tilde{m}^2 - m_B^2 - \Sigma(\tilde{m}) = 0.$$

- We can now expand the complete propagator around $\tilde{m}$,

$$G^{(2)}(p) = \frac{iZ}{p^2 - \tilde{m}^2} + \text{terms regular at } p^2 = \tilde{m}^2.$$

- Z is a probability amplitude and we should normalize this probability to 1.
- This can be done by rescaling the field and considering the renormalized field ϕ_r with

$$\phi = Z^{1/2}\phi_r.$$

- We can now include this new renormalized field into the Lagrangian and find

$$\mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_B^2 Z \phi_r^2 - \frac{\lambda_B}{4!} Z^2 \phi_r^4 .$$

- Still, bare quantities are appearing in our Lagrangian. By defining

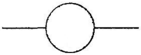
$$\delta_Z = Z - 1, \qquad \delta_m = m_B^2 Z - m^2, \qquad \delta_\lambda = \lambda_B Z^2 - \lambda,$$

we can write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4 \\ & + \frac{1}{2} \delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2} \delta_m \phi_r^2 - \frac{\delta_\lambda}{4!} \phi_r^4 . \end{aligned}$$

- $$\begin{aligned}\mathcal{L} = & \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ & + \frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_m \phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 .\end{aligned}$$
- First line: Same as in the original Lagrangian but now with the physical quantities
- Second line: *counterterms* of the same form.
- So far, we just split up the original terms in terms containing the physical observables and counterterms.
- We have not defined these quantities, so far. The conditions to define them are called *renormalization conditions*.

- First renormalization condition:
  We define the renormalized full propagator as



$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2)$$

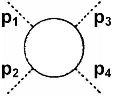
  which means that
  - we define the physical mass as the location of the pole of the propagator
  - we fix the residuum at this pole

- Second renormalization condition:
- Two definitions:
- The *4-point vertex function* is the full 4-point function with amputated legs:

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = G_c^{(4)}(p_1, p_2, p_3, p_4)(G_c^{(2)}(p_1))^{-1} (G_c^{(2)}(p_2))^{-1}(G_c^{(2)}(p_3))^{-1}(G_c^{(2)}(p_4))^{-1}.$$

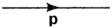
- It is useful to define the *Mandelstam variables*
  - $s = (p_1 + p_2)^2$
  - $t = (p_1 + p_3)^2$
  - $u = (p_1 + p_4)^2$

- Second renormalization condition:
  In contrast to the definition of the physical mass m, the definition of the coupling constant λ is not unique. We define a condition at a certain triple of Mandelstam variables (s, t, u), but we could do this also at any other triple:
-

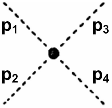


$$= -i\lambda \qquad \text{at } s = 4m^2,\ t = u = 0,$$

- New Feynman rules:
  Together with the new Lagrangian, we have now new Feynman rules:

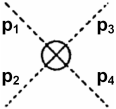


$$=\frac{i}{p^2-m^2}$$



$$=-i\lambda$$

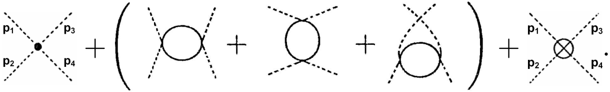
$$=i(p^2\delta_Z-\delta_m)$$



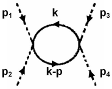
$$=-i\delta_\lambda$$

- If we now evaluate 2-point and 4-point functions with these Feynman rules, we will still find divergent diagrams.
- However, we can now adjust our counterterms in such a way that they cancel these diverging contributions and that the renormalization conditions are fulfilled.
- This procedure of using counterterms to renormalize a theory is known as *renormalized perturbation theory*.
- We will now just see how renormalization of ϕ^4 theory is done to one-loop order.

- Let us start with the second renormalization condition and consider the 4-point vertex function:

$\Gamma^{(4)} =$ 

- Using $p = p_1 + p_2$, we can write the second (diverging) contribution as



$$= \frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

$$\equiv (-i\lambda)^2 \cdot iV(p^2).$$

- Using the Mandelstam variables, we can write

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta_\lambda.$$

- Comparing this equation,

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta_\lambda,$$

  to the second renormalization condition $\Gamma^{(4)}(s = 4m^2, t = 0, u = 0) = -i\lambda$, we find

$$\delta_\lambda = -\lambda^2[V(4m^2) + 2V(0)].$$

- The divergent quantity $V(p^2)$ may be calculated using *dimensional regularization*, which means that we calculate the integral in d dimensions and then consider the limit $d \to 4$. It turns out that $V(p^2)$ has a simple pole in 4-d.
- However, it is important to see that the divergences in $V(s)$, $V(t)$, $V(u)$ and in δ_λ just cancel each other such that the 4-point vertex function is finite and fulfills the renormalization condition.

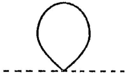
- To determine the counterterms δ_m and δ_Z, we consider the first renormalization condition:
  As the full two-point function may be written as

$$G^{(2)}(p) = \frac{i}{p^2 - m^2 - \Sigma(p^2)},$$

  the first renormalization condition (containing two conditions) is equal to the following two conditions:
  - $\Sigma(p^2)\Big|_{p^2=m^2} = 0$
  - $\frac{d}{dp^2}\Sigma(p^2)\Big|_{p^2=m^2} = 0.$

- Considering the self-energy to one-loop order

$$\frac{\Sigma(p^2)}{i} =$$



$$+$$

$$= -\frac{i\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m),$$

we see that the renormalization conditions are fulfilled, if

- $\delta_Z = 0$
- $\delta_m = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2}$,

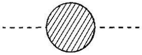
such that $\Sigma(p^2) = 0$ for all p^2 to one-loop order.

- Non-zero contributions to δ_Z and $\Sigma(p^2)$ and also further contributions to δ_m and δ_λ will appear in higher-loop order.
- The procedure is totally self-consistent:
  - In higher order perturbation theory, we will always include the counterterms according to the Feynman rules, on the one side.
  - And on the other side, we will have to add additional contributions to the counterterms in each order.
- It is proven that a theory is renormalizable if all divergencies can be canceled by counterterms of the same form as the original terms of the Lagrangian in every order of perturbation theory.

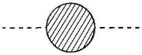
Callan-Symanzik equation

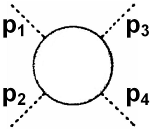
- We have now seen how the ϕ^4 theory can be renormalized.
- We will now derive a differential equation for the coupling constant, which determines how the coupling constant changes with changing momentas.
- We will again consider the ϕ^4 theory, however this differential equation will be valid for all dimensionless coupling theories.

- For simplicity, we assume that the mass has been adjusted to zero: $m^2 = 0$.
- In this case, we have to choose new renormalization conditions:



$= 0 \qquad$ at $p^2 = -M^2$

$\frac{d}{dp^2}\Big($  $\Big) = 0 \qquad$ at $p^2 = -M^2$



$= -i\lambda \qquad$ at $s = t = u = -M^2$

- The parameter M is called the *renormalization scale*.

- We have chosen M arbitrary, we could define the same theory at another scale M'. Same theory means that we have the same bare, unrenormalized Green's functions

$$\langle\Omega|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|\Omega\rangle.$$

- The renormalized Green's functions are related to the bare Green's function by ($\phi = Z^{1/2}\phi_r$)

$$\langle\Omega|T(\phi_r(x_1)\phi_r(x_2)\cdots\phi_r(x_n))|\Omega\rangle$$
$$= Z^{-n/2}\langle\Omega|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|\Omega\rangle.$$

- A shift in the renormalization scale would lead to a shift in the renormalized coupling constant and to a new rescaling factor:

$$\begin{aligned} M &\rightarrow M + \delta M, \\ \lambda &\rightarrow \lambda + \delta\lambda, \\ Z &\rightarrow Z + \delta Z, \end{aligned}$$

from which follows

$$\phi_r \rightarrow \left(1 + \frac{\delta Z}{Z}\right)\phi_r \rightarrow (1 + \delta\eta)\phi_r,$$

- As the Green's function $G^{(n)}$ contains n field components:

$$G^{(n)} \rightarrow (1 + \delta\eta)^n G^{(n)} \approx (1 + n\delta\eta) G^{(n)}$$

- All these shifts are related:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)},$$

- From this we can derive the Callan-Symanzik equation

$$\boxed{\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] G^{(n)}(\{x_i\}; M, \lambda) = 0}$$

with the dimensionless parameters

$$\beta = \frac{M}{\delta M}\delta\lambda \qquad \gamma = -\frac{M}{\delta M}\delta\eta.$$

- The Callan-Symanzik equation is very useful.
  - We can apply it to Green's function of a certain order.
  - From this, we can determine the β- and γ-function.
- The β-function is of huge importance, because it determines the development of the coupling constant with changing momentum scale.
- Without derivation, we state that the β-function of the ϕ^4 to leading order is given as

$$\boxed{\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)}$$

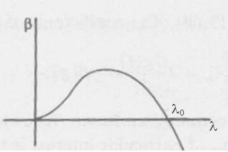
β-function and triviality of ϕ^4 theory

- Let us reconsider the definition of the β-function,

$$\beta(\lambda) = M \frac{\partial \lambda}{\partial M},$$

  which determines the behavior of the coupling constant with changing momentum scale M.
- This *flow* of the coupling constant is the reason for speaking from a *running coupling*.
- Let us first consider two examples of possible behaviors of the running coupling.

- Suppose that $\beta(\lambda)$ has the form



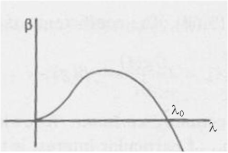
- There are two zeros of the β-function at 0 and λ_0.
- Considering a value of λ below λ_0, we have

$$M\frac{\partial\lambda}{\partial M} > 0$$

  and λ moves towards λ_0 with increasing momentum.
- Considering a coupling $\lambda > \lambda_0$, we have

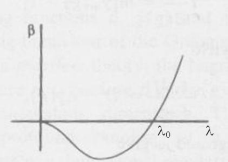
$$M\frac{\partial\lambda}{\partial M} < 0$$

  and λ decreases towards λ_0 with increasing momentum.



- Thus, λ_0 is an *ultra-violet stable fixed point*.
- On the other hand, $\lambda = 0$ is an *infra-red stable fixed point*.

- If we consider the following behavior



we again have two fixed points.

- If we consider a coupling λ between zero and λ_0, we have for decreasing momentum an increasing coupling and thus an infra-red stable fixed point at λ_0.
- On the other side, for increasing momentum λ tends towards zero and thus for large momenta the coupling constant vanishes.
- This behavior is known as *asymptotic freedom*, which will be of interest later on.

- These two exemplary considerations have the purpose to show how the β-function determines the running coupling.
- We now turn back to the situation in ϕ^4 theory, where we found

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$

- For small λ the behavior of the β-function is determined by the quadratic λ term and the coupling constant is increasing with increasing momenta.
- Question: Is there a nontrivial zero of the β-function?

- Question: Is there a nontrivial zero of the β-function?
- This cannot be examined in perturbation theory for increasing λ.
- Possibility: Consider the ϕ^4 theory on a lattice and do numerical calculations.
- From this examination, we can conclude that there is no ultra-violet fixed point in ϕ^4 theory.
- This is known as the *triviality* of ϕ^4 theory: coupling constant λ grows with growing momentas.

Nonlinear Sigma Model

- We consider N scalar fields ϕ^i with a Lagrangian
$$\mathcal{L} = f_{ij}(\{\phi^l\})\partial_\mu\phi^i\partial^\mu\phi^j.$$
- Dimensional analysis $\rightarrow$ dimensionless coupling constants $\rightarrow$ theory renormalizable for any possible function $f_{ij}(\{\phi^l\})$.
- Restrict the scalar fields ϕ^i form a N-dim. unit vector, $\phi^i = n^i(x)$ with a $O(N)$-symmetry of the field components.
- Restricted by those conditions the most general choice of $f_{ij}(\{\phi^l\})$ is a constant and the most general Lagrangian is
$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu\vec{n}|^2,$$
where g is the coupling constant.

- We can parametrize our field by

$$
\begin{aligned}
n^i =& (\pi^1, \ldots, \pi^{N-1}, \sigma) \\
\text{where} \quad \sigma =& (1 - \vec{\pi})^{1/2}.
\end{aligned}
$$

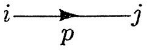
- Configuration with $\pi^k = 0$
  $\rightarrow$ state of spontaneous symmetry breaking in N direction.
- Using this parametrization, we find

$$
\mathcal{L} = \frac{1}{2g^2} \left[|\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{1 - \vec{\pi}^2} \right],
$$

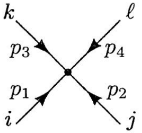
  which can be expanded in powers of π^k

$$
\mathcal{L} = \frac{1}{2g^2} |\partial_\mu \vec{\pi}|^2 + \frac{1}{2g^2} (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \frac{1}{2g^2} \vec{\pi}^2 (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \ldots .
$$

Feynman rules



$$= \frac{ig^2}{p^2}\delta^{ij}$$



$$= -\frac{i}{g^2}\big[(p_1+p_2)\cdot(p_3+p_4)\delta^{ij}\delta^{kl} + (p_1+p_3)\cdot(p_2+p_4)\delta^{ik}\delta^{jl} + (p_1+p_4)\cdot(p_2+p_3)\delta^{il}\delta^{jk}\big]$$

and additional vertices for all even numbers of π^k fields.

Callan-Symanzik equation and β-function

- As we have dimensionless coefficients in our Lagrangian, this theory can be made finite by renormalization of the coupling constant g and rescaling of the fields π^k and σ.
- Instead of going through the whole renormalization procedure, we can make use of the fact that our renormalizable theory has to fulfill the Callan-Symanzik equation for some functions β and γ.
- But let us just write down the β-function and discuss its significance.

- The β-function for $d = 2$ is given as

$$\beta(g) = -\frac{(N-2)g^3}{4\pi} + \mathcal{O}(g^5).$$

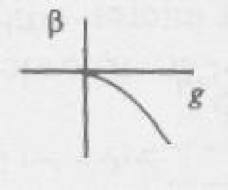
- Obviously the β-function depends on N.
- For $N = 2$ the β-function vanishes exactly (not only to order g^3).
  This is obvious, because in this case we can parametrize $\pi^1 = \sin\theta$ and thus $\sigma = \cos\theta$ and the Lagrangian simplifies considerably:

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu \sin\theta|^2 + \frac{1}{2g^2}\frac{(\sin\theta \cdot \partial_\mu \sin\theta)^2}{1-(\sin\theta)^2} = \frac{1}{2g^2}(\partial_\mu\theta)^2.$$

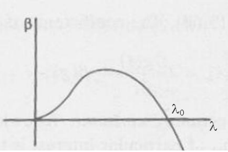
- For $N > 2$, the β-function

$$\beta(g) = -\frac{(N-2)g^3}{4\pi} + \mathcal{O}(g^5)$$

is negative and the theory is asymptotically free, which means that the coupling constant goes to zero as the momentum becomes large.



- Asymptotic freedom of the nonlinear σ-model is restricted $d = 2$.
- For higher dimensions ($2 < d < 4$), there is an ultra-violet stable fixed point which tends towards zero for $d \to 2$.



QCD - Parton model

- Quantum chromodynamics describes the strong interactions between the constituents of the nuclei, which are responsible for nuclear bonding.
- At first, strong interactions showed mysterious properties which could not be described by common field theories (before the developement of QCD).
- For example, interactions turn themselves off for large momentas (small displacements).
- It was recognized that this requires asymptotic freedom.
- As non-Abelian gauge theories are asymptotically free in four-dimensional space-time, they are possible candidates for theories describing strong interactions.

- *Parton model* is a model put forward by Bjorken and Feynman:
  Describes the proton as a loosely bound assemblage of a small number of constituents, called *partons*.
- If one compares the parton model to QCD, partons are *quarks*, charged fermions, and also neutral species responsible for the binding, called *gluons*.
- These gluons are included in the description by QCD as vector gauge bosons.

β-function of QCD

- The Lagrangian of QCD is the famous Yang-Mills Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2,$$

with the field strength tensor of the gauge bosons

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu,$$

where A^a_μ is a component of the gauge boson field (gluon field, $a \in \{1, \cdots, 8\}$) and f^{abc} is the structure constant of the gauge symmetry, in the case of QCD $SU(3)$.

- Without going in any details of calculation, we state that the β-function in the case of a $SU(N)$ gauge theory with n_f different fermions is given to leading order as

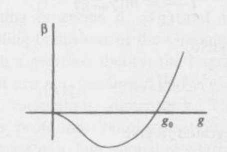
$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right).$$

- Sign of the β-function depends on the ratio of the number of fermions n_f and N (from the symmetry $SU(N)$), thus we have $N = 3$ for QCD.
- For a small enough number n_f, β is negative and the theory is asymptotically free, which is for QCD the case if $n_f \leq 16$. (there are 6 flavours: up, down, strange, charm, bottom, top)

- Using the β-function, the running coupling can be calculated to

$$g^2(k) = \frac{g_0^2}{1 + \frac{g_0^2}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)\log\left(\frac{k^2}{M^2}\right)},$$

which tends to zero at large momentum.
- This means that the theory is asymptotically free.
- As we have mentioned before, this is an necessary condition to describe the strong interactions.



- Let us discuss the behavior of the running coupling in more detail.
- In electrodynamics:
- The vacuum behaves as a dielectric medium due to electron-positron pair creation, which decreases the effective charge of the electron and thus the coupling at large distances.
- In non-Abelian gauge theories, the fermions still produce such an effect (positive contribution to the β-function)
- However, the non-Abelian gauge bosons produce a dominating antiscreening effect.

- To understand this effect, we study a simplified example:
- Coulomb gauge: $\partial_i A^{ai} = 0$
- Considering the Coulomb potential of the field A^{a0} described by an analogue of Gauss's law in this non-Abelian case with covariant form

$$D_i E^{ai} = g\rho^a,$$

where the covariant derivative acting on a field in adjoint representation is defined as

$$(D_\mu \phi)^a = \partial_\mu \phi^a + gf^{abc} A^b_\mu \phi^c,$$

$E^{ai} = F^{a0i}$ and ρ^a is the charge density of the fermions, where a is a index for the *color* of charge.

- To make a further simplification, we choose $SU(2)$-symmetry, because in this case the structure constant simplifies to $f^{abc} = \epsilon^{abc}$:

$$(D_\mu \phi)^a = \partial_\mu \phi^a + g\epsilon^{abc} A_\mu^b \phi^c.$$

- We now want to compute the Coulomb potential of a point charge of magnitude $+1$ with orientation (color) $a = 1$.

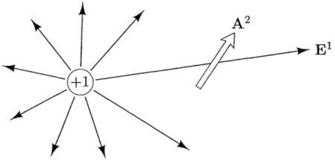
- We want to solve iteratively for E^{ai}.
- First we rewrite the equation as

$$\partial_i E^{ai} = g\delta^{(3)}(x)\delta^{a1} + g\epsilon^{abc}A^{bi}E^{ci}$$

- In this non-Abelian theory,
  - not only a charge density,
  - but also the common presence of a vector potential and a electric field

  is a source of electric fields.

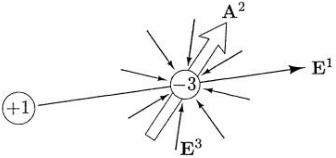
- The first term implies a $1/r^2$ electric field of color $a = 1$ radiating from $x = 0$.
- We now consider a point in space where this field crosses a bit of vector potential A^{bi} arising as fluctuation of the vacuum, assume A^{2i} which points in some diagonal direction to the electric field.



- If we now consider $a = 3$,

$$\partial_i E^{3i} = g\epsilon^{321} A^{2i} E^{1i} = -gA^{2i} E^{1i} < 0,$$

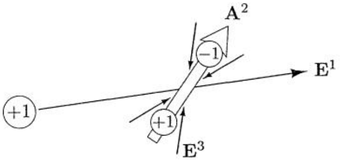
we find a sink of the field E^{3i} at this location:



- Considering now the influence of this field E^{3i} on the field E^{1i}, we find

$$\begin{aligned}\partial_i E^{1i} =& g\delta^{(3)}(x) + g\epsilon^{123}A^{2i}E^{3i} \\ =& g\delta^{(3)}(x) + gA^{2i}E^{3i}.\end{aligned}$$

- We have to consider the orientation of A^{2i} and E^{3i} in more detail: We see that closer to the origin the fields are parallel and thus there is a source for E^{1i}, farther away, the fields are antiparallel and thus there is a sink.



- This is an induced electric dipole which is oriented with the positive charge towards the original charge.
- Thus, this amplifies the original charge instead of screening it and therefore the effect of the charge gets stronger at larger distances.
- Comparing screening and antiscreening effects, one can find that antiscreening is 12 times larger.
- This simplified example should just show how such antiscreening can occur.
- Antiscreening leads to the effect of confinement.

- This antiscreening originates from the second term of the covariant derivative which is peculiar for a non-Abelian gauge theory.
- So the coupling constant grows at large distances for non-Abelian gauge theories.
- This is the one direction of coupling constant flow, in the other direction asymptotic freedom occurs.

Acknowledgment

- I want to thank first of all my tutor Michael Fromm for his extensive assistance and some cups of coffee.
- I also want to thank Dr. Philippe de Forcrand for his support and his advices.
- And last but not least, I want to thank you all for your attention.