Outline
Introduction
Abelian gauge fields on the lattice (QED)
Non abelian gauge fields on the lattice (QCD)
The continuum limit
Strong coupling expansion

Lattice formulation of Yang-Mills theory and confinement

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Lattice QCD

Perturbation theory

- full solution A, known solution (without perturbation) A_0

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- full solution A, known solution (without perturbation) A_0
- series: $A = \epsilon^0 A_0 + \epsilon^1 A_1 + \epsilon^2 A_2 + \dots$

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- QCD: $\epsilon \approx 1$ (highly dependent on the energy)

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- QED: $\epsilon = \alpha \approx \frac{1}{137} \ll 1$
- QCD: $\epsilon \approx 1$ (highly dependent on the energy)
- no reasonable perturbative approach for QCD!

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- QED: $\epsilon = \alpha \approx \frac{1}{137} \ll 1$
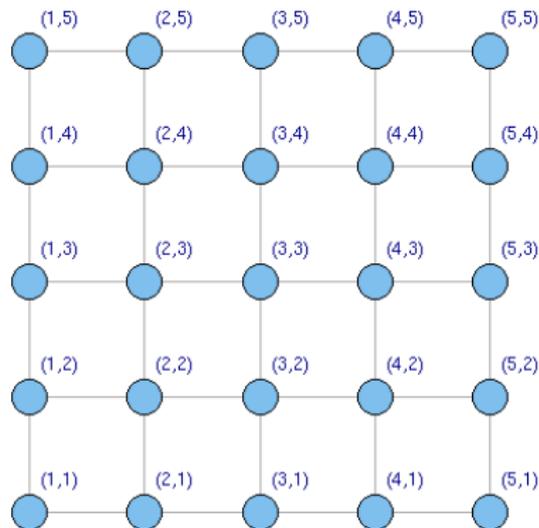- QCD: $\epsilon \approx 1$ (highly dependent on the energy)
- no reasonable perturbative approach for QCD!
- non-perturbative approach: Lattice QCD

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Lattice QCD - a two dimensional example

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From continuum QCD to lattice QCD...

- ▶ introduce a hypercubic lattice in your theory

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- ▶ introduce a hypercubic lattice in your theory
- ▶ all quantities are only defined on lattice points

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- ▶ quantities may depend on lattice spacing a

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From continuum QCD to lattice QCD...

- ▶ introduce a hypercubic lattice in your theory
- ▶ all quantities are only defined on lattice points
- ▶ quantities may depend on lattice spacing a
- ▶ partition function $Z = \int D\phi \, e^{-S}$
  integration over infinitely many space-points

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From continuum QCD to lattice QCD...

- ▶ introduce a hypercubic lattice in your theory
- ▶ all quantities are only defined on lattice points
- ▶ quantities may depend on lattice spacing a
- ▶ partition function $Z = \int D\phi \, e^{-S}$
  integration over infinitely many space-points
- ▶ by introducing a lattice only finitely many integrations

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...and back to continuum QCD

▶ continuum QCD should be recovered for $a \to 0$

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...and back to continuum QCD

- ▶ continuum QCD should be recovered for $a \rightarrow 0$
- ▶ problems may arise:
  - symmetry breaking terms
  - fermion doublers

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QED action and gauge symmetry

- action of QED given by
  $S_{QED} = \int d^4x \left[\bar{\psi}(x)(i\gamma_M^\mu D_\mu - m)\psi(x) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]$

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- covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ promotes global gauge symmetry to local one

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- covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ promotes global gauge symmetry to local one

- local $U(1)$ transformations:
  $\psi(x) \rightarrow G(x)\psi(x)$
  $\bar{\psi}(x) \rightarrow \bar{\psi}(x)G^{-1}(x)$
  $A_\mu(x) \rightarrow G(x)A_\mu(x)G^{-1}(x) - \frac{i}{e}G(x)\partial_\mu G^{-1}(x)$
  $G(x) = e^{i\Lambda(x)} \in U(1)$

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Euclidean action for QED

- $x^0 \rightarrow -ix_4$
  $D_0 \rightarrow +iD_4$

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Euclidean action for QED

- $x^0 \to -ix_4$
  $D_0 \to +iD_4$
- $S_{QED} \to iS_E = i \int d^4x \left[\bar{\psi}(x)(\gamma^\mu D_\mu + m)\psi(x) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]$

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Substitutions in the lattice formulation

▶ quantities on the lattice are usually denoted with a hât

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- quantities on the lattice are usually denoted with a h$\hat{a}$t
- $x_\mu \rightarrow n_\mu a$

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Substitutions in the lattice formulation

- quantities on the lattice are usually denoted with a $\hat{\text{hat}}$
- $x_\mu \rightarrow n_\mu a$
- $\int d^4 x \rightarrow a^4 \sum_n$

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- quantities on the lattice are usually denoted with a $\hat{\text{hat}}$

- $x_\mu \rightarrow n_\mu a$

- $\int d^4 x \rightarrow a^4 \sum_n$

- $\psi_\alpha(x) \rightarrow \frac{1}{a^{3/2}} \hat{\psi}_\alpha(n)$
  $\bar{\psi}_\alpha(x) \rightarrow \frac{1}{a^{3/2}} \hat{\bar{\psi}}_\alpha(n)$

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- ▶ $\partial_\mu \psi(x) \rightarrow \frac{1}{a^{5/2}} \hat{\partial}_\mu \hat{\psi}(n) = \frac{1}{a^{3/2}} \cdot \frac{1}{2a} [\hat{\psi}(n + \hat{\mu}) - \hat{\psi}(n - \hat{\mu})]$

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▶ $m \rightarrow \frac{1}{a} \hat{m}$

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- $S_1 = \int d^4x \, \bar{\psi}(x) \gamma^\mu D_\mu \psi(x)$

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- action on the lattice is not unique
  naïve limit has to recover continuum action
  action can be modified to avoid difficulties (will be done to avoid problems with fermions)

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The link variables

- $\bar{\psi}(x)\psi(y) \rightarrow \bar{\psi}(x)G^{-1}(x)G(y)\psi(y)$ is not gauge invariant!

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- ▶ Dirac action on the lattice introduced link variables $U_\mu(n)$
- ▶ $U_\mu(x) = U(x, x+\mu) = e^{ie \int_x^{x+\mu} dx' \, A_\mu(x')}$

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- $U(x, y)$ transforms to $G(x)U(x, y)G^{-1}(y)$

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- $U_\mu(n) = e^{iaeA_\mu(n)} \approx 1 + iaeA_\mu(n)$

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- $U_\mu(n)$ are directed quantities

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\begin{aligned}
S_1 &= \sum_{n,\mu} \frac{1}{2a} \bar{\psi}(n) \gamma_\mu [U_\mu(n)\psi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})] \\
&= \sum_{n,\mu} \frac{1}{2a} \bar{\psi}(n) \gamma_\mu [(1 + iaeA_\mu(n) + \dots)(\psi(n) + a\partial_\mu \psi(n) + \dots) - \\
& \quad (1 - iaeA_\mu(n) + \dots)(\psi(n) - a\partial_\mu \psi(n) + \dots)] \\
&= \sum_{n,\mu} \bar{\psi}(n) \gamma_\mu (\partial_\mu + \frac{a^2}{6}\partial_\mu^3 + \dots)\psi(n) + \\
& \quad ie\bar{\psi}(n)\gamma_\mu [A_\mu + \frac{a^2}{2}(\frac{1}{4}\partial_\mu^2 A_\mu + (\partial_\mu A_\mu)\partial_\mu + A_\mu \partial_\mu^2) + \dots]\psi(n) \\
&= \sum_{n,\mu} \bar{\psi}(n) \gamma_\mu (\partial_\mu + ieA_\mu)\psi(n) + O(a^2) \, .
\end{aligned}
$$

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- $S_2 = \int d^4x \, \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

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- $\hat{S}_2 = \frac{1}{e^2} \sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \frac{1}{2}(P_{\mu\nu}(n) + P^\dagger_{\mu\nu}(n))]$

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The 1×1 plaquette

- $P_{\mu\nu}$ is a 1×1 plaquette, the simplest Wilson loop

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The 1×1 plaquette

▶ $P_{\mu\nu}$ is a 1×1 plaquette, the simplest Wilson loop

▶
$$
\begin{aligned}
P_{\mu\nu}(n) &= U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu^\dagger(n + \hat{\nu}) U_\nu^\dagger(n) \\
&= e^{iea^2 F_{\mu\nu}(n)}
\end{aligned}
$$

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&= e^{iea^2 F_{\mu\nu}(n)}
\end{aligned}
$$

▶ discretized field strength tensor:
$$
F_{\mu\nu}(n) = \frac{1}{a}[(A_\nu(n+\hat{\mu}) - A_\nu(n)) - (A_\mu(n+\hat{\nu}) - A_\mu(n))]
$$

The 1×1 plaquette

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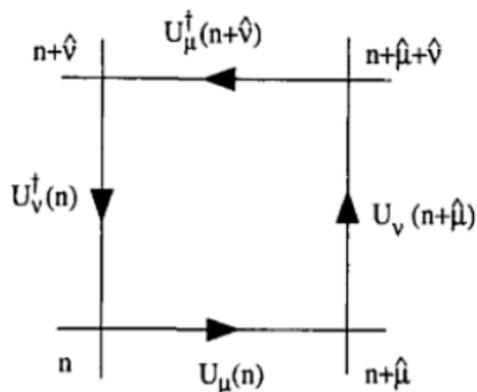
The gauge action on the lattice

$$
\begin{aligned}
S_2 &= \frac{1}{e^2} \sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \frac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))] \\
&= \frac{1}{e^2} \sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \frac{1}{2}(1 + iea^2 F_{\mu\nu(n)} - \frac{e^2 a^4}{2} F_{\mu\nu}^2(n) + \dots \\
&\quad +1 - iea^2 F_{\mu\nu}(n) - \frac{e^2 a^4}{2} F_{\mu\nu}^2(n) + \dots)] \\
&= \frac{1}{e^2} \sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \frac{1}{2}(2 - e^2 a^4 F_{\mu\nu}^2(n))] + O(a^6) \\
&\approx \frac{1}{2} \sum_n \sum_{\mu,\nu,\mu<\nu} [a^4 F_{\mu\nu}^2(n)] \\
&= \frac{1}{4} \sum_{n,\mu,\nu} a^4 F_{\mu\nu}(n) F_{\mu\nu}(n) \, .
\end{aligned}
$$

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The gauge action on the lattice
The action on the lattice

The simplest action on the lattice

$$
\begin{aligned}
S \quad = \quad & \frac{1}{2a}\bar{\psi}(n)\sum_\mu \gamma_\mu[U_\mu(n)\psi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})] \\
& + \frac{1}{e^2}\sum_n \sum_{\mu,\nu,\mu<\nu}[1 - \tfrac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))] \\
& + \hat{m}\sum_n \bar{\psi}(n)\psi(n)
\end{aligned}
$$

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The Wilson action on the lattice

$$
\begin{aligned}
S_{QED}[U, \psi, \bar{\psi}] \;=\; & \tfrac{1}{e^2} \sum_n \sum_{\mu, \nu, \mu < \nu} [1 - \tfrac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^{\dagger}(n))] \\
& + (\hat{m} + 4r) \sum_n \bar{\psi}(n)\psi(n) - \\
& \tfrac{1}{2} \sum_{n,\mu} [\bar{\psi}(n)(r \cdot \mathbb{1}_4 - \gamma_\mu) U_\mu(n) \psi(n + \hat{\mu}) + \\
& \bar{\psi}(n + \hat{\mu})(r \cdot \mathbb{1}_4 + \gamma_\mu) U_\mu^{\dagger}(n) \psi(n)]
\end{aligned}
$$

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&\quad + (\hat{m} + 4r) \sum_n \bar{\psi}(n)\psi(n) - \\
&\quad \frac{1}{2} \sum_{n,\mu} [\bar{\psi}(n)(r \cdot \mathbb{1}_4 - \gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \\
&\quad \bar{\psi}(n+\hat{\mu})(r \cdot \mathbb{1}_4 + \gamma_\mu)U_\mu^\dagger(n)\psi(n)]
\end{aligned}
$$

► Added a term $r \cdot \Delta$ to the action

► vanishes for $a \to 0$

► r is called Wilson parameter

► breaks chiral symmetry

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From $U(1)$ to $SU(N)$

- $\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix}$, $\bar{\psi} = \begin{pmatrix} \bar{\psi}^1, & \ldots, & \bar{\psi}^N \end{pmatrix}$

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- $G(n) = e^{i\Lambda(n)}$ with $G(n) \in SU(N)$

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- $G(n) = e^{i\Lambda(n)}$ with $G(n) \in SU(N)$

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- $G(n) = e^{i\Lambda(n)}$ with $G(n) \in SU(N)$

- $U_\mu(n) = e^{igA_\mu(n)}$ with $U_\mu(n) \in SU(N)$

- $G(n)$ and $U_\mu(n)$ not abelian anymore!

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The fermion action

$$
\begin{aligned}
S_F \;=\;& (\hat{m} + 4r) \sum_n \bar{\psi}(n)\psi(n) \\
& -\tfrac{1}{2} \sum_{n,\mu} [\bar{\psi}(n)(r - \gamma_\mu)U_\mu(n)\psi(n + \hat{\mu}) \\
& + \bar{\psi}(n + \hat{\mu})(r + \gamma_\mu)U_\mu^\dagger(n)\psi(n)]
\end{aligned}
$$

▶ same action as in the abelian case

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The gauge action

$$S_G = \frac{2}{g^2} Tr \sum_{n,\mu<\nu} [\mathbb{1}_3 - \frac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))]$$

▶ similar as before, but now we take the trace

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- similar as before, but now we take the trace
- simplest way to achieve a gauge invariant action

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The gauge action

$S_G = \frac{2}{g^2} Tr \sum_{n,\mu<\nu} [\mathbb{1}_3 - \frac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))]$

- ▶ similar as before, but now we take the trace
- ▶ simplest way to achieve a gauge invariant action
- ▶ action holds for $SU(3)$

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The link variables in $SU(3)$

- remember the definition:
  $P_{\mu\nu}(n) = U_\mu(n)U_\nu(n + \hat{\mu})U_\mu^\dagger(n + \hat{\nu})U_\nu^\dagger(n)$

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▶ remember the definition:
$$P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)$$

▶ $P_{\mu\nu}(n) = e^{iga^2\mathcal{F}_{\mu\nu}(n)}$

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- $P_{\mu\nu}(n) = e^{iga^2 \mathcal{F}_{\mu\nu}(n)}$

- Baker-Campbell-Hausdorff: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+...}$

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- remember the definition:
  $P_{\mu\nu}(n) = U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu^\dagger(n + \hat{\nu}) U_\nu^\dagger(n)$

- $P_{\mu\nu}(n) = e^{iga^2 \mathcal{F}_{\mu\nu}(n)}$

- Baker-Campbell-Hausdorff: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+...}$

- $\mathcal{F}_{\mu\nu} \xrightarrow{a \to 0} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

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The non abelian action

$$
\begin{aligned}
S &= (\hat{m} + 4r) \sum_n \bar{\psi}(n)\psi(n) \\
&\quad -\tfrac{1}{2} \sum_{n,\mu} [\bar{\psi}(n)(r - \gamma_\mu) U_\mu(n)\psi(n + \hat{\mu}) \\
&\quad +\bar{\psi}(n + \hat{\mu})(r + \gamma_\mu) U_\mu^\dagger(n)\psi(n)] \\
&\quad +\tfrac{2}{g^2} Tr \sum_{n,\mu<\nu} [\mathbb{1}_3 - \tfrac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))] \,,
\end{aligned}
$$

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The naïve limit

- naïve limit: $a \to 0$

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The naïve limit

- naïve limit: $a \to 0$
- Θ observable, coupling $g(a)$ may depend on a
  $\Theta(g(a), a) \to \Theta_{PHYS}$

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The Wilson loop

- Wilson loop: $W_C[A] = P e^{ig \int dz_\mu A_\mu(z)}$

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- Wilson loop: $W_C[A] = P e^{ig \int dz_\mu A_\mu(z)}$
- consider static quark-antiquark potential (no kinetic energy)

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- Wilson loop: $W_C[A] = Pe^{ig \int dz_\mu A_\mu(z)}$
- consider static quark-antiquark potential (no kinetic energy)
- $\langle W_C[A] \rangle \equiv W(R, T) \xrightarrow{T \to \infty} F(R)e^{-V(R)T}$

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- Wilson loop: $W_C[A] = Pe^{ig \int dz_\mu A_\mu(z)}$
- consider static quark-antiquark potential (no kinetic energy)
- $\langle W_C[A] \rangle \equiv W(R, T) \stackrel{T \to \infty}{\longrightarrow} F(R)e^{-V(R)T}$
- $V(R, g, a) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W_C[A] \rangle =$
  $-\lim_{T \to \infty} \frac{1}{T} \ln \left[Pe^{ig \int dz_\mu A_\mu(z)} \right]$

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The Wilson loop

- Wilson loop: $W_C[A] = P e^{ig \int dz_\mu A_\mu(z)}$
- consider static quark-antiquark potential (no kinetic energy)
- $\langle W_C[A] \rangle \equiv W(R, T) \overset{T \to \infty}{\longrightarrow} F(R) e^{-V(R)T}$
- $V(R, g, a) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W_C[A] \rangle =$
  $-\lim_{T \to \infty} \frac{1}{T} \ln \left[P e^{ig \int dz_\mu A_\mu(z)} \right]$
- perturbative result:
  $V(R, g, a) = \frac{C}{4\pi R} \left[g^2 + \frac{22}{16\pi^2} g^4 \ln \frac{R}{a} + O(g^6) \right]$

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Diagrams contributing to potential in order g^4

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Callan-Symanzik β function

▶ renormalization group equation:
$[a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}]V(R,g,a) = 0$

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Callan-Symanzik β function

- renormalization group equation:
  $[a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}]V(R, g, a) = 0$

- Callan-Symanzik β function: $\beta(g) = -a\frac{\partial g}{\partial a}$

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Callan-Symanzik β function

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- Callan-Symanzik β function: $\beta(g) = -a\frac{\partial g}{\partial a}$

- with result from last slide: $\beta(g) \approx -\frac{11}{16\pi^2}g^3$

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Asymptotic freedom

- $a = \frac{1}{\Lambda_L} e^{-\frac{16\pi^2}{22g^2}}$

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Asymptotic freedom

- $a = \frac{1}{\Lambda_L} e^{-\frac{16\pi^2}{22g^2}}$
- $a \to 0 \Rightarrow g \to 0$
  asymptotic freedom!

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Asymptotic freedom

- $a = \frac{1}{\Lambda_L} e^{-\frac{16\pi^2}{22g^2}}$
- $a \to 0 \Rightarrow g \to 0$
  asymptotic freedom!
- $g \ll 1$; can therefore also be studied in perturbation theory, no lattice needed

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The Wilson loop operator

- remember:
  $S_G = \frac{6}{g^2} \sum_P [1 - \frac{1}{6} Tr(P + P^\dagger)] = -\frac{6}{g^2} \sum_P S_P + const.$
  $S_P = \frac{1}{6} Tr(P + P^\dagger)$

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- partition function: $Z = \int DU \, e^{\beta \sum_P S_P}$

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  $S_P = \frac{1}{6} Tr(P + P^\dagger)$

- partition function: $Z = \int DU \, e^{\beta \sum_P S_P}$

- Wilson loop operator: $W_C[U] = Tr \prod_{l \in C_L} U_l$

- $\langle W_C[U] \rangle = \frac{\int DU \, W_C[U] \prod_P e^{\beta S_P}}{\int DU \prod_P e^{\beta S_P}}$

  with Wilson loop C given by spatial extension $\hat{R}$ and temporal extension $\hat{T}$

- $\beta = \frac{6}{g^2}$

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Leading contribution

- expand $\prod_P e^{\beta S_P} = \prod_P \left[\sum_n \frac{\beta^n}{n!} (S_P)^n \right]$

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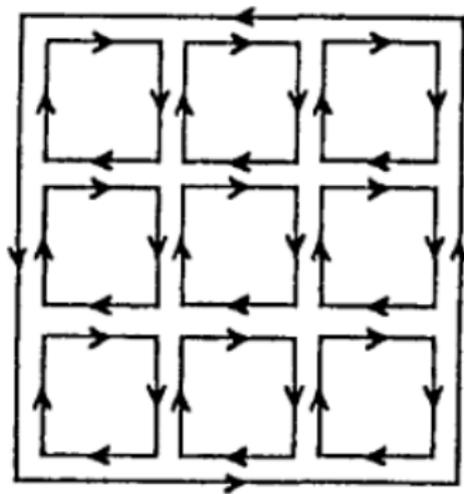# Leading contribution

- expand $\prod_P e^{\beta S_P} = \prod_P \left[\sum_n \frac{\beta^n}{n!} (S_P)^n \right]$

- leading contribution in β (strong coupling expansion): every plaquette costs a factor of β
  therefore we are interested in the Wilson loop with the smallest number of elementary plaquettes yielding a non-vanishing integral

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Leading contribution

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- integration rules for unitarian group: $\int dU \, U = 0$,
  $\int dU \, U^\dagger U = \int dU = 1$

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Leading contribution

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- leading contribution in β (strong coupling expansion): every plaquette costs a factor of β
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- integration rules for unitarian group: $\int dU\, U = 0$,
  $\int dU\, U^\dagger U = \int dU = 1$

- every link needs a counterpart link pointing in the other direction in order to have a non-vanishing integral!

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Smallest number of elementary plaquettes with non-vanishing integral

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Area law

- numerator of Wilson loop operator expectation value
  proportional to β^N with N the number of plaquettes

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Area law

- numerator of Wilson loop operator expectation value proportional to β^N with N the number of plaquettes
- $N = \hat{A} = \hat{R}\hat{T}$

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Area law

- numerator of Wilson loop operator expectation value proportional to β^N with N the number of plaquettes
- $N = \hat{A} = \hat{R}\hat{T}$
- Area law: $\langle W_C[U] \rangle \approx 3 \left(\frac{\beta}{18} \right)^{\hat{R}\hat{T}}$

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- $\hat{V}(\hat{R}) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W_C[U]\rangle = \hat{\sigma}(g)\hat{R}$

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Confinement

- $\hat{V}(\hat{R}) = -\lim_{T \to \infty} \frac{1}{T} \ln\langle W_C[U] \rangle = \hat{\sigma}(g)\hat{R}$

- static $q\bar{q}$-potential rises linearly with $\hat{R}$! Confinement!

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- static $q\bar{q}$-potential rises linearly with $\hat{R}$! Confinement!

- particle not in color-singlet state are impossible to separate

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Confinement

- $\hat{V}(\hat{R}) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W_C[U]\rangle = \hat{\sigma}(g)\hat{R}$

- static $q\bar{q}$-potential rises linearly with $\hat{R}$! Confinement!

- particle not in color-singlet state are impossible to separate

- calculations hold for small β (large g), numerical calculations indicate that it is also true for large β
  no phase transition (true for zero temperature)

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Abelian gauge fields on the lattice (QED)
Non abelian gauge fields on the lattice (QCD)
The continuum limit
Strong coupling expansion

Leading contribution of Wilson loop operator
Confinement

Confinement

- $\hat{V}(\hat{R}) = -\lim_{T \to \infty} \frac{1}{T} \ln\langle W_C[U] \rangle = \hat{\sigma}(g)\hat{R}$

- static $q\bar{q}$-potential rises linearly with $\hat{R}$! Confinement!

- particle not in color-singlet state are impossible to separate

- calculations hold for small β (large g), numerical calculations indicate that it is also true for large β
  no phase transition (true for zero temperature)

- caution: proof of confinement holds only on the lattice, not on continuum QCD