Exercises for "Phenomenology of Particle Physics II"

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Exercise 3 Local Gauge Invariance and Gauge Field Self-Interactions (corrected)

This exercise follows chapter 15.2 in Peskin/Schroeder closely.

In this exercise we derive the necessity of gauge field self-interactions from gauge invariance. Our gauge group G is a simple compact Lie group. The fermions transform in a unitary representation of G, the elements of this representation can be written as $e^{i\alpha^a T^a}$ where the T^a are the generators of the representation and the summation over the repeated index a is understood. The properties of the T^a we need are $\left[T^a, T^b\right] = if^{abc}T^c$ (where the f^{abc} are antisymmetric under interchange of any two indices) and $\mathrm{Tr}(T^aT^b) = C(r)\delta^{ab}$. Our starting point is the gauge symmetry of $\mathcal{L}$ under local gauge transformations

$$\psi(x) \to V(x)\psi(x)$$

where $\psi(x)$ has several components which we supress in our notation. We can see immediately that the combination $\overline{\psi}(x)\psi(x)$ is a gauge-invariant quantity. If our Lagrange density is to contain terms with derivatives (as in $\overline{\psi}(i\partial_\mu\gamma^\mu - m)\psi$, we encounter the problem that

$$n^\mu\partial_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\psi(x+\epsilon n) - \psi(x)\right)$$

does obviosly not have a well-defined transformation under the local gauge transformation $V(x)$ because the gauge transformation is in general different at x and $x + \epsilon n$. We define a derivative which transforms the same as $\psi(x)$ by introducing the comparator $U(y, x)$. The properties we assume are $U(x, x) = 1$ and

$$U(y, x) \to e^{i\alpha^a(y)T^a}U(y, x)e^{-i\alpha^b(x)T^b}$$

under a local gauge transformation. Using the comparator we define the covariant derivative as

$$n^\mu D_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right)$$

which does have the transformation property $D_\mu\psi(x) \to e^{i\alpha^a(x)T^a}D_\mu\psi(x)$ by construction (therefore terms like $\overline{\psi}(iD_\mu\gamma^\mu - m)\psi$ are candidates for the Lagrange density). The gauge field is defined as the expansion coefficient of the comparator

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A^a_\mu(x)T^a + O\left(\epsilon^2\right)$$

giving us the covariant derivative

$$D_\mu = \partial_\mu - igA^a_\mu(x)T^a.$$

Derive the transformation of the gauge field A under an infinitesimal gauge transformation $\alpha(x)$ from the transformation of the comparator U by first expanding to first order in ϵ and then to first order in the gauge transformation $\alpha(x)$. You should arrive at

$$A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g}(\partial_\mu \alpha^a(x)) + f^{abc}A^b_\mu \alpha^c(x).$$

Now give a short argument why $[D_\mu, D_\nu]\,\psi(x)$ transforms like $\psi(x)$, then compute $[D_\mu, D_\nu]$ explicitly to see that is in fact not a differential operator but a multiplicative factor which is therefore gauge invariant. We define the field tensor $F^a_{\mu\nu}$ by

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T^a$$

and compute $F^a_{\mu\nu}F^{a\mu\nu}$, the simplest gauge-invariant combination of the field tensor, to see that it contains terms that are cubic and quartic in A.

Exercise 4 Higgs couplings in the standard model

Starting from the Lagrange density

$$\mathcal{L} = (D_\mu\phi)^+(D^\mu\phi) - V(\phi), \qquad D_\mu = \partial_\mu - igT^aW^a_\mu - ig'\frac{Y}{2}B_\mu, \quad V(\phi) = \mu^2\phi^2 - \frac{\lambda}{4}\phi^4$$

for the scalar dublet

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix},$$

find the couplings hWW,$hhWW$, hZZ and $hhZZ$. You may follow the steps below.

- Specialise the Lagrange density to $Y = 1$, $T^a = \frac{1}{2}\sigma^a$ and $\phi^T = (0, \phi_2)$ and get rid of the Pauli matrices by inserting them explicitly ($\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$).
- Diagonalise the quadratic terms by introducing the physical fields

$$W^+_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu - iW^2_\mu\right) = (W^-_\mu)^\dagger \tag{1}$$

$$Z_\mu = \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}} \tag{2}$$

$$A_\mu = \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2 + g'^2}}. \tag{3}$$

- Now you can read off the coefficients in the expansion

$$\begin{aligned}(D_\mu\phi)^+(D_\mu\phi) &= (\partial_\mu h)^2 + M_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu - iV_{hWW}hW^+_\mu W^{-\mu} \\ &\quad -iV_{hhWW}hhW^+_\mu W^{-\mu} - iV_{hZZ}hZ_\mu Z^\mu - iV_{hhZZ}hhZ_\mu Z^\mu.\end{aligned}$$