Asymptotic freedom and the beta-function: ϕ^4, 2d σ-model, QCD

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Contents

References

- M. E. Peskin and D. V. Schroeder, An Introduction To Quantum Field Theory, Reading, USA: Addison-Wesley (1995)
- L. H. Ryder, Quantum Field Theory, Cambridge, Uk: Univ. Pr. (1985)

1 Introduction

Renormalization is one of the most important concepts of quantum field theories. Many quantum field theories contain divergences and to obtain any reasonable results, one has to find a way to handle these divergences. Renormalization of a quantum field theory is the procedure of dealing with these divergences to get finite results for observable physical quantities occurring in the theory. We will investigate the concept of renormalization in the framework of ϕ^4 theory. Along the way, we will encounter the concept of a *running coupling* $\lambda(p)$, which means that λ depends on the momentum scale, p. This dependence is not arbitrary but happens in a well-defined way through the *β-function* $\beta(\lambda)$. The development of the β-function not only determines the behavior of $\lambda(p)$ with varying p but also the limits $p \to \infty$ and $p \to 0$. If a coupling constant goes to zero for large momentas, this is called *asymptotic freedom*, which is a further very important concept. Afterwards, we will consider the nonlinear σ-model as an example of an asymptotically free theory. In the end, we will point out the consequences of asymptotic freedom and renormalization on quantum chromodynamics. Let us start with an introduction of some formalism and definitions.

1.1 Generating functionals and n-point functions

A free scalar field ϕ describing the free propagation of particles is described by the Klein-Gordon Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

As we have seen in the first talk 'Euclidean path integral formalism'[1], in this case of a free field the generating functional of n-point functions is given by

$$Z_0[J] = \exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\Big], \tag{2}$$

where $J(z)$ is a source of the field $\phi(z)$ and Δ_F is the Feynman propagator, obeying

$$(\Box + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x). \tag{3}$$

This equation can easily be calculated inverted in momentum space which leads to

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} \tag{4}$$

[1] But we are considering Minkowski space expressions here instead of Euclidean ones.

We can expand Eq. (2)

$$
\begin{aligned}
Z_0[J] = 1 &+ \Big(-\frac{i}{2}\Big) \int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y \\
&+ \frac{1}{2!}\Big(-\frac{i}{2}\Big)^2 \Big[\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\Big]^2 \\
&+ \frac{1}{3!}\Big(-\frac{i}{2}\Big)^3 \Big[\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\Big]^3 + \cdots ,
\end{aligned}
\tag{5}
$$

which can represented diagrammatically using the rules

$$
\mathsf{x}\,\rule[0.5ex]{5em}{0.4pt}\,\mathsf{y} \;= i\Delta_F(x-y), \tag{6}
$$

$$
\times \;= iJ(z), \tag{7}
$$

as

$$
\begin{aligned}
Z_0[J] =1 &+ \Big(\frac{1}{2}\Big) \int \times\!\rule[0.5ex]{5em}{0.4pt}\!\times \; \mathrm{d}^4x_1\mathrm{d}^4y_1 \\
&+ \frac{1}{2!}\Big(\frac{1}{2}\Big)^2 \int \begin{matrix}\times\!\rule[0.5ex]{5em}{0.4pt}\!\times\\ \times\!\rule[0.5ex]{5em}{0.4pt}\!\times\end{matrix} \; \mathrm{d}^4x_1\mathrm{d}^4y_1\mathrm{d}^4x_2\mathrm{d}^4y_2 \\
&+ \frac{1}{3!}\Big(\frac{1}{2}\Big)^3 \int \begin{matrix}\times\!\rule[0.5ex]{5em}{0.4pt}\!\times\\ \times\!\rule[0.5ex]{5em}{0.4pt}\!\times\\ \times\!\rule[0.5ex]{5em}{0.4pt}\!\times\end{matrix} \; \mathrm{d}^4x_1\mathrm{d}^4y_1\mathrm{d}^4x_2\mathrm{d}^4y_2\mathrm{d}^4x_3\mathrm{d}^4y_3 + \cdots ,
\end{aligned}
\tag{8}
$$

where x_i and y_i label the external points (sources). Having this diagrammatic representation, it is straightforward to evaluate the n-point functions defined as

$$
\tau(x_1,\ldots,x_n) := \frac{1}{i^n}\frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}\Big|_{J=0}. \tag{9}
$$

These functional derivatives can be calculated from Eq. (2). But we can also see the impact of these derivatives in the diagrammatic expansion as canceling one cross and evaluating the external point at a certain point, which can be seen in the following most simple example:

$$
\Big(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big)\Big(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}^4x\mathrm{d}^4y\Big) = \int i\Delta_F(x_1-z)iJ(z)\mathrm{d}^4z
$$

$$
\Big(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big)\Big(\frac{1}{2}\int \times\!\rule[0.5ex]{5em}{0.4pt}\!\times \; \mathrm{d}^4x\mathrm{d}^4y\Big) = \int {\scriptstyle \mathsf{x}_1}\,\rule[0.5ex]{5em}{0.4pt}\!\times \mathrm{d}^4z
$$

Let us consider the example of the two-point function $\tau(x,y)$, we do two derivativens and set then $J=0$. Therefore, each diagram which contains still a cross labelling a source is zero. As we have two derivatives, only the diagram with one line and two crosses gives a contribution. If we take symmetry into account, we find directly

$$
\tau(x,y) = \mathbf{x}\,\rule[0.5ex]{4em}{0.4pt}\,\mathbf{y} \;= i\Delta_F(x-y). \tag{10}
$$

So the 2-point function for free particles is a propagation of a particle between the two points. To see this in more detail, we may use that

$$\tau(x_1, \ldots, x_n) = \langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle, \tag{11}$$

which was also introduced in the first talk 'Euclidean path integral formalism'. Using this identity for the 2-point function, we see that the 2-point function is equal to the vacuum-expectation value of the time-ordered (T) product of the field at x and y. This can be written as

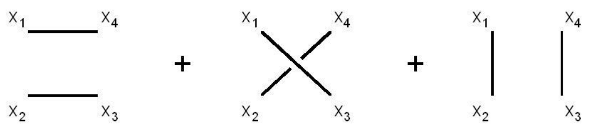
$$\tau(x, y) =\theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi(y)\phi(x)|0\rangle \tag{12}$$

and decomposing the field into a creating ($\phi^{(-)}$) and an annihilating part ($\phi^{(+)}$), $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, as

$$\tau(x, y) =\theta(x_0 - y_0)\langle 0|\phi^{(+)}(x)\phi^{(-)}(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi^{(+)}(y)\phi^{(-)}(x)|0\rangle, \tag{13}$$

from this we can interpret that the particle gets created at the earlier time at the corresponding coordinate and propagates to the later space-time point and gets annihilated there.
If we consider the one-point function $\tau(x)$, diagrammatically one cross gets cancelled and by setting $J = 0$ everything vanishes because in every term there is still a cross present.
Considering the 4-point function, we can find

$$\tau(x_1, x_2, x_3, x_4) = \tag{14}$$



$$\begin{aligned}
&=(i\Delta_F(x_1 - x_4))(i\Delta_F(x_2 - x_3)) + (i\Delta_F(x_1 - x_3))(i\Delta_F(x_2 - x_4)) \\
&\quad + (i\Delta_F(x_1 - x_2))(i\Delta_F(x_3 - x_4)) \\
&=\tau(x_1, x_4)\tau(x_2, x_3) + \tau(x_1, x_3)\tau(x_2, x_4) + \tau(x_1, x_2)\tau(x_3, x_4).
\end{aligned}$$

We see that in the free particle case the 4-point function is a sum of products of 2-point function, this is generalized by *Wick's theorem*:

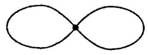
$$\tau(x_1, \ldots, x_{2n}) = \sum_{perms} \tau(x_{p_1}, x_{p_2}) \cdots \tau(x_{p_{2n-1}}, x_{p_{2n}}). \tag{15}$$

So far, we did only consider the free field case with free propagation of particles. However, more interesting things appear not until interactions are present. If we have interactions described by a Lagrangian $\mathcal{L}_{int}(\phi)$, one can derive[2] that the generating functional is given by

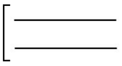
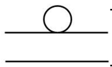
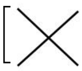
$$Z[J] = N \exp\left(i \int \mathcal{L}_{int}\Big(\frac{1}{i}\frac{\delta}{\delta J(x)}\Big) \mathrm{d}^4x \right) Z_0[J], \tag{16}$$

[2] Lewis H. Ryder, Quantum field theory, p.196ff

where N is a normalization constant and the generating functional is constructed by functional derivatives acting on the free generating functional $Z_0[J]$. We do not investigate this now any further but refer to the following section, Sec. 2.1, where we consider this generating functional for the ϕ^4 interaction Lagrangian.
However, it is important to point out the following: A not normalized generating functional contains *vacuum diagrams*, which are diagrams without external legs as for example:



In contrast, in a normalized generating functional all these vacuum diagrams cancel out. Therefore, a normalized generating functional contains only diagrams with external legs. If we consider the 4-point function of ϕ^4 theory to first order, which we will reconsider in the next section, we recognize that there are two kinds of diagrams:

$\tau(x_1, x_2, x_3, x_4) = 3\,[$  $] + 3(-i\lambda)\,[$  $] + (-i\lambda)\,[$  $]$. (17)

While in the third diagram all external points are connected (*connected diagram*), in the first two diagrams not all external points are connected with each other and thus such diagrams are called *disconnected diagrams.*
The disconnected diagrams are not that important as they describe the independent propagation of two particles without scattering. As we are more interested in connected diagrams, it is useful to introduce another generating functional W, which only generates the connected contributions to the n-point functions, defined by

$$Z[J] = e^{iW[J]} \quad \text{or} \quad W[J] = -i \ln Z[J]. \tag{18}$$

We can define in an analogous way to the n-point functions $\tau(x_1, \dots, x_n)$, Eq. (9), the *irreducible n-point functions* $\phi(x_1, \dots, x_n)$ by

$$\phi(x_1, \dots, x_n) = \frac{1}{i^n} \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)}\Big|_{J=0}. \tag{19}$$

It can be proven generally[3] that $\phi(x_1, \dots, x_n)$ contains exactly the irreducible (connected) contributions from $\tau(x_1, \dots, x_n)$. Let us just consider the 2-point and 4-point functions to make this reasonable:

$$\phi(x_1, x_2) = \frac{1}{i^2} \frac{\delta^2 W[J]}{\delta J(x_1) \delta J(x_2)}\Big|_{J=0} = \left(\frac{i}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} - \frac{i}{Z[J]^2} \frac{\delta Z[J]}{\delta J(x_1)} \frac{\delta Z[J]}{\delta J(x_2)} \right)\Big|_{J=0},$$

as we have $\quad \dfrac{\delta Z[J]}{\delta J(x)}\Big|_{J=0} = 0 \quad$ and $\quad Z[0] = 1$,

$$\phi(x_1, x_2) = i \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)}\Big|_{J=0} = -i\tau(x_1, x_2). \tag{20}$$

[3]J. Zinn-Justin, (1997), section 6.1.1.

Thus, we find $i\phi(x_1, x_2) = \tau(x_1, x_2)$, which means that the 2-point functions are equal what is obvious as there are no disconnected two-point diagrams possible. Calculating the same for the 4-point function, we find

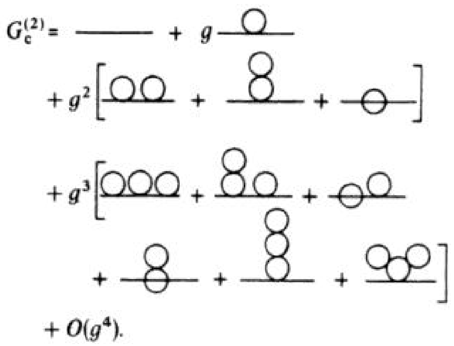
$$\begin{aligned} i\phi(x_1, x_2, x_3, x_4) =& \tau(x_1, x_2, x_3, x_4) - \tau(x_1, x_2)\tau(x_3, x_4) \\ & - \tau(x_1, x_3)\tau(x_2, x_4) - \tau(x_1, x_4)\tau(x_2, x_3), \end{aligned} \tag{21}$$

where we see now that from the whole 4-point function $\tau(x_1, x_2, x_3, x_4)$ the individual propagations of two particles (e.g. $\tau(x_1, x_2)\tau(x_3, x_4)$) have been subtracted to obtain the irreducible part. This happens generally for any n-point function such that $\phi(x_1, \ldots, x_n)$ contains exactly all connected diagrams of $\tau(x_1, \ldots, x_n)$. It is also covenient to talk about Green's functions, which have a simple relation to the n-point functions:

$$G^{(n)}(x_1, \ldots, x_n) := \tau(x_1, \ldots, x_n), \tag{22}$$

$$G_c^{(n)}(x_1, \ldots, x_n) := i\phi(x_1, \ldots, x_n), \tag{23}$$

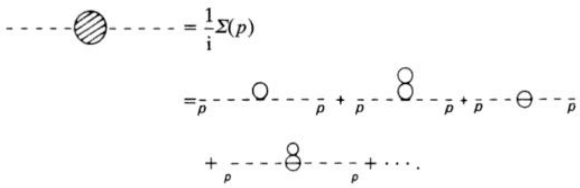
where the subscript c just stands for *connected.* We now introduce one further classification. Instead of introducing this very general, we just consider an example to see the two types of diagrams to be classified. We again use an example of ϕ^4 theory, which we will consider in the next section in detail, namely the connected two-point function[4]



As we can see there are diagrams, which can be built up by diagrams of lower order, e.g. the first diagram of order g^2 is just twice the diagram of order g, the first diagram of order g^3 is just three times the order g diagram, the following diagrams of order g^3 are obviously also built up from lower order diagrams, while the last three diagrams of order g^3 can not be divided into lower order diagrams. We call the first ones *1-particle reducible* graphs, which have the property that they can be divided into two subdiagrams by cutting one internal line while the *1-particle irreducible* (1 PI) graphs may not be divided. It is in general true that 1-particle reducible diagrams can be built up by 1-particle irreducible diagrams. To see that, we define the *self-energy* part[5] as the sum of all 1 PI graphs:

[4]Coupling constant g instead of λ.

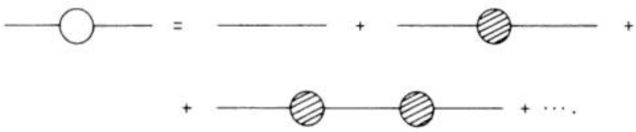
[5]The *self-energy* is a fundamental concept and will be of interest later.



Using the bare propagator $G_0(p) = i/(p^2 - m^2)$ and the self-energy function $\Sigma(p)$, we can write the complete propagator (two-point function) as

$$
\begin{aligned}
G_c^{(2)}(p) =& G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \cdots \\
=& G_0\Big(1 + \frac{\Sigma(p)}{i}G_0 + \frac{\Sigma(p)}{i}G_0\frac{\Sigma(p)}{i}G_0 + \cdots\Big) \\
=& G_0\Big(1 - \frac{\Sigma(p)}{i}G_0\Big)^{-1} \\
=& \Big[G_0^{-1}(p) - \frac{\Sigma(p)}{i}\Big]^{-1} = \frac{i}{p^2 - m^2 - \Sigma(p)}.
\end{aligned}
\tag{24}
$$

It is remarkable to see that the complete propagator is the same as the bare propagator simply corrected by the self-energy part. This shows the importance of the self-energy part, because normally the physical mass is defined as the pole of the propagator and thus the self-energy modifies the bare mass to a physical, measurable mass. To make clear again that this simple result includes all possible diagrams, we can write the full propagator diagrammatically as a power series:



where it should be easy to see that all the combinations from the sums in the self-energy part give all possible diagrams.
Now, that we have introduced the whole formalism of n-point functions, generating functionals and self-energy, let us start with the consideration of ϕ^4 theory.

2 ϕ^4 theory

2.1 General properties

The ϕ^4 theory is one of the simplest theories describing a scalar field but still it shows many interesting features occuring also in more complex theories. We will consider the concept of renormalization in detail at the ϕ^4 theory. The name of this theory comes from the form of the interaction term in the Lagrangian

$$
\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4,
\tag{25}
$$

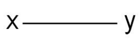
where λ is a coupling constant, the factor 4! is due to symmetry reasons and the ϕ to the fourth power (quartic[6] interaction) leads to interaction which involves four times the field (this property we will also see explicitly in the Feynman diagrams where at the interaction vertices always four lines meet).
The whole Lagrangian is given by this interaction part and the free field Lagrangian, Eq. (1)

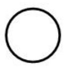
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{26}$$

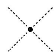
Considering now the whole Lagrangian, we see that it is important to have a positive coupling constant to have a repulsive interaction potential, because for a negative λ the generating functional Z would not converge as $\phi \to \infty$.
As we have seen in Sec. (1.1), to calculate n-point functions we need the generating functional. The normalized generating functional is given by Eq. (16) as

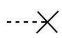
$$Z[J] = \frac{\exp[i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})\mathrm{d}z]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y]}{\{\exp[i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})\mathrm{d}z]\exp[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y]\}\Big|_{J=0}}. \tag{27}$$

This expression is calculated to order g in App. A. To write down this expression, we use the diagrammatic rules in coordinate space

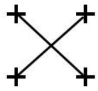
 $\to i\Delta_F(x-y),$ (28)

 $\to i\Delta_F(0) = i\Delta_F(x-x),$ (29)

 $\to -i\lambda$ and integration over z (30)

 $\to iJ(x)$ (31)

The generating functional is then given by

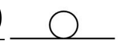
$Z[J] = \Big[1 + \frac{(-i\lambda)}{4!}\int \Big(6$  $+$  $\Big)\mathrm{d}z\Big]\exp\Big(-\frac{i}{2}\int J\Delta_F J\Big).$ (32)

The 2-point function, defined as

$$\tau(x_1, x_2) := -\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0}, \tag{33}$$

may be calculated to

$$\tau(x_1, x_2) = i\Delta_F(x_1 - x_2) + \frac{(-i\lambda)}{2}i\Delta_F(0)\int \mathrm{d}z\, i\Delta_F(z-x_1)i\Delta_F(z-x_2) + \mathcal{O}(\lambda^2)$$

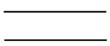
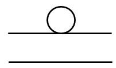
$=$ $+ \frac{(-i\lambda)}{2}$  $+ \mathcal{O}(g^2),$ (34)

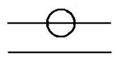
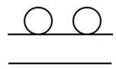
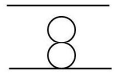
[6]quartic=bi-quadratic

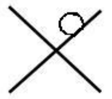
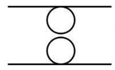
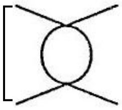
which is done in detail in App. A. Analogously, the 4-point function, defined as

$$\tau(x_1, x_2, x_3, x_4) := \left. \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \right|_{J=0}, \tag{35}$$

can be calculated as

$\tau(x_1, x_2, x_3, x_4) = 3\Big[$  $\Big] + 3(-i\lambda)\Big[$  $\Big] + (-i\lambda)\Big[$  $\Big]$ (36)

$+(-i\lambda)^2\Big[$  $\Big] + \frac{3}{2}(-i\lambda)^2\Big[$  $\Big] + \frac{3}{2}(-i\lambda)^2\Big[$  $\Big]$

$+(-i\lambda)^2\Big[$  $\Big] + \frac{3}{2}(-i\lambda)^2\Big[$  $\Big] + \frac{3}{2}(-i\lambda)^2\Big[$  $\Big]$

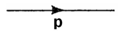
$+ \mathcal{O}(\lambda^3)$

As we see, the first two diagrams are disconnected and thus of less importance while the third one gives us the first scattering contribution, which is just the simple 4-vertex interaction characteristic for ϕ^4 theory. We will be considering the last term of second order in much details in the context of renormalization because this term is divergent.
It is usful to work in momentum space rather than in coordinate space. We consider the simple 4-vertex diagram to see how to transform to Fourier space:

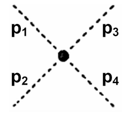
$-i\lambda$ 

$$\begin{aligned} &= -i\lambda \int \mathrm{d}^4 z\, i\Delta_F(x_1 - z) i\Delta_F(x_2 - z) i\Delta_F(x_3 - z) i\Delta_F(x_4 - z) \\ &= -i\lambda \int \mathrm{d}^4 z \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} i\Delta_F(p_1) e^{-ip_1(x_1 - z)} \int \frac{\mathrm{d}^4 p_2}{(2\pi)^4} i\Delta_F(p_2) e^{-ip_2(x_2 - z)} \\ &\qquad \int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} i\Delta_F(p_3) e^{-ip_3(x_3 - z)} \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} i\Delta_F(p_4) e^{-ip_4(x_4 - z)} \\ &= -i\lambda \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \frac{\mathrm{d}^4 p_2}{(2\pi)^4} \frac{\mathrm{d}^4 p_3}{(2\pi)^4} \frac{\mathrm{d}^4 p_4}{(2\pi)^4}\, i\Delta_F(p_1) e^{-ip_1 x_1}\, i\Delta_F(p_2) e^{-ip_2 x_2} \\ &\qquad i\Delta_F(p_3) e^{-ip_3 x_3}\, i\Delta_F(p_4) e^{-ip_4 x_4}\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \end{aligned}$$

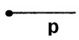
Using the expression for $\Delta_F(p)$, Eq. (4), we can read off the Feynman rules in momentum space:

for each propagator 

$\to \frac{i}{p^2 - m^2 + i\epsilon}$ and $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$,

for each vertex 

$\to -i\lambda(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4)$, (37)

for each external point 

$\to e^{-ipx}$,

where the delta function takes care of energy-momentum conservation at each vertex. These delta functions together with the integrations over all momentas reduce to integrations over independent momentas.

2.2 Symmetry properties

Before analyzing a physical system any further, it is always important to consider the symmetry properties. If we consider the Lagrangian of our ϕ^4 theory, we recognize that it has a global Z_2 symmetry mapping

$$\phi \to -\phi. \tag{38}$$

One consequence of this is that all n-point functions for odd n vanish:

$$\begin{aligned}\tau(x_1, \ldots, x_n) =& \langle 0|T(\phi(x_1)\ldots\phi(x_3))|0\rangle \\ =& \langle 0|T((-\phi(x_1))\ldots(-\phi(x_3)))|0\rangle \\ =& (-1)^n \langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3))|0\rangle = 0 \quad \text{for odd } n.\end{aligned} \tag{39}$$

The generalization from one scalar field to a set of N real scalar field, has a Lagrangian of the form

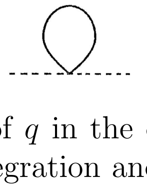
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 - \frac{1}{2}m^2(\phi^i)^2 - \frac{\lambda}{4!}[(\phi^i)^2]^2. \tag{40}$$

It is obvious that this Lagrangian has a further symmetry, namely $O(N)$ symmetry.

2.3 n-point functions

2.3.1 Primitive divergences

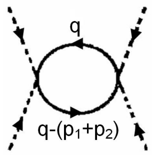
We will now consider two important diagrams which contribute to the 2- resp. 4-point function. As we will see, their contribution is divergent and we will examine how to deal with those divergences. We call these divergences *primitive* as those diagrams diverge per se. In contrast, we will see diagrams which diverge only because they contain diverging primitive subdiagrams and would not diverge without these diverging subdiagrams.
Starting with the 2-point function, Eq. (34), we see that the first modification from the free particle propagation is the one-loop diagram



$$= \lambda \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2}, \tag{41}$$

which has two powers of q in the denominator and obtains four powers of q in the numerator from integration and therefore is quadratically diverging at large

q (ultra-violet divergence).
Considering the 4-point function, Eq. (36), we recognize that there is no divergent diagram to order λ, but one can find the first diverging diagram of order λ^2, also given by a divergent one-loop diagram



$$
\begin{aligned}
&= \lambda^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\delta^{(4)}(q_1 + q_2 - p_1 - p_2)}{(q_1^2 - m^2)(q_2^2 - m^2)} \\
&= \lambda^2 \int \frac{d^4 q}{(2\pi)^8} \frac{1}{(q^2 - m^2)((p_1 + p_2 - q)^2 - m^2)}.
\end{aligned} \tag{42}
$$

This expression is logarithmically diverging at large q because there are four powers of q both in the denominator and in the numerator (from integration).
We considered exactly these two diagrams because they are the divergent diagrams which are lowest order in the coupling constant λ. Although higher order diagrams also diverge, their contribution is small compared to the ones above due to the higher order in the coupling constant λ.

2.3.2 Loop-Expansion

However, we should now remark the following: If we compare the additional contributions we have calculated, we see that they are of different order in λ. The contribution to the 2-point function is of order λ, which means that the correction is $(1 + \mathcal{O}(\lambda))$. The additional contribution to the 4-point function is of order λ^2, but the correction is also $(1 + \mathcal{O}(\lambda))$. But although we are considering different order λ contribution, we are considering identical order contribution in the number of loop. It is important to note that an expansion in the number of loops has more physical relevance than an expansion in powers of λ. As could be shown, a diagram with L loops would be of order $\hbar^{L-1}$ if we would not set $\hbar = 1$. So an expansion in loops is equal to an expansion in powers of $\hbar$ which is just an expansion around the classical theory.

2.3.3 Superficial degree of divergence

It is important to analyze the degree of divergence of a particular diagram in general. As we will see, only 2-point and 4-point functions can be primitively divergent while all divergences in higher n-point functions just appear if they contain divergent 2-point or 4-point functions. Thus, if we are able to renormalize the divergent 2-point and 4-point function, also all divergencies in higher n-point functions disappear. If each n-point function would have its own primitive divergencies, we would not be able to renormalize the theory.
Because of this internal divergencies we are not able to determine a general degree of divergence but a superficial degree of divergence which will confirm the

statement above. From the Feynman rules of ϕ^4 theory, Eq. (37), we see that each propagator (internal line) contributes q^2 to the denominator. Each vertex contributes an integration over d-dimensional momentum space (in d space-time dimensions) together with a delta function for momentum conservation. Thus, integration will only take place over independent momentas, which is equal to the number of loops of the particular diagram. Summing up the superficial degree of divergence of a diagram with L loops and I internal lines is

$$D = dL - 2I. \tag{43}$$

This formula gives $D = 2$ and $D = 0$ for the diagrams above.
It is our goal to express D in terms of the number of external lines E and the order of the diagram n (number of vertices). We have I internal momentas which are constrained by n momentum conservation conditions and an overall momentum conservation. Thus, we have $I - n + 1$ independent momentas which is equal to the number of loops:

$$L = I - n + 1. \tag{44}$$

In ϕ^4 theory, each vertex has 4 legs, we have 4n legs in total which are connected by internal and external lines:

$$4n = E + 2I. \tag{45}$$

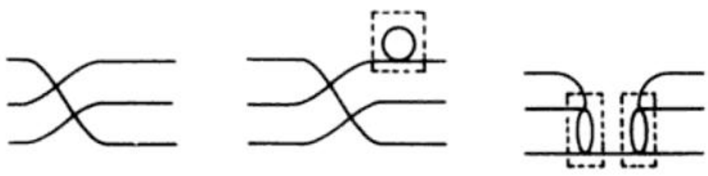
Using eqs. (43), (44), and (45), we find

$$D = d - \Big(\frac{d}{2} - 1\Big)E + n\Big(d - 4\Big) \tag{46}$$

which simplifies in the case $d = 4$ to

$$D = 4 - E. \tag{47}$$

This is in accordance with the statement above that all diagrams with $n > 4$ are not diverging superficially. However, to make this point more clear, let us consider three examples of 6-point functions:



(a) (b) (c)

The first diagram (a) is converging as could be shown by writing down the corresponding expression, while the others (b and c) are diverging because they contain internal 2-point or 4-point functions which let them diverge.
After this analysis, it should be clear why we will only consider 2-point and 4-point functions with the goal to renormalize them.

2.3.4 Dimensional analysis

Furthermore, it is important to see that space-time dimensionality has important consequences to the renormalizability of our theory. Thus, we analyze the dimensionality of the quantities we are dealing with. Starting with the action

$$S = \int \mathrm{d}^d x \, \mathcal{L} \tag{48}$$

which is dimensionless (in units with $\hbar = 1$), we find that

$$[\mathcal{L}] = L^{-d} \quad \text{(L is a length)}. \tag{49}$$

Considering the kinetic energy term ($\partial^\mu \phi \partial_\mu \phi$) and using $[\partial_\mu] = L^{-1}$, we have

$$[\phi] = L^{1-d/2}. \tag{50}$$

If we now consider the interaction term $\lambda\phi^4$ and suppose $[\lambda] = L^{-\delta}$, we find that

$$\delta = 4 - d. \tag{51}$$

We conclude that in 4 dimensions, our coupling constant λ is dimensionless. We have to remark that this is important for the following reason: If we substitute Eq. (51) into Eq. (46), we have

$$D = d - \left(\frac{d}{2} - 1\right) E - n\delta, \tag{52}$$

from which we see that a coupling constant of a renormalizable theory has to have a mass dimension[7] $\delta \geq 0$ because otherwise higher order diagrams would have increasing divergences and we could forget about renormalization[8]. For our ϕ^4 theory, this is the case for $d > 4$ and the theory is non-renormalizable.

2.4 Renormalization

2.4.1 Renormalized Perturbation Theory

As we have seen in Sec. 2.3.1, in ϕ^4 theory there are diverging 2-point and 4-point diagrams. We will now see how we can renormalize the theory to give finite results for all measurable quantities: mass m, coupling constant λ, and field-strength ϕ. In the following, it is important to distinguish between bare quantities, which we consider so far and which we will indicate by a subscript B, and measurable quantities. If we consider the complete propagator, Eq. (24),

$$G^{(2)}(p) = \frac{i}{p^2 - m_B^2 - \Sigma(p)}, \tag{53}$$

[7] It is more common to talk in mass units (mass^{-1} = length for $\hbar = 1$)

[8] This is the reason, why e.g. a term $\frac{1}{m^4}(F_{\mu\nu}F^{\mu\nu})^2$ may not occur in the QED Lagrangian, what we have seen in the last talk 'Yang-Mills theory and the QCD Lagrangian'.

we see that the pole of this propagator is not at m anymore but at $\tilde{m}$ defined by,

$$\tilde{m}^2 - m_B^2 - \Sigma(\tilde{m}) = 0. \tag{54}$$

We can now expand the complete propagator around $\tilde{m}$,

$$G^{(2)}(p) = \frac{iZ}{p^2 - \tilde{m}^2} + \text{terms regular at } p^2 = \tilde{m}^2, \tag{55}$$

where Z is the residuum of the pole. This Z is a probability amplitude[9] and we should normalize this probability amplitude to 1, which can be done by considering a renormalized field ϕ_r:

$$\phi = Z^{1/2}\phi_r \tag{56}$$

Using this definition of the renormalized field, we can write

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}m_B^2 Z\phi_r^2 - \frac{\lambda_B}{4!}Z^2\phi_r^4. \tag{57}$$

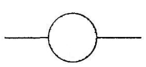
Still, bare mass m_B and bare coupling constant λ_B appear in the Lagrangian. To introduce the physically measurable mass m and coupling constant λ, we define

$$\delta_Z = Z - 1, \qquad \delta_m = m_B^2 Z - m^2, \qquad \delta_\lambda = \lambda_B Z^2 - \lambda, \tag{58}$$

and rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} = \; & \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ & + \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4, \end{aligned} \tag{59}$$

where in the first line, we have now the same Lagrangian as in Eq. (26), but now with the physical quantities. In the second line, there are *counterterms* of the same form as the original terms of the Lagrangian.
It is important to note that we did not add these counterterm but rather split the bare quantities in the physical quantities and in additional parts which appear as counterterms. However, so far we just introduced quantities m and λ which have to be defined properly. These necessary conditions are called *renormalization conditions*:
We define the renormalized full propagator, which we considered above, as



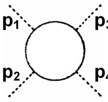
$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2) \tag{60}$$

[9] $Z = |\langle\lambda_0|\phi(0)|\Omega\rangle|^2$, where Ω is the vacuum state and λ_0 is an exact one-particle state.

which includes two conditions: First of all, the pole of the propagator defines the physical mass and additionally the residuum at this pole is fixed. The other condition concerns the 4-point function, more precisely the full 4-point function with amputated external legs, the *4-point vertex function*

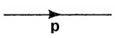
$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = G^{(4)}(p_1, p_2, p_3, p_4)\, G^{(2)^{-1}}(p_1)\, G^{(2)^{-1}}(p_2)\, G^{(2)^{-1}}(p_3)\, G^{(2)^{-1}}(p_4).$$

This condition is not unique, as we are free to choose a condition certain momentas p_i. It is convenient to introduce the *Mandelstam variables* $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, and $u = (p_1 + p_4)^2$, which are important variables in experiments. Using this, we can postulate the second condition

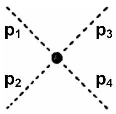


$$= -\, i\lambda \qquad \text{at } s = 4m^2,\ t = u = 0, \tag{61}$$

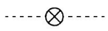
Together with the new Lagrangian we have new Feynman rules



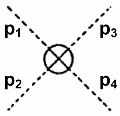
$$= \frac{i}{p^2 - m^2} \tag{62}$$



$$= -\, i\lambda \tag{63}$$



$$= i(p^2\delta_Z - \delta_m) \tag{64}$$

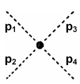
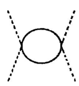
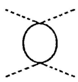
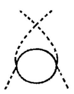
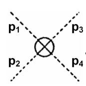


$$= -\, i\delta_\lambda \tag{65}$$

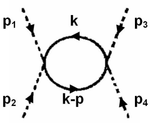
where we instead of demanding integrations over all momentas and a delta function for energy-momentum conservation at each vertex, we just demand integration over each independent momentum. The first two rules have the same form as the original rules, but now they contain the physical mass and the physical coupling constant.
If we evaluate diagrams contributing to the 2- and 4-point functions, we still find divergent contributions. However, we are now able to compensate them by contributions originating from the counterterms. The counterterms are chosen in such a way that the renormalization conditions are fulfilled.
This procedure, using Feynman rules with counterterms, is known as *renormalized perturbation theory*. Let us now make use of this procedure by renormalization of ϕ^4 theory to one-loop order.

2.4.2 Renormalization to one-loop order

We start the renormalization by considering the 4-point vertex function

$\Gamma^{(4)} =$  $+ \Big($  $+$  $+$  $\Big) +$  . (66)

If we define $p = p_1 + p_2$, the second contribution can be written as



$$
\begin{aligned}
&= \frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \\
&\equiv (-i\lambda)^2 \cdot iV(p^2).
\end{aligned} \tag{67}
$$

The next two diagrams are identical and using the Mandelstam variables, the whole 4-point function to one loop order can be expressed as

$$
\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta_\lambda. \tag{68}
$$

Reconsidering the renormalization condition $\Gamma^{(4)}(s = 4m^2, t = 0, u = 0) = -i\lambda$, we find

$$
\delta_\lambda = -\lambda^2 [V(4m^2) + 2V(0)]. \tag{69}
$$

$V(p^2)$ is a divergent quantity in 4 space-time dimensions as we have already seen. However, it turns out that it can be computed explicitly using dimensional regularization, which means that we perform the momentum integral in d dimensions, where it is not divergent. Then, we consider the limit $d \to 4$ and recognize that the diagram has a simple pole in $4 - d$. The result (calculated in App. B) is

$$
V(p^2) = -\frac{1}{32\pi^2} \Big(\frac{2}{\epsilon} - \gamma - \int_0^1 \mathrm{d}x \log \Big[\frac{m^2 - x(1-x)p^2}{4\pi} \Big] \Big), \tag{70}
$$

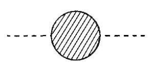
where $\epsilon = 4 - d$ and γ is the Euler-Mascheroni constant[10]. Using Eq. (69), we have

$$
\delta_\lambda = \frac{\lambda^2}{32\pi^2} \Big(\frac{6}{\epsilon} - 3\gamma - \int_0^1 \mathrm{d}x \Big(\log \Big[\frac{m^2 - x(1-x)4m^2}{4\pi} \Big] + 2 \log \Big[\frac{m^2}{4\pi} \Big] \Big) \Big) \tag{71}
$$

Now, both δ_λ and $V(p^2)$ are divergent but using Eq. (68) we find the finite result

$$
\begin{aligned}
\Gamma^{(4)}(p_1, p_2, p_3, p_4) = &- i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 \mathrm{d}x \Big(\log \Big[\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} \Big] \\
&+ \log \Big[\frac{m^2 - x(1-x)t}{m^2} \Big] + \log \Big[\frac{m^2 - x(1-x)u}{m^2} \Big] \Big),
\end{aligned} \tag{72}
$$

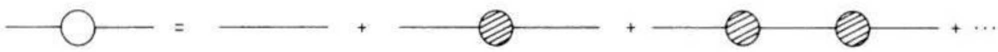
which is valid to one loop order. So far, δ_Z and δ_m are not determined and we have to compute the two-point function to determine them. Making use of the self-energy Σ,



$$
= \frac{\Sigma(p^2)}{i}, \tag{73}
$$

[10] $\gamma = 0.577$

the full two-point function is given by a geometric series



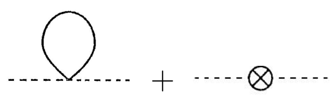
$$= \frac{i}{p^2 - m^2 - \Sigma(p^2)} \tag{74}$$

To maintain the renormalization conditions for the propagator, Eq. (60), this is equivalent to fulfill the two conditions

$$\Sigma(p^2)\Big|_{p^2=m^2} = 0 \qquad \text{and} \qquad \frac{d}{dp^2}\Sigma(p^2)\Big|_{p^2=m^2} = 0. \tag{75}$$

To maintain these conditions, we calculate the self-energy to one loop order and adjust δ_Z and δ_m as required,

$$\frac{\Sigma(p^2)}{i} = \tag{76}$$

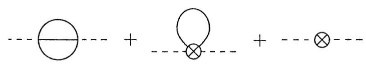


$$= -\frac{i\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m). \tag{77}$$

As the first term is independent of p^2, we have $\delta_Z = 0$. To determine δ_m, we have to calculate the first term using dimensional regularization and find a result similar to the one for the 4-point diagram and by setting

$$\delta_m = -\frac{\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} = -\frac{\lambda m^2}{32\pi^2}\Big(\frac{2}{\epsilon} + (1-\gamma) - \log\Big[\frac{m^2}{4\pi}\Big]\Big), \tag{78}$$

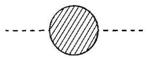
we have $\Sigma(p^2) = 0$ for all p^2. The first nonzero contribution to $\Sigma(p^2)$ and δ_Z are of order λ^2, which occur when we consider second order diagrams.



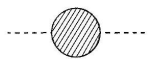
It is important to note that at each order of loops all divergencies can be canceled by adjustment of counterterms to render the renormalization condition valid. The procedure is totally selfconsistent: On the one side, in each order we have to include the counterterm contributions according to the Feynman rules. On the other side, in each order additional contributions to the counterterms occur.
A theory is renormalizable if all divergencies can be canceled in each order of perturbation theory by counterterms of the same form as the original Lagrangian.
So far, we have only considered 2-point and 4-point functions and renormalized them. What concerns higher n-point functions, we have discussed that their divergences originate from internal diverging 2-point and 4-point functions. As we have renormalized these divergences, higher n-point functions include renormalized 2-point and 4-point functions which are not divergent. Thus, we do not have to worry about divergences in higher n-point functions.

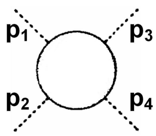
2.5 Callan-Symanzik equation

We have now seen how the ϕ^4 theory can be renormalized. We will now derive a flow equation for the coupling constant, which determines how the coupling constant changes with changing momentas.
We again consider the situation of our ϕ^4 theory. However, the flow equation we will determine is valid for all dimensionless coupling theories. For simplicity, we assume that the mass term m^2 has been adjusted to zero. In the last section, we used a set of renormalization condition, eqs. (60) and (61), which for $m^2 = 0$ lead to singularities.[11] As we have the freedom to choose an appropriate renormalization condition. We can choose the conditions at a spacelike momentum p with $p^2 = -M^2$:



$$= \quad 0 \qquad \text{at } p^2 = -M^2 \tag{79}$$

$$\frac{d}{dp^2}\Big($$  $$\Big) = \quad 0 \qquad \text{at } p^2 = -M^2 \tag{80}$$



$$= -\, i\lambda \qquad \text{at } s = t = u = -M^2 \tag{81}$$

The parameter M is called the *renormalization scale*[12]. It is important to note that we have chosen the renormalization scale M arbitrary. The same theory could have been defined at another scale M'. Same theory here means that the bare, unrenormalized Green's functions

$$\langle\Omega|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|\Omega\rangle \tag{82}$$

are the same. The renormalized Green's functions are related to the bare Green's function by

$$\langle\Omega|T(\phi_r(x_1)\phi_r(x_2)\cdots\phi_r(x_n))|\Omega\rangle = Z^{-n/2}\langle\Omega|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|\Omega\rangle. \tag{83}$$

Another renormalization scale M' would lead other renormalized Green's functions with a new renormalized coupling constant λ' and a new rescaling factor Z'. However, the bare Green's functions do not change, obviously. Let us now consider the consequences of a shift of M by δM. This is connected with shifts in the coupling constant and in the rescaling factor:

$$\begin{aligned} M \to& M + \delta M, \\ \lambda \to& \lambda + \delta\lambda, \\ Z \to& Z + \delta Z, \end{aligned} \tag{84}$$

[11] Consider for example the limit $m^2 \to 0$ in Eq. (71) and Eq. (72).

[12] In the case of nonzero mass, we could choose the same renormalization conditions with a large momentum scale M, such that m could be treated as perurbation.

from which follows

$$\phi_r \rightarrow (1+\delta\eta)\phi_r, \tag{85}$$

where $\delta\eta = \delta Z/Z$. As we know, the Green's function $G^{(n)}$ contains n field components, from which follows

$$G^{(n)} \rightarrow (1+\delta\eta)^n G^{(n)} \approx (1+n\delta\eta)G^{(n)}. \tag{86}$$

We know that the Green's functions depend on M and λ and thus we can connect the shifts by calculating the shift in the Green's function

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)}, \tag{87}$$

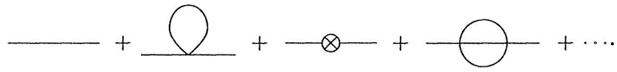
where we used in the last step the result of Eq. (86). Defining the dimensionless parameters

$$\beta = \frac{M}{\delta M}\delta\lambda \quad \text{and} \quad \gamma = -\frac{M}{\delta M}\delta\eta, \tag{88}$$

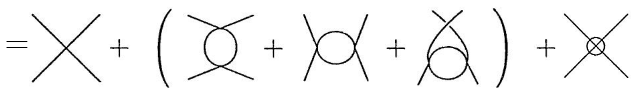
we find the famous Callan-Symanzik equation

$$\Big[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\Big]G^{(n)}(\{x_i\}; M, \lambda) = 0, \tag{89}$$

where the dependence of our dimensionless parameters only on λ comes from dimensional analysis.
The functions β and γ can be calculated to desired order in λ by considering two n-point functions to this order in λ. Let us first consider the two-point function

$G^{(2)}(p) =$  (90)

In a massless ϕ^4 theory the one vertex contribution vanishes and thus the first nontrivial contributions to $G^{(2)}$ are of order λ^2. This has the important consequence that γ is at least of order λ^2.
Considering now the 4-point function, we have

$G^{(4)} =$  (91)

$$= \Big[i\lambda + (-i\lambda)^2\Big(iV(s) + iV(t) + iV(u)\Big) - i\delta_\lambda\Big] \cdot \prod_{i=1,\dots,4}\frac{i}{p_i^2} \tag{92}$$

The counterterm δ_λ has to be determined to fulfill the renormalization condition and one can find

$$\begin{aligned}\delta_\lambda =& (-i\lambda)^2 \cdot 3V(-M^2) && (93)\\ =& \frac{3\lambda^2}{2(4\pi)^2}\Big[\frac{1}{2-d/2} - \log(M^2) + \text{finite}\Big]. && (94)\end{aligned}$$

To find the β-function, we have to apply the Callan-Symanzik equation to the renormalized 4-point function and consider only the leading terms.

$$\begin{aligned}\left[M\frac{\partial}{\partial M}+\beta(\lambda)\frac{\partial}{\partial\lambda}+4\gamma(\lambda)\right]G^{(4)}(p_1,p_2,p_3,p_4)=&0\\ \frac{3i\lambda^2}{16\pi^2}+\beta(\lambda)(-i)+4\gamma(\lambda)(-i\lambda)=&0\end{aligned} \tag{95}$$

As $\gamma(\lambda)$ is at least of order λ^2, this equation can only be satisfied if

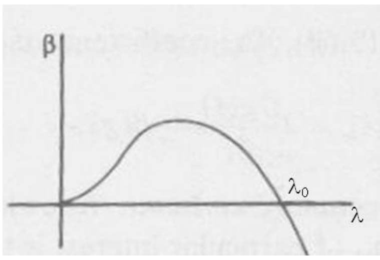
$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+\mathcal{O}(\lambda^3). \tag{96}$$

2.6 β-function and triviality of ϕ^4 theory

From the definition of the β-function, Eq. (88), we have

$$\beta(\lambda)=M\frac{\partial\lambda}{\partial M}, \tag{97}$$

which determines the behavior of the coupling constant with changing momentum scale M. This *flow* of the coupling constant is the reason for speaking from a *running coupling*. Let us first consider two examples of possible behaviors of the running coupling. If we suppose that $\beta(\lambda)$ has the form



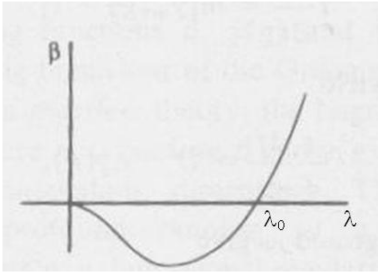
we see that there are two zeros of the β-function at 0 and λ_0. Considering a value of λ below λ_0, we have

$$M\frac{\partial\lambda}{\partial M}>0 \tag{98}$$

and λ moves towards λ_0 with increasing momentum. If we consider a coupling $\lambda>\lambda_0$, we have

$$M\frac{\partial\lambda}{\partial M}<0 \tag{99}$$

and λ decreases towards λ_0 with increasing momentum. Thus, λ_0 is an *ultra-violet stable fixed point*. On the other hand, $\lambda=0$ is an *infra-red fixed point*: Consider a small λ, in this case for decreasing momentum, λ decreases to zero. If we consider the following behavior



we again have two fixed points. If we consider a coupling λ between zero and λ_0, we have for decreasing momentum an increasing coupling and thus an infrared stable fixed point λ_0. On the other side, for increasing momentum λ tends towards zero and thus for large momenta the coupling constant vanishes. This behavior is known as *asymptotic freedom*, which will be of interest later on.
These two exemplary considerations have the purpose to show how the β-function determines the running coupling. We now turn back to the situation in ϕ^4 theory, where we found

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3). \tag{100}$$

For small λ the behavior of the β-function is determined by the quadratic λ term and therefore the coupling constant is increasing with increasing momenta. The question now is if there is a nontrivial zero of the β-function which would lead to a stable point as in the first example considered above. However, with increasing λ the quadratic term is not dominating anymore and the behavior of the β-function can not be calculated in perturbation theory anymore. A possibility to examine the behavior of the coupling constant for large momentas is to consider the ϕ^4 theory on a lattice and to do numerical calculations. From this examination, we can conclude that there is no ultra-violet fixed point in ϕ^4 theory. This is known as the *triviality* of ϕ^4 theory which describes that the coupling constant grows with growing momentas.

3 Nonlinear Sigma Model

We will now consider the nonlinear σ-model as another example of a scalar field, whose structure is very different from the structure of ϕ^4 theory. We will make use of the concepts which we have introduced in discussing the ϕ^4 theory. As we will see, the nonlinear σ-model is asymptotically free for $d = 2$ what has certain consequences. We will also see that the asymptotic freedom of the nonlinear σ-model is restricted to the case $d = 2$.
Considering scalar fields ϕ^i in two space-time dimensions with a Lagrangian

$$\mathcal{L} = f_{ij}(\{\phi^l\})\partial_\mu\phi^i\partial^\mu\phi^j, \tag{101}$$

we recognize that the fields are dimensionless ($[\mathcal{L}] = (\text{mass})^2$, $[\partial_\mu \phi^i \partial^\mu \phi^j] = (\text{mass})^2$, $[\phi] = 1$) and that thus all coupling constants are dimensionless. Therefore, any theory of this form is renormalizable, for any possible function $f_{ij}(\{\phi^l\})$. To restrict these possibilities to choose, we impose that the scalar fields ϕ^i form a N-dimensional unit vector, $\phi^i = n^i(x)$ with

$$\sum_{i=1}^{N} |n^i(x)|^2 = 1 \tag{102}$$

together with a $O(N)$-symmetry of the field components. Restricted by those conditions, the most general choice for f is a constant. All possible interactions would be restricted by $O(N)$-symmetry to a form $c|\vec{n}|^r$ and as $|\vec{n}| = 1$, this would be just a constant c which does not change any n-point function. Thus, the most general Lagrangian with $\vec{n}(x)$ and $O(N)$-symmetry is

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu \vec{n}|^2. \tag{103}$$

Although, these Lagrangian is very different from the ϕ^4-Lagrangian, there is a connection between the two theories: If we consider the mass term m in the ϕ^4-Lagrangian of N scalar fields ϕ^i as a parameter and allow $m^2 < 0$, we find, using $m^2 = -\mu^2$, the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\mu^2[\phi^i]^2 - \frac{\lambda}{4!}[(\phi^i)^2]^2, \tag{104}$$

which is the Lagrangian of the linear σ-model, which is a good model to describe spontaneous symmetry breaking. This Lagrangian can be rewritten (rescale $\lambda \to 6\lambda$) as

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \frac{\lambda}{4}\Big((\phi^i)^2 - \frac{\mu^2}{\lambda}\Big)^2 + \frac{\mu^4}{4\lambda} \\ =& \frac{1}{2}\kappa^2 \partial_\mu \tilde{\phi}^i \partial^\mu \tilde{\phi}^i - \frac{\lambda}{4}\kappa^2\Big((\tilde{\phi}^i)^2 - 1\Big)^2 + \text{const}, \end{aligned} \tag{105}$$

where we defined $\kappa = \mu^2/\lambda$ and $\phi^i = \kappa\tilde{\phi}^i$. If we now let the mass parameter μ go to infinity, $\mu \to \infty$, while keeping κ constant, also $\lambda \to \infty$ and if we now consider the potential term in our Lagrangian

$$V(\{\phi^i\}) = \frac{\lambda}{4}\Big((\phi^i)^2 - \frac{\mu^2}{\lambda}\Big)^2 + \mu^2\frac{\kappa}{4}, \tag{106}$$

we see that to minimize the potential, the field is forced to the unit sphere, $\sum_i |\tilde{\phi}^i|^2 = 1$, which then leads to the nonlinear σ-model.
Let us make here one more remark about the connection between the ϕ^4 model and the nonlinear σ-model: In the most general Lagrangian for the nonlinear

σ-model, Eq. (103), we included a constant g by a factor $1/g^2$ in front of the kinetic term, which should be our coupling constant in this model. To see, why we introduced g by a factor $1/g^2$, let us consider Eq. (105): if we consider the definition of κ, we recognize that deriving the nonlinear σ-model from the ϕ^4 Lagrangian, the coupling constant λ which appeared in the interaction term $\lambda\phi^4$ appears now in front of the kinetic term in the form $1/\lambda^2$. This is the reason why g in our model is a coupling constant.
Coming back to the nonlinear σ-model, we can parametrize $\vec{n}$ by π^k with

$$n^i = (\pi^1, \dots, \pi^{N-1}, \sigma) \tag{107}$$

$$\text{where} \quad \sigma = (1 - \vec{\pi})^{1/2}. \tag{108}$$

The configuration with $\pi^k = 0$ corresponds to a uniform state of spontaneous symmetry breaking in the N direction. From the parametrization follows

$$|\partial_\mu \vec{n}|^2 = |\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\vec{\pi}^2} \tag{109}$$

and the Lagrangian takes the form

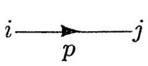
$$\mathcal{L} = \frac{1}{2g^2}\left[|\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\vec{\pi}^2}\right], \tag{110}$$

which can be expanded in powers of π^k

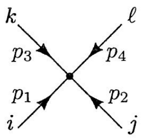
$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu\vec{\pi}|^2 + \frac{1}{2g^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \frac{1}{2g^2}\vec{\pi}^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \dots . \tag{111}$$

3.1 Feynman rules

This Lagrangian leads to the following Feynman rules



$$= \frac{ig^2}{p^2}\delta^{ij} \tag{112}$$



$$= -\frac{i}{g^2}[(p_1+p_2)\cdot(p_3+p_4)\delta^{ij}\delta^{kl} + (p_1+p_3)\cdot(p_2+p_4)\delta^{ik}\delta^{jl}$$
$$+ (p_1+p_4)\cdot(p_2+p_3)\delta^{il}\delta^{jk}] \tag{113}$$

and additional vertices for all even numbers of π^k fields. To motivate these rules, let us remark the following: The first rule just originates from the kinetic term and is very similar to the propagator in ϕ^4 theory except that there is no mass term in the Lagrangian and thus there is no mass term in the denominator of the propagator. The 4-vertex is also similar to the 4-vertex in ϕ^4 except the appearance of momentas. An interaction term $(1/2g^2)(\vec{\pi}\cdot\vec{\pi})^2 = (1/2g^2)(\pi^l\cdot\pi^l)(\pi^k\cdot\pi^k)$ would just give the same vertex but without momentas, which would be the same as a ϕ^4 4-vertex. The real interaction term $(1/2g^2)(\pi^l\cdot\partial_\mu\pi^l)(\pi^k\cdot\partial_\mu\pi^k)$ contains two additional derivatives which just leads to the the presence of the momentum factors.

3.2 Callan-Symanzik equation and β-function

As we have dimensionless coefficients in our Lagrangian, this theory can be made finite by renormalization of the coupling constant g and rescaling of the fields π^k and σ.
Instead of going through the whole renormalization procedure, we make use of the fact that our renormalizable theory has to fulfill the Callan-Symanzik equation for some functions β and γ. Thus, we will now calculate these functions β and γ to show that the nonlinear σ-model is asymptotically free. The Callan-Symanzik equation

$$\Big[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\Big]G^{(n)} = 0 \tag{114}$$

has to be fulfilled by all Green's functions $G^{(n)}$ with n fields π^k and σ. We will calculate β and γ to leading order in perturbation theory (one-loop order).
The first Green's function to consider is

$$G^{(1)} = \langle\sigma(x)\rangle, \tag{115}$$

which is nonvanishing because we do not have the Z_2 symmetry which we found when we considered the ϕ^4 theory. Due to translation invariance, we find

$$\langle\sigma(x)\rangle = \langle\sigma(0)\rangle = 1 - \frac{1}{2}\langle\vec{\pi}^2(0)\rangle + \ldots = 1 - \frac{1}{2}$$  $$+ \ldots . \tag{116}$$

Thus, we have to calculate

$$\langle\pi^k(0)\pi^l(0)\rangle =$$  $$= \int\frac{d^dk}{(2\pi)^d}\frac{ig^2}{k^2-\mu^2}\delta^{kl}$$

$$= \int\frac{d^dk_E}{(2\pi)^d}\frac{g^2}{k_E^2+\mu^2}\delta^{kl}, \tag{117}$$

where we have introduced a little mass μ as infrared cutoff and in a second step transformed to a Euclidean momentum integral. We can make use of the identity

Eq. (150) and find ($d = 2 - \epsilon$)

$$
\begin{aligned}
\langle \pi^k(0)\pi^l(0)\rangle &= \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(\mu^2)^{1-d/2}} \delta^{kl} \\
&= \frac{g^2}{(4\pi)^{1-\epsilon/2}} \frac{\Gamma(\frac{\epsilon}{2})}{(\mu^2)^{\epsilon/2}} \delta^{kl} \\
&\overset{\epsilon \to 0}{=} \frac{g^2}{4\pi}\Big(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\Big)\Big(1 - \frac{\epsilon}{2}\log(\mu^2)\Big)\delta^{kl} \\
&= \frac{g^2}{4\pi}\Big(\frac{2}{\epsilon} - \gamma - \log(\mu^2)\Big)\delta^{kl}.
\end{aligned} \tag{118}
$$

As we can see, this expression is diverging for $\epsilon \to 0$. We now have to perform a renormalization. Instead of using the renormalization scheme which we used above, we can use the *modified minimal subtraction scheme* ($\overline{MS}$ scheme), where we subtract from the result a term of the same form with an arbitrary mass M (renormalization scale) instead of μ and find

$$
\begin{aligned}
\langle \pi^k(0)\pi^l(0)\rangle =& \left((-1)^d \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(\mu^2)^{1-d/2}} \delta^{kl}\right) - \left((-1)^d \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(M^2)^{1-d/2}} \delta^{kl}\right) \\
=& \frac{g^2}{4\pi}\left[\Big(\frac{2}{\epsilon} - \gamma - \log(\mu^2)\Big) - (\frac{2}{\epsilon} - \gamma - \log(M^2)\Big)\right]\delta^{kl}.
\end{aligned} \tag{119}
$$

Thus we find

$$
\langle \pi^k(0)\pi^l(0)\rangle = \frac{g^2}{4\pi}\log\Big(\frac{M^2}{\mu^2}\Big)\delta^{kl} \tag{120}
$$

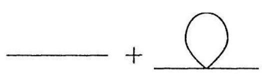
and inserting this into Eq. (116)

$$
\langle \sigma \rangle = 1 - \frac{1}{2}\frac{(N-1)g^2}{4\pi}\log\Big(\frac{M^2}{\mu^2}\Big) + \mathcal{O}(g^4). \tag{121}
$$

Applying the Callan-Symanzik equation to these one-point function, we obtain to leading order

$$
-\frac{(N-1)g^2}{4\pi} - \beta(g)\frac{(N-1)g}{4\pi}\log\big(\frac{M^2}{\mu^2}\big) + \gamma(g) = 0. \tag{122}
$$

To find the functions β and γ to lowest order, we have to consider a second n-point function: For simplicity, we consider the two-point function

$\langle \pi^k(p)\pi^l(-p)\rangle =$  $+\cdots$

$$
= \frac{ig^2}{p^2}\delta^{kl} + \frac{ig^2}{p^2}(-i\Pi^{kl})\frac{ig^2}{p^2} + \cdots, \tag{123}
$$

where Π^{kl} is the contribution of the loop. This can be calculated (using the Feynman rules) in a very similar way as in Eq. (118). Here, we have to apply again the $\overline{MS}$ scheme and find

$$\langle\pi^k(p)\pi^l(-p)\rangle = \frac{i}{p^2}\delta^{kl}\Big(g^2 - \frac{g^4}{4\pi}\log\Big(\frac{M^2}{\mu^2}\Big) + \mathcal{O}(g^6)\Big). \tag{124}$$

Applying now the Callan-Symanzik equation to this two-point function, we find

$$\begin{aligned} 0 =& \Big[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)\Big]\langle\pi^k(p)\pi^l(-p)\rangle \\ =& \frac{i\delta^{kl}}{p^2}\Big[-\frac{g^4}{2\pi} + \beta(g)\cdot 2g + 2\gamma(g)\cdot g^2 + \mathcal{O}(g^6)\Big]. \end{aligned} \tag{125}$$

Combining eqs. (122) and (125), we can eliminate $\gamma(g)$ and find the equation

$$-\frac{(N-1)g^4}{2\pi} + \frac{g^4}{2\pi} - \beta(g)\frac{(N-1)g^3}{2\pi}\log\big(\frac{M^2}{\mu^2}\big) - 2\beta(g)g = 0, \tag{126}$$

which determines $\beta(g)$ to leading order:

$$\beta(g) = -\frac{(N-2)g^3}{4\pi} + \mathcal{O}(g^5). \tag{127}$$

Inserting this in Eq. (122), we find $\gamma(g)$ to leading order as

$$\gamma(g) = \frac{g^2(N-1)}{4\pi} + \mathcal{O}(g^4). \tag{128}$$

It is important to note now the dependence of $\beta(g)$ on N. At $N = 2$, the β-function vanishes, not only to order g^3 but to all orders. This is easy to see if we consider the Lagrangian of this case in detail. As we have only two components, we can parametrize them by $\pi^1 = \sin\theta$ and thus $\sigma = \cos\theta$. Thus, the Lagrangian (Eq. (110)) simplifies considerably:

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu \sin\theta|^2 + \frac{1}{2g^2}\frac{(\sin\theta\cdot\partial_\mu\sin\theta)^2}{1-(\sin\theta)^2} = \frac{1}{2g^2}(\partial_\mu\theta)^2. \tag{129}$$

It is easy to see that the case $N = 2$ (called XY-model) does not need a renormalization and thus the β-function is zero. In the case $N > 2$, we have a negative β-function. Thus our theory is asymptotically free, which means that the coupling constant goes to zero as the momentum becomes large. This consideration just includes first orders of perturbation theory. To show the asymptotic freedom in general, we would also have to consider regions of large coupling which can not be treated by perturbation theory.
It is interesting to note that the asymptotic freedom of the nonlinear σ-model is restricted to the case $d = 2$. For dimensions higher than 2, $2 < d < 4$, there is an ultra-violet stable fixed point which tends towards zero for $d \to 2$.

4 QCD

So far, we have investigated two models of field theories: the ϕ^4 theory and the nonlinear σ-model. The motivation for doing that was the following: We studied the ϕ^4 theory and its renormalization as the ϕ^4 theory is the simplest example of a scalar field theory and thus the best candidate to introduce the formalism of renormalization and the concepts of β-function and running coupling constant. Then the nonlinear σ-model was presented as a simple example of an asymptotically free theory. In fact, we are interested in studying the asymptotic freedom of *quantum chromodynamics*, its β-function and the running coupling flow. However, to perform this task is accompanied by mathematical challenges due to the non-Abelian nature of QCD. Thus, our approach had the advantage that we could consider more easy examples and thus focus on the ideas behind the techniques. Concerning QCD, we will now just consider the result of the whole renormalization calculations and the physical importance of asymptotic freedom for strong interactions.

4.1 Parton model

Quantum chromodynamics describes the strong interactions between the constituents of the nuclei, which are responsible for nuclear bonding. In fact, it is interesting to see that, at first, strong interactions showed mysterious properties which could not be described by common field theories (before the developement of QCD). Then, it was recognized that asymptotic freedom was a requirement for a theory to be able to describe these properties. As non-Abelian gauge theories are asymptotically free in four-dimensional space-time, they were candidates for theories describing strong interactions.
Let us investigate these mysterious properties and the connected models in more details as this will also give us a physical unterstanding of these interactions.
The *parton model* was put forward by Bjorken and Feynman and describes the proton as a loosely bound assemblage of a small number of constituents, called *partons*. These partons are *quarks*, charged fermions, and also neutral species responsible for the binding. One of the obscure properties was that these quarks do not exist as isolated species. Also, there are some complications due to the fact that strong interactions are strong as it is not possible to work perturbatively in this case. However, the most important point is that it is important that strong interactions turn themselves off, when the momentum transfer is large.
This feature is exactly provided by a theory with asymptotic freedom. Because asymptotic freedom means that the coupling for large momentas is small which is equivalent to the vanishing of large momentum transfer. Therefore, only a asymptotically free theory could describe strong interactions and quantum chromodynamics was developed as a non-Abelian gauge theory, where the quarks of the model are bound together by interacting vector bosons, called *gluons*.

4.2 β-function of QCD

The Lagrangian of QCD is the famous Yang-Mills Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2, \tag{130}$$

with the field strength tensor of the gauge bosons

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{131}$$

where A^a_μ is a component of the gauge boson field (gluon field, $a \in \{1, \cdots, 8\}$) and f^{abc} is the structure constant of the gauge symmetry, in the case of QCD $SU(3)$.
Without going in any details of calculation, we state that the β-function in the case of a $SU(N)$ gauge theory with n_f different fermions is given as

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\Big(\frac{11}{3}N - \frac{2}{3}n_f\Big). \tag{132}$$

It is interesting to note that the sign of the β-function depends on the ratio of the number of fermions n_f and N (from the symmetry $SU(N)$), where we have $N = 3$ for QCD. For a small enough number n_f, β is negative and the theory is asymptotically free, which is for QCD the case if $n_f \leq 16$. Using the β-function, the running coupling can be calculated to

$$g^2(k) = \frac{g_0^2}{1 + \frac{g_0^2}{(4\pi)^2}(\frac{11}{3}N - \frac{2}{3}n_f)\log(\frac{k^2}{M^2})}, \tag{133}$$

which tends to zero at large momentum. Experimental measurements show that coupling constant gets small for values of k of about 1 GeV. This means that the strong interactions disappear for distances smaller than about 0.1 fm, which is roughly the size of light hadrons. Which means that the constituents of hadrons cannot interact via strong interactions.
As we have discussed above, the asymptotic freedom is a necessary condition for a theory to be able to describe strong interactions. However, this asymptotic freedom and the effect on the running coupling for long distances is worth to be discussed in more details because it is rather different from electrodynamics.
In electrodynamics, it is easy to understand the direction of the coupling constant flow: The vacuum behaves as a dielectric medium due to electron-positron pair creation, which decreases the effective charge of the electron at large distances.
In non-Abelian gauge theories, the fermions still produce such an effect as can be seen from the positive contribution to the β-function originating from the fermions. However, the non-Abelian gauge bosons produce a dominating antiscreening effect. To understand this effect, we study a simplified example:

Working in Coulomb gauge, $\partial_i A^{ai} = 0$, we have transversely polarized photons as field quantas. Considering the Coulomb potential of the field A^{a0} which is described by an analogue of Gauss's law in this non-Abelian case with covariant form

$$D_i E^{ai} = g\rho^a, \tag{134}$$

where the covariant derivative acting on a field in adjoint representation is defined as

$$(D_\mu \phi)^a = \partial_\mu \phi^a + g f^{abc} A^b_\mu \phi^c, \tag{135}$$

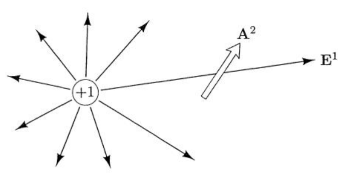
$E^{ai} = F^{a0i}$ and ρ^a is the charge density of the fermions, where a is a index for the *color* of charge. To make a further simplification, we choose $SU(2)$-symmetry, because in this case the structure constant simplifies to $f^{abc} = \epsilon^{abc}$:

$$(D_\mu \phi)^a = \partial_\mu \phi^a + g \epsilon^{abc} A^b_\mu \phi^c. \tag{136}$$

We now want to compute the Coulomb potential of a point charge of magnitude $+1$ with orientation (color) $a = 1$. We want to solve Eq. (134) for E^{ai} and do this iteratively. First we rewrite the equation as

$$\partial_i E^{ai} = g\delta^{(3)}(x)\delta^{a1} + g\epsilon^{abc} A^{bi} E^{ci} \tag{137}$$

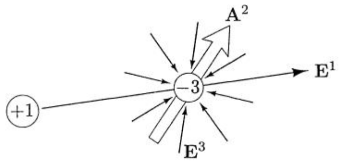
It is important to see that in this non-Abelian theory not only a charge density is a source of the electric field (first term), but also the common presence of a vector potential and an electric field (second term) is a source of electric fields. The first term implies a $1/r^2$ electric field of color $a = 1$ radiating from $x = 0$. We now consider a point in space where this field crosses a bit of vector potential A^{bi} arising as fluctuation of the vacuum. Let us consider a vector potential A^{2i} which points in some diagonal direction to the electric field as shown in the figure below.



If we now consider $a = 3$,

$$\partial_i E^{3i} = g\epsilon^{321} A^{2i} E^{1i} = -g A^{2i} E^{1i} < 0, \tag{138}$$

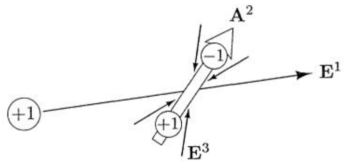
we find a sink of the field E^{3i} at this location, as shown in figure below.



Considering now the influence of this field E^{3i} on the field E^{1i}, we find

$$\begin{aligned}\partial_i E^{1i} =& g\delta^{(3)}(x) + g\epsilon^{123} A^{2i} E^{3i} \\ =& g\delta^{(3)}(x) + gA^{2i}E^{3i}.\end{aligned} \tag{139}$$

This means that we have to consider the orientation of A^{2i} and E^{3i} in more detail. We see that closer to the origin the fields are parallel and thus there is a source for E^{1i}, farther away, the fields are antiparallel and thus there is a sink, as indicated in the next figure.



This is an induced electric dipole which is oriented with the positive charge towards the original charge. Thus, this amplifies the original charge instead of screening it and therefore the effect of the charge gets stronger at larger distances.
The detailed balance between this antiscreening effect and screening effects has to be investigated in more details and it can be found that the antiscreening effect is 12 times larger. This simplified example should just show how such antiscreening can occur. As we have seen, this antiscreening originates from the second term of the covariant derivative which is peculiar for a non-Abelian gauge theory. So the coupling constant grows at large distances for non-Abelian gauge theories. This leads to the effect of *confinement* as charges cannot be separated as the coupling between them grows with distance. On the other hand, for large momentas the coupling constant decreases towards zero which leads to asymptotic freedom.

A Generating functional of ϕ^4 theory

To obtain the normalized generating functional of ϕ^4 theory to order λ, we have to expand the expression

$$\exp\Big[i\int \mathcal{L}_{int}\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)\mathrm{d}z\Big]\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big] \tag{140}$$

to first order in λ:

$$\Big[1-\frac{i\lambda}{4!}\int\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)^4\mathrm{d}z+\mathcal{O}(\lambda^2)\Big]\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big].\quad(141)$$

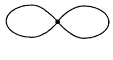
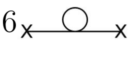
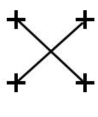
To order λ^0, we just have the free particle generating functional $Z_0[J]$. To calculate the order λ contribution, we have to perform four times a functional derivative:

$$\begin{aligned}
&\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big]\\
&\quad=-\int\Delta_F(z-x)J(x)\mathrm{d}x\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big],\\
&\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)^2\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big]\\
&\quad=\Big\{i\Delta_F(0)+\Big[\int\Delta_F(z-x)J(x)\mathrm{d}x\Big]^2\Big\}\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big],
\end{aligned}$$

$$\begin{aligned}
&\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)^3\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big]\\
&\quad=\Big\{-3i\Delta_F(0)\int\Delta_F(z-x)J(x)\mathrm{d}x-\Big[\int\Delta_F(z-x)J(x)\mathrm{d}x\Big]^3\Big\}\\
&\qquad\qquad\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big],\\
&\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)^4\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big]\\
&\quad=\Big\{-3[\Delta_F(0)]^2+6i\Delta_F(0)\Big[\int\Delta_F(z-x)J(x)\mathrm{d}x\Big]^2+\Big[\int\Delta_F(z-x)J(x)\mathrm{d}x\Big]^4\Big\}\\
&\qquad\qquad\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big].
\end{aligned}$$

This can be written using the diagrammatic rules as

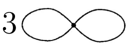
$$\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)^4\exp\Big[-\frac{i}{2}\int J\Delta_F J\Big]$$

$= \Big\{3$  $+\,6$  $+$  $\Big\}\exp\Big[-\frac{i}{2}\int J\Delta_F J\Big].\quad(142)$

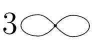
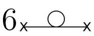
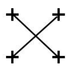
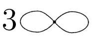
We have now found the contributions to order g, however we should now normalize our functional which is done by the expression in the denominator:

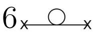
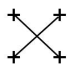
$$\exp\Big[i\int\mathcal{L}_{int}\Big(\frac{1}{i}\frac{\delta}{\delta J(z)}\Big)\mathrm{d}z\Big]\exp\Big[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\mathrm{d}x\mathrm{d}y\Big]\Big|_{J=0}.\quad(143)$$

This expression can be evaluated to first order by just setting any source terms to zero and we are left with terms without external legs and thus without sources (vacuum diagrams):

$$1+\frac{(-i\lambda)}{4!}\int\Big(3$$  $$\Big)\mathrm{d}z. \quad (144)$$

We can now use eqs. (142) and (144) to obtain the normalized generating functional to order λ:

$$Z[J]=\frac{\Big[1+\frac{(-i\lambda)}{4!}\int\Big\{3$$  $$+6$$  $$+$$  $$\Big\}\mathrm{d}z+\mathcal{O}(\lambda^2)\Big]\exp\Big[-\frac{i}{2}\int J\Delta_F J\Big]}{1+\frac{(-i\lambda)}{4!}\int\Big(3$$  $$\Big)\mathrm{d}z+\mathcal{O}(\lambda^2)}$$

$$=\Big[1+\frac{(-i\lambda)}{4!}\int\Big\{6$$  $$+$$  $$\Big\}\mathrm{d}z+\mathcal{O}(\lambda^2)\Big]\exp\Big[-\frac{i}{2}\int J\Delta_F J\Big], \quad (145)$$

where the vacuum diagram has disappeared which is always the case for normalized generating functionals.
We now can calculate the 2-point function of ϕ^4 theory from this generating functional. From the definition , Eq. (9), we find

$$\tau(x_1,x_2)=-\frac{\delta^2}{\delta J(x_2)\delta J(x_1)}\Big|_{J=0}$$

and recognize that we have to differentiate the generating functional two times with respect to the source terms and then set the sources to zero. Thus, we see that the term containing four sources may not contribute to the 2-point function and we are left with two terms: The first is just the free generating functional and thus contributes a free propagator $i\Delta_F(x_1-x_2)$. The second one, of order λ, can be written as

$$\frac{\lambda}{4}\Delta_F(0)\int \mathrm{d}x\mathrm{d}y\Delta_F(z-x)J(x)\Delta_F(z-y)J(y)\exp\Big(-\frac{i}{2}\int J\Delta_F J\Big) \qquad (146)$$

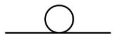
and we have to perform two differentiations (as we will set $J=0$ after two differentiations, we see that differentiations of the exponential functions vanish in the end):

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big(\quad\Big)=-\frac{i\lambda}{2}\Delta_F(0)\int \mathrm{d}y\mathrm{d}z\Delta_F(z-x_1)\Delta_F(z-y)J(y)\exp\Big(-\frac{i}{2}\int J\Delta_F J\Big)$$
$$+\text{ terms which give no contribution,} \qquad (147)$$

$$\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big(\quad\Big)=-\frac{\lambda}{2}\Delta_F(0)\int \mathrm{d}z\Delta_F(z-x_1)\Delta_F(z-x_2)\exp\Big(-\frac{i}{2}\int J\Delta_F J\Big)$$
$$+\text{ terms which give no contribution.} \qquad (148)$$

Summing up, we find

$$\tau(x_1, x_2) = i\Delta_F(x_1 - x_2) + \frac{(-i\lambda)}{2} i\Delta_F(0) \int \mathrm{d}z\, i\Delta_F(z - x_1) i\Delta_F(z - x_2) + \mathcal{O}(\lambda^2)$$

$$=$$ $$+ \frac{(-i\lambda)}{2}$$  $$+ \mathcal{O}(\lambda^2) \tag{149}$$

In an analogous way, we can also calculate the 4-point function and any other n-point function.

B Calculation of $V(p^2)$

We calculate $V(p^2)$ in d space-time dimensions:

$$\begin{aligned} V(p^2) =& \frac{i}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} \frac{1}{((k+p)^2 - m^2)} \\ & \frac{i}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{1}{[k^2 + 2xkp + xp^2 - m^2]^2}, \end{aligned}$$

where we used $\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2}$. We make the substitution $l = k + xp$ and find

$$V(p^2) = \frac{i}{2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{[l^2 + x(1-x)p^2 - m^2]^2}.$$

We can now change from d-dim. Minkowski space to d-dim. Euclidean space by substituting $l_E^0 = -il^0$ and obtain

$$V(p^2) = -\frac{1}{2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{[l_E^2 - x(1-x)p^2 + m^2]^2},$$

where we can make use of the identity

$$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}} \tag{150}$$

which leads to

$$V(p^2) = -\frac{1}{2} \int_0^1 \mathrm{d}x \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \frac{1}{[m^2 - x(1-x)p^2]^{2-d/2}}. \tag{151}$$

We now want to consider the limit $d \to 4$. Putting $\epsilon = 4 - d$, $\Gamma(2 - d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$, where $\gamma = 0.5772$ is the Euler-Mascheroni constant. On the other hand, we have a term $a^{\epsilon/2} \approx 1 + \frac{\epsilon}{2}\log(a) + \ldots$ and therefore

$$V(p^2) = -\frac{1}{32\pi^2} \Big(\frac{2}{\epsilon} - \gamma - \int_0^1 \mathrm{d}x \log\Big[\frac{m^2 - x(1-x)p^2}{4\pi} \Big] \Big), \tag{152}$$