Finite temperature field theory

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Finite temperature in the Euclidean path integral
- Partition function of the grand canonical ensemble
- Path integral representation of partition function for bosons
- Fermions
- Thermal Green's functions

Center Symmetry and Polyakov loop
- Refresh: Action and links
- Center symmetry
- Polyakov loop
- Physical meaning of Polyakov loop
- Simulations

Grand canonical ensemble

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- $\longrightarrow$ use grand canonical ensemble

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$$\mathrm{Tr}\hat{A} = \sum_n \langle n|\hat{A}|n\rangle$$

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- All thermodynamic properties like pressure, particle number etc. can be determined

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$$
\begin{aligned}
P &= \frac{\partial(T \ln Z)}{\partial V} \\
N_i &= \frac{\partial(T \ln Z)}{\partial \mu_i} \\
S &= \frac{\partial(T \ln Z)}{\partial T} \\
E &= -PV + TS + \mu_i N_i \\
F &= -T \ln Z
\end{aligned}
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- Set $\beta = \epsilon N$

$$Z = \int dq \langle q| \underbrace{e^{-\epsilon H} e^{-\epsilon H} \cdots e^{-\epsilon H}}_{N\,\text{times}} |q\rangle$$

- Set $q = q^{(0)}$ and insert a complete set of states
  $\int \prod_{i=0}^{N-1} dq^{(i)} \langle q^{(N)} | e^{-\epsilon H} | q^{(N-1)} \rangle \cdots \langle q^{(1)} | e^{-\epsilon H} | q^{(0)} \rangle |_{q^{(0)} = q^{(N)}}$

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$$Z = \int Dq \int Dp\, e^{\sum_{i=0}^{N-1}\sum_\alpha ip_\alpha^{(i)}(q_\alpha^{(i+1)}-q_\alpha^{(i)})-\epsilon H(q^{(i)},p^{(i)})}\big|_{q^{(N)}=q^{(0)}}$$

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- Coordinates q are identified at "time" 0 and β
- $\rightarrow$ coordinates satisfy periodic boundary conditions
- Formula holds for any Hamiltonian of the form $H(\hat{p}, \hat{q}) = P_1(\hat{p}) + P_2(\hat{q})$ where P_i are polynomials

Field theory

We can immediately generalize to field theory. Replace

- $\alpha \to \vec{x}$
- $q_\alpha(\tau) \to \phi(\vec{x}, \tau)$
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Noether's theorem $\rightarrow \mathcal{N}$ conserved charge density

$\mathcal{H}$ quadratic in π

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Gaussian integral $\rightarrow$

$$Z = N \int_{periodic} [\mathrm{d}\phi] e^{\left(-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L}_E\right)} := N \int_{periodic} [\mathrm{d}\phi] e^{-S_E}$$

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$$\phi(\vec{x},\tau) = \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3}\tilde{\phi}(\omega_n,\vec{p})e^{i\vec{k}\vec{x}+i\omega_n\tau}$$

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With $\omega_n = \frac{2\pi n}{\beta}$ the Matsubara frequencies

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- Introduce Grassmann variables with $\{a, a^*\} = \{a, a\} = \{a^*, a^*\} = 0$
- Replace operators by Grassmann variables to get normal form $\tilde{A}(a^*, a) = K_{00} + K_{10}a^* + K_{01}a + K_{11}a^*a$

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- Product:
$C(a^*, a) = \int \mathrm{d}b^* \mathrm{d}b \, e^{-b^* b} A(a^*, b) B(b^*, a)$

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  $\mathrm{Tr}\left(e^{-\epsilon(E-\mu)\hat{a}^\dagger\hat{a}}\right)^N := \mathrm{Tr}\,\hat{\rho}_\epsilon^N$

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Matrix form of $\hat{\rho}_\epsilon$

$$\rho_\epsilon(a^*, a) = e^{a^* a}\tilde{\rho}_\epsilon(a^*, a) = e^{a^* a}e^{(e^{-\epsilon(E-\mu)}-1)a^* a} \approx e^{a^* a}e^{-\epsilon(E-\mu)a^* a}$$

$$\rho_\epsilon(a^*, a) \approx e^{a^* a(1-\epsilon(E-\mu))}$$

$$\rho(a_N^*, a_N) = \int \prod_{i=1}^{N-1} da_i^* da_i e^{-a_i^* a_i} \rho_\epsilon(a_N^*, a_{N-1}) \rho_\epsilon(a_{N-1}^*, a_{N-2}) \cdots \rho_\epsilon(a_1^*, a_0)\big|_{a_0=a_N}$$

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$$Z = \mathrm{Tr}\hat{\rho} = \int da_N^* da_N e^{-a_N^* a_N} \rho(a_N^*, -a_N)$$
$$= \int \prod_{i=1}^{N} da_i^* da_i e^{a_i^* a_{i-1}(1-\epsilon(E-\mu)) - a_i^* a_i}\big|_{a_0=-a_N}$$

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Contiuum

$$\int_{antip} [da^*][da] e^{-\int_0^\beta d\tau (a^* \frac{\partial a}{\partial \tau} + H - \mu N)}$$

Field theory

$$\int_{antiper} [d\psi^*][d\psi] e^{-\int_0^\beta d\tau \int d^3x \left(\psi^*(x)\frac{\partial}{\partial\tau}\psi(x) + \mathcal{H}(\psi,\psi^*) - \mu\mathcal{N}(\psi,\psi^*)\right)}$$

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$$\psi(\vec{x},0) = -\psi(\vec{x},\beta)$$

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- Obtain Green's function by derivation with respect to J at $J=0$
- Expectation value of an operator

$$\langle\mathcal{O}\rangle = \frac{\int_{periodic}[\mathrm{d}\phi]\,\mathcal{O}(\phi)e^{-S_E}}{Z[J=0]}$$

Propagator

Generating functional can be integrated for free field

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$$\Delta(x) = \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3} \tilde{\Delta}(\omega_n, \mathbf{p}) e^{\mathrm{i}(\omega_n\tau + \mathbf{p}\cdot\vec{x})}$$

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- Propagator in frequency momentum space

$$\tilde{\Delta}(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}$$

Get expressions for finite T from those at $T = 0$ by replacing:

$$\begin{aligned} p_4 &\rightarrow \omega_n \quad \omega_n = 2\pi n/\beta \\ \int \frac{\mathrm{d}p_4}{2\pi} f(p_4) &\rightarrow \frac{1}{\beta} \sum_n f(\omega_n) \\ \int \mathrm{d}^4 x &\rightarrow \int_\beta \mathrm{d}^4 x \end{aligned}$$

Links and action on lattice

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$$U_\mu(n) = \mathrm{P} e^{\mathrm{i}g \int_n^{n+\hat{\mu}} \mathrm{d}z_\nu A_\nu(z)}$$

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- SU(N) action

$$S_G = \beta \sum_{n,\mu<\nu} [1 - \mathrm{Tr}(U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n))/2N]$$

$$\beta = 2N/g^2$$

Center

- The center $\mathcal{C} \subset G$ of a group G consists of all elements $z \in G$ for which $zgz^{-1} = g \, \forall g \in G$

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- In other words, an element of the center commutes with all elements of the group

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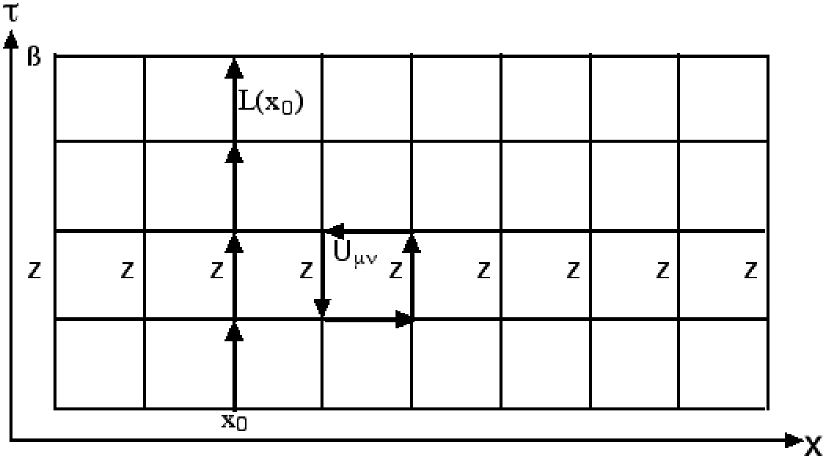
- Action invariant under these transformations, too
- For example $n_4 = 0$

$$U_4(\vec{n}, 0) \longrightarrow zU_4(\vec{n}, 0)$$

Center symmetry



Center symmetry

- Let us look at a plaquette

$$
\begin{aligned}
U_{i4}(\vec{n},0) &= U_i(\vec{n},0)U_4(\vec{n}+\hat{i},0)U_i^\dagger(\vec{n},1)U_4^\dagger(\vec{n},0) \\
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- Action is composed of plaquettes, therefore it is invariant.

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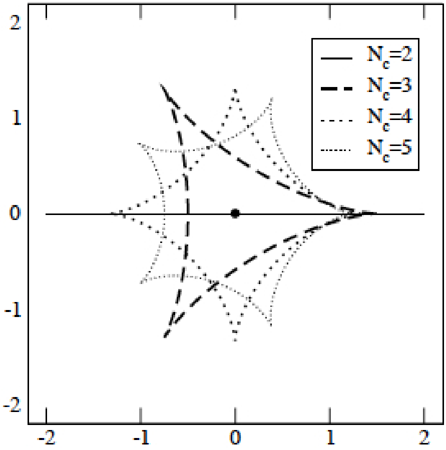
- Trace of special unitary matrix. It can get values in complex plane

Polyakov loop



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- Polyakov loop is invariant under periodic gauge transformations $G(\vec{n},0) = G(\vec{n},N_\tau)$

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\mathrm{Tr}\prod_{n_4=0}^{N_\tau-1} U_4(\vec{n},n_4) \quad &\rightarrow \quad \mathrm{Tr}\prod_{n_4=0}^{N_\tau-1} G(\vec{n},n_4)U_4(\vec{n},n_4)G^{-1}(\vec{n},n_4+1) \\
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- But not invariant under center transformations unless it is zero $L(\vec{n}) \longrightarrow zL(\vec{n})$

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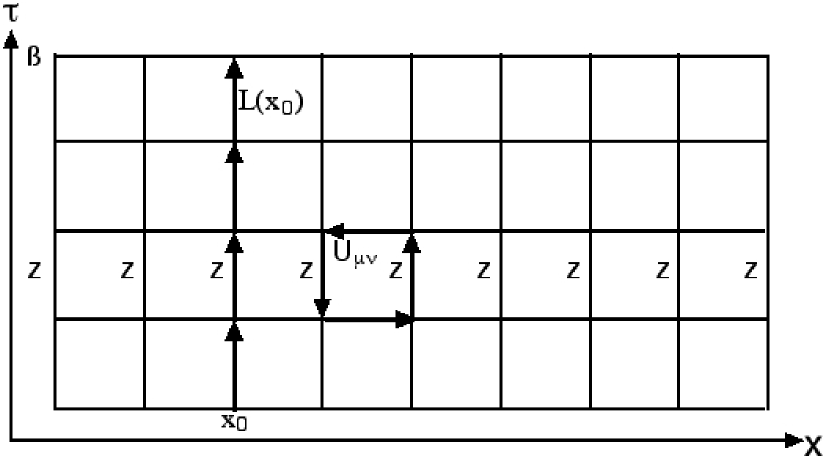

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- center symmetric confined phase if $\langle L \rangle = 0$ at low temperatures
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- phases separated by a phase transition
- Polyakov loop serves as an order parameter for distinguishing these phases

Physical meaning of Polyakov loop

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- $$Z = \sum_s \langle s|e^{-\beta H}|s\rangle$$
- Apply creation operators to states $|s'\rangle$ which do not contain the quark

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Z &= N \sum_{s'} \langle s' | \Psi(\vec{x}, 0) e^{-\beta H} \Psi^\dagger(\vec{x}, 0) | s' \rangle \\
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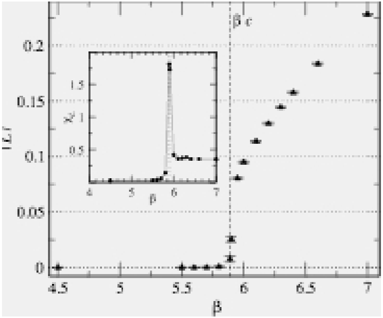
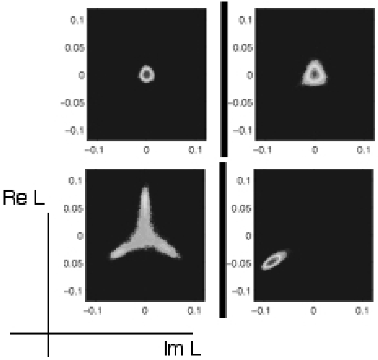
- If $\langle L\rangle = 0$ energy increases with higher separations $\to$ confinement
- If $\langle L\rangle \neq 0$ energy approaches finite constant $\to$deconfinement

Absolute value of P. loop as a function of temperature



Distribution of Polyakov loop in complex plane



Time evolution during simulation near critical Temperature