Yang-Mills Theory and the QCD Lagrangian

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Overview

Introduction

Why gauge symmetry?

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Why gauge symmetry?

Gauge symmetry

- possibility to chose freely a local parameter without changing the physics
- offers a way to describe interactions due to invariance properties of Lagrangians

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and expand to first order in α

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Try to construct a derivative since the Lagrangian $\mathcal{L}$ will eventually contain derivatives of the field:

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{\psi(x + \epsilon n) - \psi(x)}{\epsilon} \tag{3}$$

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$\Longrightarrow$ difference makes sense $\Leftrightarrow$ $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation law.
This defines the *covariant derivative*:

$$n^\mu D_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon} \tag{5}$$

and we can find an explicit expression via the expansion of the comparator

$$U(x+\epsilon n, x) = \underbrace{U(x,x)}_{:=1} - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2) \tag{6}$$

$$\begin{aligned} \Longrightarrow \quad n^\mu D_\mu \psi &= \lim_{\epsilon\longrightarrow 0} \frac{\psi(x+\epsilon n) - (1 - ie\epsilon n^\mu A_\mu(x))\psi(x)}{\epsilon} \\ &= \underbrace{\lim_{\epsilon\longrightarrow 0} \frac{1}{\epsilon}(\psi(x+\epsilon n) - \psi(x))}_{\partial_\mu \psi(x)} + ien^\mu A_\mu(x)\psi(x) \end{aligned} \tag{7}$$

$$\Rightarrow D_\mu = \partial_\mu + ieA_\mu(x) \tag{8}$$

$A_\mu(x)$ is called gauge field and its transformation law is given by

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \qquad (9)$$

needed to fix the gauge invariance of the covariant derivative:

$$\begin{aligned} D_\mu \psi(x) \rightarrow & \left(\partial_\mu + ie(A_\mu(x) - \frac{1}{\epsilon}\partial_\mu \alpha)\right) e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}\left(\partial_\mu + ieA_\mu(x)\right)\psi(x) \\ &\quad + e^{i\alpha(x)} i(\partial_\mu \alpha(x))\psi(x) - \frac{ie}{e}(\partial_\mu \alpha(x))\psi(x) \\ &= e^{i\alpha(x)}(\partial_\mu + ieA_\mu(x))\psi(x) \\ &= e^{i\alpha(x)} D_\mu \psi(x) \end{aligned} \qquad (10)$$

Our Lagrangian is a function

$$\mathcal{L} = \mathcal{L}(\psi(x), D\psi(x), A_\mu(x), t) \tag{11}$$

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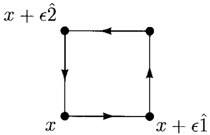
by expanding the comparator to order ϵ^2 and consider it around an infinitesimal closed rectangular loop



$$\begin{aligned} \mathbf{U}(x) := {} & U(x, x + \epsilon\hat{2})U(x + \epsilon\hat{2}, x + \epsilon\hat{1} + \epsilon\hat{2}) \\ & \times U(x + \epsilon\hat{1} + \epsilon\hat{2}, x + \epsilon\hat{1})U(x + \epsilon\hat{1}, x) \end{aligned} \tag{12}$$

using

$$U(x+\epsilon n, x) = \exp\left(-ie\epsilon n^\mu A_\mu(x+\tfrac{\epsilon}{2}n) + \mathcal{O}(\epsilon^3)\right) \tag{13}$$

we find

$$\mathbf{U}(x) = 1 - i\epsilon^2 e \underbrace{\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)}_{:=F_{\mu\nu}} + \mathcal{O}(\epsilon^3) \tag{14}$$

which depends only on $A_\mu(x) \Rightarrow$ gauge invariant

Field-strength tensor

$F_{\mu\nu} \hat{=}$ recognized as the QED field-strength tensor which is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{15}$$

where

$$A_\mu = \left(\frac{\phi}{c}, \vec{A}\right) \tag{16}$$

$$\partial_\mu = \left(\partial_0, \vec{\nabla}\right) \tag{17}$$

QED Lagrangian

requirements to the Lagrangian $\mathcal{L}$:

- 4-dimensional
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Most general $\mathcal{L}$ which fulfills requirements is given by

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi - \frac{1}{4}(F_{\mu\nu})^2 \tag{18}$$

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kinetic and potential term for the fields ψ as well as for the gauge field A_μ:

$$-\frac{1}{4}(F_{\mu\nu})^2 = \frac{1}{2}\left(E^2 - B^2\right) \tag{19}$$

Non-abelian gauge symmetry

In 1954 Yang and Mills considered the isospin-doublet and found a general gauge invariant Lagrangian.
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isospin doublet $\Leftrightarrow$ state which remains invariant under spin-transformations. $\Rightarrow$ consider the doublet of fields

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{20}$$

and the local transformation

$$\psi \longrightarrow \underbrace{\exp\left(i\alpha^j(x)\frac{\sigma^j}{2}\right)}_{:=V(x)} \psi \tag{21}$$

where the $\frac{\sigma^j}{2}$ are the Pauli matrices.

$V(x) \in SU(2) \Leftrightarrow$ the Pauli matrices are the generators of a Lie algebra $\mathrm{Lie}SU(2)$

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$V(x) \in SU(2) \Leftrightarrow$ the Pauli matrices are the generators of a Lie algebra $\mathrm{Lie}SU(2)$
They do not commute $\Rightarrow$ non-abelian gauge symmetry
Define the comparator via its transformation law

$$U(y,x) \longrightarrow V(y)U(y,x)V^{\dagger}(x) \tag{22}$$

with the expansion

$$U(x+\epsilon n, x) = \underbrace{U(x,x)}_{:=1} + ig\epsilon n^{\mu} A^{i}_{\mu}\frac{\sigma^{i}}{2} + \mathcal{O}(\epsilon^{2}) \tag{23}$$

and going through the same steps as before gives the *covariant derivative* as

$$D_{\mu} = \partial_{\mu} - igA^{i}_{\mu}\frac{\sigma^{i}}{2} \tag{24}$$

Inserting (23) in (22) and doing some algebra gives the transformation law of the gauge field up to order ϵ^2 as

$$A^i_\mu \frac{\sigma^i}{2} \longrightarrow A^i_\mu \frac{\sigma^i}{2} + \frac{1}{g}(\partial_\mu \alpha^i)\frac{\sigma^i}{2} + i\left[\alpha^i \frac{\sigma^i}{2}, A^j_\mu \frac{\sigma^j}{2}\right] + \dots \quad (25)$$

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compared to the abelian case: new term involving the commutator of the generators of our transformation group. Doing some algebra we see that again the covariant derivative of (20) transforms properly:

$$D_\mu \psi \longrightarrow \underbrace{\exp\left(i\alpha^j \frac{\sigma^j}{2}\right)}_{=V(x)} D_\mu \psi \quad (26)$$

kinetic energy term

As in the abelian case we have to find a "kinetic energy term" for the gauge field.
$\Rightarrow$ take the commutator of two covariant derivatives, which is gauge invariant since the covariant derivatives are, i.e.

$$[D_\mu, D_\nu]\,\psi \longrightarrow V(x)\,[D_\mu, D_\nu]\,\psi \tag{27}$$

and we find

$$\begin{aligned}[D_\mu, D_\nu]\psi &= -ig\left(\partial_\mu A^j_\nu \frac{\sigma^j}{2} - \partial_\nu A^i_\mu \frac{\sigma^i}{2} - ig\left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2}\right]\right)\psi \\ &\equiv -igF^i_{\mu\nu}\frac{\sigma^i}{2}\psi \end{aligned} \tag{28}$$

where $F^i_{\mu\nu} \mathrel{\hat{=}}$ field-strength tensor of the gauge field.

Field-strength tensor in the $SU(2)$

The field-strength tensor can be extracted to give

$$F^i_{\mu\nu} = \partial_\mu A^j_\nu - \partial_\nu A^i_\mu - ig\left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2}\right]\frac{2}{\sigma^i}$$
$$= \partial_\mu A^j_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu. \qquad (29)$$

where as in the transformation law for A^i_μ a commutator appears.

Lie group $\Leftrightarrow$ Lie algebra

A Lie group is:

- a continuous transformation group
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- completely captures the local group structure
- can be represented by the inverse exponential map of a Lie group

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$\Longrightarrow$ symmetry transformations as elements of the Lie group

$$\psi(x) \longrightarrow V(x)\psi(x) \tag{30}$$

$V(x) \in SU(N)$.

The exponential map has as arguments elements of the Lie algebra and can be expanded as

$$V(x) = 1 + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2) \tag{31}$$

where the t^a are the generators of the algebra. An algebra is assigned with a multiplication law.
In Lie algebras take the commutator

$$\left[t^a, t^b\right] = if_c^{ab}t^c \tag{32}$$

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The covariant derivative in general given as

$$D_\mu = \partial_\mu - igA_\mu^a t^a \tag{33}$$

infinitesimal transformation of A_μ^a is given by

$$A_\mu^a \longrightarrow A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a + f_{bc}^a A_\mu^b \alpha^c \tag{34}$$

From this and in analogy to the $SU(2)$ case the finite transformation yields

$$A_\mu^a(x)t^a \longrightarrow V(x)\left(A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{35}$$

Field strength tensor

The field strength tensor is defined by

$$[D_\mu, D_\nu]\psi = ig \underbrace{\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c\right)}_{F_{\mu\nu}^a} t^a \psi \tag{36}$$

$$= -igF_{\mu\nu}^a t^a \psi \tag{37}$$

and its transformation law by

$$F_{\mu\nu}^a t^a \longrightarrow V(x) F_{\mu\nu}^b t^b V^\dagger(x) \tag{38}$$

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$\Rightarrow$ not gauge invariant anymore!
Reasonable since it has to reflect the algebra's structure.

Field strength tensor

Construct gauge invariant terms:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\mathrm{Tr}\left[\left(F_{\mu\nu}\right)^2\right] = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \tag{40}$$

Field strength tensor

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$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\mathrm{Tr}\left[(F_{\mu\nu})^2\right] = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \tag{40}$$

as one can verify

$$-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \longrightarrow \mathrm{Tr}\left[V(x)\right]\left(-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2\right)\mathrm{Tr}\left[V^\dagger(x)\right] \tag{41}$$

$$= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 \tag{42}$$

Field strength tensor

Notice: the definition of the field strength tensor contains new terms.

$\Rightarrow$ selfinteraction of the gauge fields
$\Rightarrow$ gauge bosons carry themselves charge
$\Rightarrow$ can hypothetically form particles only by themself ("glueballs")

Lagrangian density

Now we have all ingredients for a gauge invariant Lagrangian:

- kinetic term for ψ given by the Dirac formalism
- mass term for ψ, i.e. $m\bar{\psi}\psi$
- kinetic and potential energy term for the gauge field A^a_μ

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$$\Longrightarrow \qquad \mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - m\bar{\psi}\psi \qquad (43)$$

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most general gauge inv. Lagrangian (*Yang-Mills Lagrangian*)
A mass term for the gauge field is ruled out because it can not be made gauge invariant
$\Rightarrow$ gauge bosons have to be massless!

How to find (40)?

Consider a general Lagrangian

$$\mathcal{L} = \mathcal{L}\left(\psi, D_\mu\psi, D_\mu D_\nu\psi, \ldots, F^a_{\mu\nu}, D_\sigma F^a_{\mu\nu}, D_\sigma D_\rho F^a_{\mu\nu}, \ldots\right) \quad (44)$$

and the invariance condition

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi} it^a\psi + \frac{\partial\mathcal{L}}{\partial(D_\mu\psi)} it^a(D_\mu\psi) + \cdots + \frac{\partial\mathcal{L}}{\partial F^a_{\mu\nu}}\delta F^a_{\mu\nu} \\ + \frac{\partial\mathcal{L}}{\partial D_\sigma F^a_{\mu\nu}}\delta D_\sigma F^a_{\mu\nu} + \cdots = 0\end{aligned} \quad (45)$$

How to find (40)?

Other conditions on $\mathcal{L}$:

- Parity conservation
- Lorentz invariance
- $\mathcal{L}$ must contain a factor $\propto (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$

dictates the gauge field term to be

$$\mathcal{L}_F = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} \tag{46}$$

with g_{ab} a constant, real, symmetric matrix
with right scalings it can be taken to be $g_{ab} = \delta_{ab}$

$$\Longrightarrow \qquad \mathcal{L}_F = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \tag{47}$$

Conservation laws

Euler-Lagrange equations for gauge field A^a_μ yield:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial \mathcal{L}}{\partial A^a_\nu} \tag{48}$$

$$\implies -\partial_\mu F^{a\,\mu\nu} = \underbrace{-F^{c\,\nu\mu} f_b^{ca} A^b_\mu - i \frac{\partial(\bar{\psi}(i\not{D} - m)\psi)}{\partial D_\nu \psi} t^a \psi}_{:=\mathcal{J}^{a\,\nu}} \tag{49}$$

$$\implies \partial_\mu F^{a\,\mu\nu} = -\mathcal{J}^{a\,\nu} \tag{50}$$

easy to show that

$$\partial_\nu \mathcal{J}^{a\,\nu} = 0 \tag{51}$$

$\implies \mathcal{J}^{a\,\nu}$ is conserved.

Conservation laws

BUT: partial derivatives $\Leftrightarrow$ gauge invariance not guaranteed.
Rewrite the E.-L.-equation

$$D_\lambda F^{a\,\mu\nu} = \partial_\lambda F^{a\,\mu\nu} - g f_b^{ac} A_\lambda^b F^{a\,\mu\nu} \tag{52}$$

and we conclude

$$D_\mu F^{a\,\mu\nu} = -\mathcal{J}^{a\,\nu} \tag{53}$$

but with a different conserved current

$$\Longrightarrow \mathcal{J}^{a\,\nu} = -i \frac{\partial(\bar{\psi}(i\not{D} - m)\psi)}{\partial D_\nu \psi} t^a \psi \tag{54}$$

Conservation laws

which is conserved since using

$$[D_\nu, D_\mu]\, F^{a\,\rho\sigma} = -f^a_{cb} F^c_{\nu\mu} F^{b\,\sigma\rho} \tag{55}$$

we see that

$$D_\nu \mathcal{J}^{a\,\nu} = 0 \tag{56}$$

and guaranteed gauge invariance.

Motivation

Late 1950s and beginning of the 1960s:

- more and more hadrons were discovered in collider experiments
- in Deep Inelastic Scattering it was shown that hadrons are not elementary particles

$\Rightarrow$ Solution by Gell-Mann and Zweig: hadrons are built up from two (mesons) or three (baryons) fermionic particles, called quarks.
$\Rightarrow$ min. 3 "'flavors"' were needed $\Rightarrow SU(3)_{\text{flavor}}$ taken as transformation group
In the beginning quarks were supposed to be hypothetical particles since they could not be observed

BUT: by the investigation of the Δ^{++} hadron build up of 3 up-quarks new problems arised:

The spin-statistic problem

Wave-function in the ground-state ($L = 0$):

$$\psi_{\Delta^{++}} = \psi_{\text{spin}} \otimes \psi_{\text{flavor}} \otimes \psi_{\text{spatial}} \tag{57}$$

and

$$\Delta^{++} = u(\uparrow)u(\uparrow)u(\uparrow) \tag{58}$$

Δ^{++} is symmetric $\Leftrightarrow$ needs to be antisym. ($s = \frac{3}{2}$)

Solution by Greenberg (1964) extended by Gell-Mann (1972): introduce an internal quantum number:

COLOR

and color-transformations as an exact $SU(3)$ symmetry transformation law of the $SU(3)$ color group:

$$q \longrightarrow q' = \underbrace{\exp\left[-i\alpha_k \frac{\lambda_k}{2}\right]}_{:=V(x)} q \tag{59}$$

q: fermionic quark field
where the group reads

$$SU(3) = \left\{A \in GL(3,\mathbb{C}) | A^\dagger A = 1, \det A = 1\right\} \tag{60}$$

and the generators of the corresponding Lie algebra are hermitian and

$$\text{unimodularity} \quad \Leftrightarrow \quad \mathrm{Tr}[\lambda_k] = 0 \tag{61}$$

Gell-Mann matrices

generators of the algebra represented by *Gell-Mann matrices*

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Lagrangian and gluons

Nambu, Fritsch, Gell-Mann and Leutwyler:
color charges are sources of gauge field that transfer the strong interaction
Write Yang-Mills Lagrangian for $SU(3)$:

$$\mathcal{L} = \bar{q}(x) \left(i \not{D} - m \right) q(x) - \frac{1}{2} \mathrm{Tr} \left[\left(G_{\mu\nu} \right)^2 \right] \tag{62}$$

with gluon fields $G_\mu^k(x)$ defined by

$$G_{\mu\nu} = D_\nu G_\mu - D_\mu G_\nu \tag{63}$$

$$= \partial_\nu G_\mu - \partial_\mu G_\mu - ig_s \left[G_\mu, G_\nu \right] \tag{64}$$

and covariant derivative

$$D_\mu = \partial_\mu - ig_s G_\mu^k \frac{\lambda_k}{2} \tag{65}$$

Interpretation of Lagrangian ingredients

$$\mathcal{L} = \bar{q}(x)\left(i\not{D} - m\right)q(x) - \frac{1}{2}\,\mathrm{Tr}\left[\left(G_{\mu\nu}\right)^2\right] \tag{66}$$

$$\begin{aligned} =&\bar{q}(x)\left(i\not{\partial} - m\right)q(x) - \frac{1}{2}\,\mathrm{Tr}\left[\left(\partial_\nu G_\mu - \partial_\mu G_\mu\right)^2\right] \\ &+ g_s \bar{q}(x)\not{G}^k(x)\frac{\lambda_k}{2}q(x) \\ &+ i g_s\,\mathrm{Tr}\left(\partial_\nu G_\mu - \partial_\mu G_\nu\right)\left[G_\mu, G_\nu\right] \\ &+ \frac{1}{2}g_s^2\,\mathrm{Tr}\left[G_\mu, G_\nu\right]^2 \end{aligned} \tag{67}$$

Thanks

Many thanks to

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