Proseminar FS09 in Theoretical Physics - Perturbative and non-perturbative methods for strong interactions

Lattice formulation of Yang-Mills theory and confinement at strong coupling

Basil Schneider
ETH Zürich
Dr. Philippe de Forcrand
Tutor: Dr. Marco Panero

27th April 2009

Contents

1 Introduction **2**

2 Abelian gauge fields on the lattice (QED) **3**

2.1 The lattice formulation 4

3 Non abelian gauge fields on the lattice **8**

4 The continuum limit **10**

4.1 The naïve limit 10

4.2 Asymptotic freedom 11

5 Strong coupling expansion **13**

1 Introduction

The perturbative approach in QED is very successful. Every higher order in perturbative theory comes with a higher order in the coupling constant α. With a coupling constant $\alpha \approx \frac{1}{137}$ well below 1, QED can be described in a precise way already in low orders. Now QCD is completely different. Experiments show that the coupling constant of the strong interaction gets very small at short distances. This means that particles obeying strong interactions get almost free for small distances. This behaviour is called asymptotic freedom. In the asymptotic freedom the coupling constant is of course very small and a perturbative approach is justified. But for larger separations the coupling gets larger and larger. When the coupling constant approaches 1, perturbative theory will breakdown, as one can not only look at the leading terms but would have to take the whole series. The behaviour of the coupling constant getting larger for larger distances leads to the situation that one is not able to separate particles as long as they are not color neutral. The fact that only color neutral particles as asymptotic states exist is called confinement.

Because of the large coupling we need an alternative for the perturbation theory - a non-perturbative approach. The most succesful non-perturbative approach to date for QCD is the lattice theory. First of all we would like to have a look at the partition function $Z = \int D\phi e^{-S}$. We have two infinities here: First, every field ϕ can have an infinite amount of configurations. Second, we integrate over an infinite amount of space-time points. Those two infinities make it impossible to solve this problem numerically. So we want to replace the continuum space-time by a discrete one. This we can do by introducing a lattice, where every field ϕ is only defined on a lattice point. So we replace the continuum space-time by a discrete one, having then only a finite number of space-time points as long as the lattice is stretched over a finite volume. One can prove that different lattices yield the same results in the continuum limit, therefore usually the simplest one, i.e. the hypercubic lattice, is considered.

But with the lattice theory we might get rid of an infinite space-time points and therefore an infinite amount of integrals, but do we really study a physical theory when considering it in a non-physical environment, i.e. working on the lattice instead in a continuum space-time? This question is very subtle. One might think that if we make the lattice finer and finer and in the end take the continuum limit (i.e. we let the lattice spacing a go to 0) one should end up with continuum QCD. But this is a process which involves more work than it might seem in the beginning. Several problems arise and probably the most important one is, that introducing a lattice breaks certain symmetries, for example rotational symmetry. If we work on a lattice the theory can only be invariant under rotations of a multiple of $\frac{\pi}{2}$. Also the translation symmetry is broken, as only translations of a multiple of the lattice spacing a are valid. But calculations show, that those symmetry breaking terms come in higher order with the lattice spacing a, so if we let $a \to 0$ we really end up with a gauge theory having the desired rotational and translational symmetry.

Another problem which can arise is due to fermions on the lattice. Because of the spin-statistics properties and their mathematical description as Grassmann variables we encounter some problems when dealing with fermions on the lattice. There exist several solutions to this problem, while we will use one of those solutions in this talk, we will not go into any further detail about the problem and for the time being just ignore it.

2 Abelian gauge fields on the lattice (QED)

In this section, we want to study how an abelian gauge field behaves on the lattice, to do so we consider QED as an example. In the next section we go a step further and look at a non-abelian gauge field.

The action of QED is given by

$$S_{QED} = \int d^4x \left[\bar{\psi}(x)(i\gamma_M^\mu D_\mu - m)\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right] , \tag{1}$$

where the last term is the kinetic term familiar from classical electrodynamics and γ_M are the Dirac-matrices in Minkowski space-time, m is the mass of the particle and $D_\mu = \partial_\mu + ieA_\mu$ denotes the covariant derivative which was introduced to obtain an action which is invariant under the *local* $U(1)$ transformations

$$\psi(x) \to G(x)\psi(x) , \tag{2}$$
$$\bar{\psi}(x) \to \bar{\psi}(x)G^{-1}(x) , \tag{3}$$
$$A_\mu(x) \to G(x)A_\mu(x)G^{-1}(x) - \frac{i}{e}G(x)\partial_\mu G^{-1}(x) \tag{4}$$

with

$$G(x) = e^{i\Lambda(x)} \in U(1) . \tag{5}$$

Note, that $\Lambda(x)$ depends on x, so we have invariance under local transformations, not only global transformations. Using the covariant derivative D_μ instead of the "normal" derivative ∂_μ we promote the invariance from a global one to a local one.

Also we could have written (2) as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\Lambda$ - which is probably more familiar -, since A_μ and G commute. But this will not be the case anymore when we look at the non-abelian theory.

As the lattice theory is formulated in euclidean spacetime (i.e. $x^0 \to -ix_4$), we would like to express the action (1) in the euclidean version as well. From $x^0 \to -ix_4$ it follows $\partial_0 \to -\frac{1}{i}\partial_4 = +i\partial_4$, so in order to D_μ transform properly (namely into itself) we demand the transformation of A to be $A^0 \to +iA^4$.

Hence the action (1) transforms as (as we are now only dealing with the euclidean action S_E we will omit the index E for simplicity for the rest of this paper)

$$S_{QED} \rightarrow iS_E = i\int d^4x \left[\bar{\psi}(x)(\gamma^\mu D_\mu + m)\psi(x) + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right], \tag{6}$$

where we have introduced a new set of γ-matrices:

$$\begin{aligned} \gamma_4 &= \gamma_M^0 \\ \gamma_i &= -i\gamma_M^i \end{aligned} \tag{7}$$

Of course, usually the γ_M-matrices are just called γ and the new defined matrices which hold in Euclidean space-time have an index, rather than those used in Minkowski space-time. As we are here dealing only in Euclidean space-time and don't want to carry unnecessary indexes along every calculation we simply define the Euclidean γ-matrices as those without index. In analogy to the Clifford algebra for Minkowski γ_M-matrices $\{\gamma_M^\mu, \gamma_M^\nu\} = 2g^{\mu\nu}$, we have an even simpler anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\ , \tag{8}$$

while we are always working with the signature $g = diag(1, -1, -1, -1)$.

As a sidenote and to draw a connection to the first talk in this series we also denote here the generating functional of the QED action:

$$Z[J,\eta,\bar{\eta}] = \int (DA)(D\bar{\psi})(D\psi)e^{iS_0 + i\int d^4x J^\mu A_\mu + i\int d^4x(\bar{\eta}\psi + \bar{\psi}\eta)}\ . \tag{9}$$

Remember that one has to differentiate with respect to the sources J^μ, η and $\bar{\eta}$ to compute the Green functions.

2.1 The lattice formulation

If we want to formulate the theory from the above section on a lattice we will not have anymore a continuous spacetime but a discrete one. Therefore we suggest the following substitutions to be made in a lattice formulation:

$$x_\mu \to n_\mu a \,, \tag{10}$$

$$\int d^4x \to a^4 \sum_n \,, \tag{11}$$

$$\psi_\alpha(x) \to \frac{1}{a^{3/2}}\hat{\psi}_\alpha(n) = \frac{1}{a^{3/2}} a\psi_\alpha(an) \,, \tag{12}$$

$$\bar{\psi}_\alpha(x) \to \frac{1}{a^{3/2}}\hat{\bar{\psi}}_\alpha(n) = \frac{1}{a^{3/2}} a\bar{\psi}_\alpha(an) \,, \tag{13}$$

$$\partial_\mu\psi(x) \to \frac{1}{a^{5/2}}\hat{\partial}_\mu\hat{\psi}(n) = \frac{1}{a^{3/2}} \cdot \frac{1}{2a}[\hat{\psi}_\alpha(n+\hat{\mu}) - \hat{\psi}_\alpha(n-\hat{\mu})] \,, \tag{14}$$

$$m \to \frac{1}{a}\hat{m} \,, \tag{15}$$

$$\tag{16}$$

where $\hat{\mu}$ is a unit vector pointing along a lattice direction.

For simplicity we forget about the mass-term (i.e. we set $m = 0$) and split up the above action to treat the two terms separately, for this we write

$$S = \int d^4x \, \bar{\psi}(x)\gamma^\mu D_\mu \psi(x) + \int d^4x \, \frac{1}{4} F^{\mu\nu}F_{\mu\nu} = S_1 + S_2 \,. \tag{17}$$

Now we just write down the action, we will check later that it indeed leads to the continuum action S_1 as written above when taking the continuum limit.

$$S_1 = \frac{1}{2a}\bar{\psi}(n) \sum_\mu \gamma_\mu [U_\mu(n)\psi(n+\hat{\mu}) - U^\dagger_\mu(n-\hat{\mu})\psi(n-\hat{\mu})] \,. \tag{18}$$

where we have introduced a factor $U(x,y) = U(x, x+\epsilon) =: U_\epsilon(x)$. This factor is included because according to (2) and (3) $\bar{\psi}(x)\psi(y)$ transforms to $\bar{\psi}(x)G^{-1}(x)G(y)\psi(y)$ and one can show that $U(x,y)$ transforms to $G(x)U(x,y)G^{-1}(y)$ so if we modify

$$\bar{\psi}(x)\psi(x+\epsilon) \to \bar{\psi}(x)U_\epsilon(x)\psi(x+\epsilon) \,, \tag{19}$$

$$\bar{\psi}(x+\epsilon)\psi(x) \to \bar{\psi}(x+\epsilon)U^\dagger_\epsilon(x)\psi(x) \,, \tag{20}$$

we achieve gauge invariance. Now one can show that the so called Schwinger line integral can be written as

$$\begin{aligned} U_\mu(x) &= e^{ie\int_x^{x+\mu} dx' A_\mu(x')} \\ \Rightarrow \quad U_\mu(n) &= e^{iaeA_\mu(n)} \approx 1 + iaeA_\mu(n) \,. \end{aligned} \tag{21}$$

Note that the $U_\mu(n)$ are elements of the $U(1)$ gauge group. This is why they are usually referred to link variables or simply as links. They are directed quantities and we have

$$U_\mu(n) = U^\dagger_{-\mu}(n+\hat{\mu}). \tag{22}$$

Now one can easily check that (18) recovers the Dirac action in the limit $a \to 0$

$$\begin{aligned} S_1 &= \tfrac{1}{2a}\bar{\psi}(n)\gamma_\mu[U_\mu(n)\psi(n+\hat{\mu}) - U^\dagger_\mu(n-\hat{\mu})\psi(n-\hat{\mu})] \\ &= \tfrac{1}{2a}\bar{\psi}(n)\gamma_\mu[(1+iaeA_\mu(n)+\dots)(\psi(n)+a\partial_\mu\psi(n)+\dots)- \\ &\quad (1-iaeA_\mu(n)+\dots)(\psi(n)-a\partial_\mu\psi(n)+\dots)] \\ &= \bar{\psi}(n)\gamma_\mu(\partial_\mu + \tfrac{a^2}{6}\partial^3_\mu+\dots)\psi(n)+ \\ &\quad ie\bar{\psi}(n)\gamma_\mu[A_\mu + \tfrac{a^2}{2}(\tfrac{1}{4}\partial^2_\mu A_\mu + (\partial_\mu A_\mu)\partial_\mu + A_\mu\partial^2_\mu)+\dots]\psi(n) \\ &= \bar{\psi}(n)\gamma_\mu(\partial_\mu + ieA_\mu)\psi(n) + O(a^2)\,. \end{aligned} \tag{23}$$

In the $O(a^2)$ all symmetry breaking terms are included. As shortly mentioned above rotational and translational symmetry are broken when a lattice is introduced. By taking the naïve continuum limit $a \to 0$ those additional terms vanish fast enough so that the original symmetry is achieved again.

For the lattice gauge action we introduce the 1×1 plaquette (see figure 1), which represents the simplest Wilson loop

$$P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n)\,. \tag{24}$$

Using (21) one finds that

$$P_{\mu\nu}(n) = e^{iea^2F_{\mu\nu}(n)}\,, \tag{25}$$

where

$$F_{\mu\nu}(n) = \frac{1}{a}[(A_\nu(n+\hat{\mu}) - A_\nu(n)) - (A_\mu(n+\hat{\nu}) - A_\mu(n))] \tag{26}$$

is a discretized version of the continuum field strength tensor.

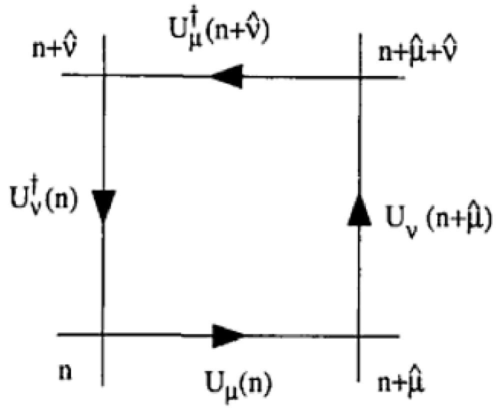

Figure 1: The 1×1 plaquette

It then follows for small a that

$$
\begin{aligned}
S_2 &= \tfrac{1}{e^2}\sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \tfrac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}(n)^\dagger)] \\
&= \tfrac{1}{e^2}\sum_n \sum_{\mu,\nu,\mu<\nu} \\
&\quad [1 - \tfrac{1}{2}(1 + iea^2 F_{\mu\nu(n)} - \tfrac{e^2a^4}{2}F^2_{\mu\nu}(n) + \dots + 1 - iea^2 F_{\mu\nu}(n) - \tfrac{e^2a^4}{2}F^2_{\mu\nu}(n) + \dots)] \\
&= \tfrac{1}{e^2}\sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \tfrac{1}{2}(2 - e^2a^4F^2_{\mu\nu}(n))] + O(a^6) \\
&\approx \tfrac{1}{2}\sum_n \sum_{\mu,\nu,\mu<\nu} [a^4 F^2_{\mu\nu}(n)] \\
&= \tfrac{1}{4}\sum_{n,\mu,\nu} a^4 F_{\mu\nu}(n) F_{\mu\nu}(n)\,.
\end{aligned}
\tag{27}
$$

Note that the last sum extends over all μ and ν and the double counting is taking care of by an extra factor of $\frac{1}{2}$. If we then take the limit $a \to 0$ and remember (11) we indeed get the continuum gauge action.

So we end up with the simplest formulation of the lattice action which reads

$$
\begin{aligned}
S &= \tfrac{1}{2a}\bar{\psi}(n)\sum_\mu \gamma_\mu [U_\mu(n)\psi(n+\hat{\mu}) - U^\dagger_\mu(n-\hat{\mu})\psi(n-\hat{\mu})] \\
&\quad + \tfrac{1}{e^2}\sum_n \sum_{\mu,\nu,\mu<\nu} [1 - \tfrac{1}{2}(P_{\mu\nu}(n) + P^\dagger_{\mu\nu}(n))] \\
&\quad + \hat{m}\sum_n \bar{\psi}(n)\psi(n)\,.
\end{aligned}
\tag{28}
$$

But this equation is not very useful for the formulation of the lattice action in QCD. The problem is, that the fermions on the lattice have some subtle difficulties: In particular, the naïve fermion discretization yields too many fermionic modes (doublers). One can avoid those problems in different ways, one of them is using so called Wilson fermions.

By using Wilson fermions the derivation of the action gets a bit more difficult but the idea stays the same (namely that the action has to reproduce the continuum action in the limit $a \to 0$). We will not go into any details of the derivation but instead just write down the lattice action for an abelian gauge theory with Wilson fermions:

$$\begin{aligned} S_{QED}[U,\psi,\bar{\psi}] &= \tfrac{1}{e^2}\textstyle\sum_n \sum_{\mu,\nu,\mu<\nu}[1-\tfrac{1}{2}(P_{\mu\nu}(n)+P^\dagger_{\mu\nu}(n))] + (\hat{m}+4r)\sum_n \bar{\psi}(n)\psi(n)- \\ & \tfrac{1}{2}\textstyle\sum_{n,\mu}[\bar{\psi}(n)(r\cdot \mathbb{1}_4-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu})+\bar{\psi}(n+\hat{\mu})(r\cdot \mathbb{1}_4+\gamma_\mu)U^\dagger_\mu(n)\psi(n)] \,. \end{aligned} \tag{29}$$

The gauge action stays the same as before, as there are no fermions involved, but the fermion action looks now a bit different. We have introduced an additional term $r \cdot \Delta$ compared to (28), where r is the Wilson parameter. With this additional term one can avoid the problems arising with fermions on the lattice. This additional factor can be added as long as the continuum limit $a \to 0$ will not change and one can show that this is indeed the case.

3 Non abelian gauge fields on the lattice

For the non abelian case we can proceed similarly to the abelian case but we have to consider that the abelian group $U(1)$ is replaced by the non-abelian group $SU(N)$, where in QCD N is 3. So we have to replace the fields ψ with

$$\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix} \,, \; \bar{\psi} = \begin{pmatrix} \bar{\psi}^1, & \ldots, & \bar{\psi}^N \end{pmatrix} \,. \tag{30}$$

The fermion action for the non-abelian case then reads the same as the one for the abelian case

$$\begin{aligned} S_F &= (\hat{m}+4r)\textstyle\sum_n \bar{\psi}(n)\psi(n) \\ & -\tfrac{1}{2}\textstyle\sum_{n,\mu}[\bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu})+\bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U^\dagger_\mu(n)\psi(n)] \,, \end{aligned} \tag{31}$$

and this equation is also invariant under the same local transformations

$$\psi(n) \to G(n)\psi(n) \,, \tag{32}$$
$$\bar{\psi}(n) \to \bar{\psi}(n)G^{-1}(n) \,, \tag{33}$$
$$U_\mu(n) \to G(n)U_\mu(n)G^{-1}(n+\hat{\mu}) \,, \tag{34}$$
$$U^\dagger_\mu(n) \to G(n+\hat{\mu})U^\dagger_\mu(n)G^{-1}(n) \,, \tag{35}$$

but now we have to be careful, as $G(n)$ and $U_\mu(n)$ are elements of $SU(N)$ and they can be written as

$$G(n) = e^{i\Lambda(n)} \,, \tag{36}$$
$$U_\mu(n) = e^{i\phi_\mu(n)} = e^{igaA_\mu(n)} \,, \tag{37}$$

where $\Lambda(n)$ and $\phi_\mu(n)$ are elements of the Lie algebra of $SU(N)$. For the Wilson loop (24) the same relation holds

$$P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n) \,. \tag{38}$$

This quantity is path ordered and this time the ordering is important as the link variables are not abelian anymore. In the abelian case the link variables were just some complex numbers, but in the non-abelian case we have to deal with matrices. We form the simplest gauge-invariant quantity which is just the trace of the path ordered product of link variables along the boundary of an elementary plaquette

$$S_G = \frac{2}{g^2} Tr \sum_{n,\mu<\nu} [\mathbb{1}_3 - \frac{1}{2}(P_{\mu\nu}(n) + P_{\mu\nu}^\dagger(n))] \,, \tag{39}$$

where we have a constant factor in front of the trace to end up with the right gauge action.

As before we write

$$U_{\mu\nu}(n) = e^{iga^2\mathcal{F}_{\mu\nu}(n)} \,, \tag{40}$$

but this time it will be harder to get the relation between $\mathcal{F}_{\mu\nu}(n)$ and $A_\mu(n)$ as the link variables appearing in (38) do not commute anymore and one has to use the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \,. \tag{41}$$

One then finds that

$$\mathcal{F}_{\mu\nu} \to F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{42}$$

for $a \to 0$ and this is the expression for the field strength tensor in continuum QCD: the gluon field tensor.

Let us briefly discuss this tensor: In the abelian case the commutator vanishes of course, $F_{\mu\nu}^2$ leads then only to two interacting A-fields. This is different for the non-abelian gluon field tensor, here also A-fields to the order 3 and 4 are possible, which corresponds to 3- and 4-gluon interactions. Gluons can therefore interact with themself, they also carry a color charge!

If we apply the local transformation (34) to the action (39) and make use of the expression

$$U_\mu(n) = e^{igaA_\mu(n)} \,, \tag{43}$$

we get the transformation law for A_μ in the continuum limit:

$$A_\mu(x) \to G(x)A_\mu(x)G^{-1}(x) - \frac{i}{g}G(x)\partial_\mu G^{-1}(x) \,, \tag{44}$$

which is of course the same as (3).

So now we have constructed our non-abelian action for QCD, it reads as

$$\begin{aligned} S &= (\hat{m}+4r)\sum_n \bar{\psi}(n)\psi(n) \\ &\quad -\tfrac{1}{2}\sum_{n,\mu}[\bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U_\mu^\dagger(n)\psi(n)] \\ &\quad +\tfrac{2}{g^2}Tr\sum_{n,\mu<\nu}[\mathbb{1}_3 - \tfrac{1}{2}(P_{\mu\nu}(n)+P_{\mu\nu}^\dagger(n))] \,, \end{aligned} \tag{45}$$

4 The continuum limit

4.1 The naïve limit

In the preceding section we constructed a lattice gauge action which reproduces the QCD action in the continuum limit. But we could also have constructed another action which would have led to the right QCD action in the limit. And also just because we might recover the right action it is not sure if our construction really leads to a meaningful theory. By taking the limit for the lattice spacing $a \to 0$ we never thought about the consequences and just worked with the naïve limit. This worked good for the action but to get a physical result now we have to be more careful and be sure, that the theory we get when a goes to 0 does not have any remnants of the lattice. For example we may consider an observable Θ, which depends on the lattice spacing a and another quantity which we denote with g as it will turn out that it will be the coupling constant. But g may also depend on a and as we will see later it really does and this is an important feature of QCD. Now for $a \to 0$ we want to get the physical value of Θ (which we denote with Θ_{PHYS})

$$\Theta(g(a), a) \to \Theta_{PHYS} \,. \tag{46}$$

So when $a \to 0$ in order to have $\Theta \to \Theta_{PHYS}$, $g(a)$ will be tuned this way, so that the relation (46) will be fulfilled. So if Θ_{PHYS} and also its corresponding lattice quantity is known one can determine $g(a)$. Now one may argue that the function $g(a)$ may depend on the observable Θ one chooses but for sufficiently small a, when the lattice theory really describes continuum physics then does a universal function $g(a)$ exist.

4.2 Asymptotic freedom

We've seen that we can use any observable to determine $g(a)$, which will be the goal of this chapter. We will use the static $q\bar{q}$ potential which is

$$V(r,g,a) = \frac{1}{a}\hat{V}(\frac{R}{a},g)\,, \tag{47}$$

where R is the physical separation and the hat on the V indicates that this quantity is to be understood as the lattice quantity corresponding to the continuum (i.e. physical) potential. (47) holds for a finite, but small lattice spacing. Now for a sufficiently small lattice spacing (47) must become independent of a. Thus the potential must satisfy the renormalization group equation

$$[a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}]V(R,g,a) = 0\,, \tag{48}$$

where

$$\beta(g) = -a\frac{\partial g}{\partial a} \tag{49}$$

is the so called Callan-Symanzik β-function.

Note that we can rewrite (48) as

$$\begin{aligned} & a\tfrac{\partial V}{\partial a} - \beta(g)\tfrac{\partial V}{\partial g} &=& \; 0 \\ \Leftrightarrow\; & a\tfrac{\partial V}{\partial a} + \tfrac{\partial g}{\partial a}\tfrac{\partial V}{\partial g} &=& \; 0 \\ \Leftrightarrow\; & a[\tfrac{\partial V}{\partial a} + \tfrac{\partial g}{\partial a}\tfrac{\partial V}{\partial g}] &=& \; 0 \\ \Leftrightarrow\; & a\cdot\tfrac{dV}{da} &=& \; 0\,. \end{aligned} \tag{50}$$

So we want to have the potential V to be independent of the lattice spacing a.

The next step would therefore be to calculate $\beta(g)$, which can be done in perturbation theory.

In the second talk the Wilson loop was introduced as

$$W_C[A] = e^{ie\int dz_\mu A_\mu(z)}\,, \tag{51}$$

where C is the contour we are integrating along. As discussed in the second talk we are interested in the ground state expectation value in the absence of the static quark-antiquark source, i.e. for large T. With the overlap of our original state with the ground state $F(R)$ and the interaction energy $V(R)$ we have

$$\langle W_C[A]\rangle \equiv W(R,T) \xrightarrow[T\to 0]{} F(R)e^{-V(R)T} \tag{52}$$

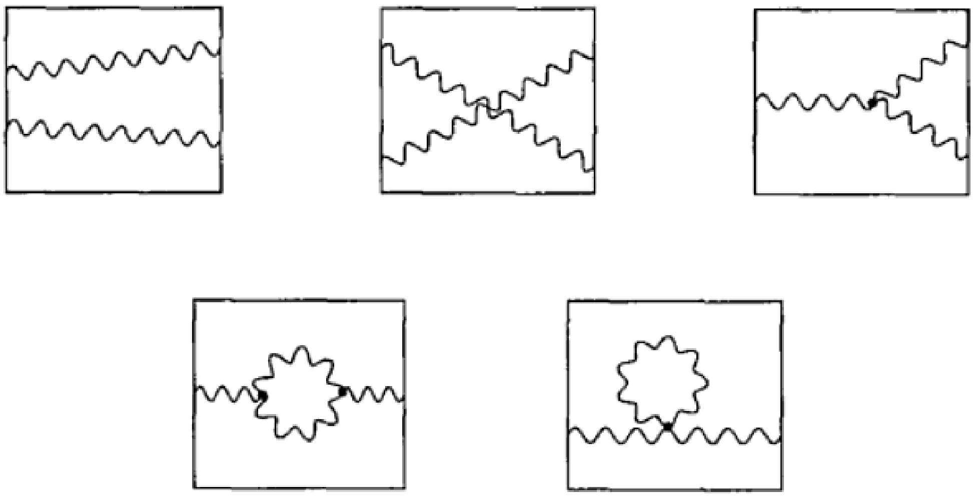

Figure 2: Some diagrams contributing to (54) up to order g^4.

and therefore the potential $V(R)$ can be calculated as

$$V(R) = -\lim_{T\to 0} \frac{1}{T} \ln \langle W_C[A]\rangle \ . \tag{53}$$

So in order to calculate $\beta(g)$, one can expand the continuum expression

$$V(R,g,a) = -\lim_{T\to\infty} \frac{1}{T} \ln[Tr(Pe^{ig\int dz_\mu A_\mu(z)})] \tag{54}$$

and one then finds that

$$V(R,g,a) = \frac{C}{4\pi R}[g^2 + \frac{22}{16\pi^2} g^4 \ln \frac{R}{a} + O(g^6)] \ , \tag{55}$$

where C is a group theoretical factor which is irrelevant for our case. Up to $O(g^4)$ the diagrams in fig. 2 contribute to the potential.

By putting (55) in (48) we readily find that

$$\beta(g) \approx -\frac{11}{16\pi^2} g^3 \ . \tag{56}$$

We expect this to be a good approximation for a sufficiently small coupling constant g.

Integrating (49) leads to

$$a = \frac{1}{\Lambda_L} e^{-\frac{16\pi^2}{22g^2}} \ , \tag{57}$$

where Λ_L is an integration constant and we see that $\beta(g)$ goes to 0 as $a \to 0$, so we have for $a \to 0$

$$g \to 0\,. \tag{58}$$

That means for vanishing lattice spacing, i.e. for the continuum limit we end up with a vanishing coupling constant for small separations. This fact is experimentally well tested and is called the asymptotic freedom. In other words the interaction between particles in this gauge theory becomes arbitrarily weak for short separations.

5 Strong coupling expansion

We now want to have a look at the $q\bar{q}$-potential $V(R)$, which we already studied in the previous chapter but now we look at large separation. For short separations we have seen that the coupling constant goes to 0, but for a small coupling constant we could also use an expansion in perturbation theory and therefore the lattice approach is not motivated. But as the theory predicts and as experiments confirm, the running coupling of QCD grows for large distances between the quarks and a treatment in the perturbation theory frame is no longer justified. (39) suggests an expansion in powers of the inverse coupling, this is the analog of the high temperature expansion in statistical mechanics. We want to concentrate on the leading strong coupling factor. We write equation (39) as

$$S_G = \frac{6}{g^2}\sum_P [1 - \frac{1}{6}Tr(P + P^\dagger)]\,, \tag{59}$$

where the summation over P just means that we sum over every elementary plaquette taken in the counterclockwise direction. We rewrite this as

$$S_G = -\frac{6}{g^2}\sum_P S_P + const.\,, \tag{60}$$

where

$$S_P = \frac{1}{6}Tr(P + P^\dagger)\,. \tag{61}$$

The corresponding partition function now is

$$Z = \int DP\, e^{\beta \sum_P S_P} \tag{62}$$

Now we also introduce the gauge invariant Wilson loop operator

$$W_C[P] = Tr\prod_{l\in C_L} P_l\,, \tag{63}$$

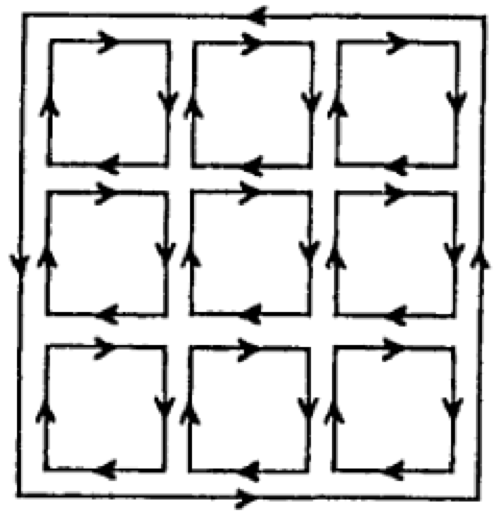

Figure 3: Smallest number of elementary plaquettes with non-vanishing integral.

where C_L is a Wilson loop. This quantity is only gauge invariant when it is path ordered.

The expectation value of the Wilson loop operator with spatial extension $\hat{R}$ and temporal extension $\hat{T}$ (see figure 4) is given by

$$\langle W_C[P] \rangle = \frac{\int DP\, W_C[P] \prod_P e^{\beta S_P}}{\int DP\, \prod_P e^{\beta S_P}}\,, \tag{64}$$

where we have written

$$\beta = \frac{6}{g^2}\,. \tag{65}$$

The exponential can be expanded as

$$\prod_P e^{\beta S_P} = \prod_P \left[\sum_n \frac{\beta^n}{n!} (S_P)^n \right]. \tag{66}$$

We're interested in the leading contribution for the expectation value of the Wilson loop operator so we want to keep β as small as possible. As each plaquette in the expansion adds a factor of β, the leading contribution is achieved by choosing the way with the smallest number of elementary plaquettes but for which the integral does not vanish.

Let us have a look at the standard group integrals involving polynomials of the link variables for any unitarian group. We have the following integrals:

$$\int dU = 1\,, \tag{67}$$

$$\int dU\, U = 0\,, \tag{68}$$

$$\int dU\, U^\dagger U = \int dU = 1\,. \tag{69}$$

Therefore we see from (69) that in order to have a non-vanishing integral every U must be paired with a $U^\dagger$, in other words, every link must be taken care of another link which points in the other direction.

Figure 3 shows an example of the leading contribution to $\langle W_C[P]\rangle$ in the strong coupling approximation for an exemplary Wilson loop C.

The numerator is proportional to $\beta^{\hat{R}\hat{T}}$ while the leading contribution to the denominator is obtained by making the replacement $e^{\beta\sum S_P} \longrightarrow 1$, therefore we have

$$\langle W_C[P]\rangle \approx c\left(\frac{\beta}{6}\right)^{\hat{R}\hat{T}}\,, \tag{70}$$

where the constant c can be calculated with the group integrals and this leads to $c = \left(\frac{1}{3}\right)^{\hat{R}\hat{T}-1}$, so we finally end up with

$$\langle W_C[P]\rangle \approx 3\left(\frac{\beta}{18}\right)^{\hat{R}\hat{T}}\,. \tag{71}$$

The $q\bar{q}$-potential in the strong coupling limit is then given by

$$\hat{V}(\hat{R}) = -\lim_{T\to\infty}\frac{1}{\hat{T}}\ln\langle W_C[P]\rangle = \hat{\sigma}(g)\hat{R}\,, \tag{72}$$

where we have defined the string tension measured in lattice units

$$\hat{\sigma} = -\ln\left(\frac{\beta}{18}\right)\,, \tag{73}$$

with this quantity we can write

$$\langle W_C[P]\rangle \approx 3e^{-\hat{\sigma}\hat{R}\hat{T}} \tag{74}$$

which holds in the leading strong coupling approximation.

Now we observe something important and it is this observation which made the lattice theory so successful: The potential $V(R) = \lim_{T\to\infty} -\frac{1}{T}\ln\langle W_C[P]\rangle$ increases linearly with

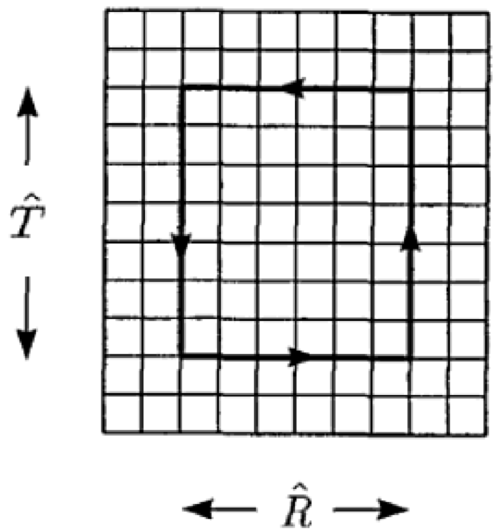

Figure 4: Integration contour on a lattice in two space-time dimensions.

$\hat{R}$. In other words, if we try to separate two quarks we would have to put an infinite amount of energy in it. It is therefore impossible to separate two quarks. This is also true for all particles which are not in a color-singlet state. This behaviour is called *confinement*.

So our lattice theory does imply confinement. But we still have hats on our quantities, that means we're still dealing with quantities on the lattice - we're still in lattice theory! If and how this confining property persists into the small coupling regime, where continuum physics should be observed is out of the scope of this talk.

References

[1] K. G. Wilson, Phys. Rev. D **10** (1974) 2445.

[2] I. Montvay and G. Münster, "Quantum fields on a lattice", Cambridge, UK: Univ. Pr. (1994) (Cambridge monographs on mathematical physics)

[3] H. J. Rothe, "Lattice gauge theories: An introduction", World Sci. Lect. Notes Phys. 74 (2005)

[4] T. DeGrand, C. DeTar, "Lattice methods for quantum chromodynamics", World Sci. (2006)

[5] R. Gupta, "Introduction to Lattice QCD" [arXiv:hep-lat/9807028v1].