QUANTUM FIELD THEORY - II

PATH INTEGRALS AND NON-ABELIAN GAUGE THEORIES

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Outline of contents.

1. Introduction (subject of course, physics and mathematics of gauge fields, particle physics, ...)
2. The Dirac-Feynman path integral in quantum mechanics
3. Path- or functional integral quantization of scalar fields; perturbation theory for $\lambda\phi^4$-theory; spontaneous symmetry breaking and Goldstone's theorem
4. Path integrals for Fermi fields
5. Path integrals for gauge (Yang-Mills) theories; Faddeev-Popov
6. BRST

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7. Effective actions; Slavnov-Taylor identities; Higgs mechanism; Coleman-Weinberg
8. Renormalization group and asymptotic freedom

9. Applications to QCD; ...

Literature

1) Ch. Anastasiou, "Quantum Field Theory II", Script, ETH – FS 2008

2) S. Coleman, "Aspects of Symmetry" (selected Erice lectures; especially lectures 1 and 3–5), Cambridge University Press 1985

3) S. Weinberg, "The Quantum Theory of Fields", Vols. I, II (III), Cambridge University Press 1995, 1996, 2000

4) J. Zinn-Justin, "Quantum Field Theory and Critical Phenomena" (4th ed.), Oxford Science Publ., Clarendon Press (Oxford) 2002

5) L. D. Faddeev and A. A. Slavnov, "Gauge Fields – Introduction to Quantum Theory", Frontiers in Physics, Benjamin 1980

6) F. Mandl and G. Shaw, "Quantum Field Th.", Wiley 1984

7) J. Fuchs and Chr. Schweigert, "Symmetries, Lie Algebras & Representations", Cambridge Univ. Press 1997

8) J.-P. Derendinger, "Théorie quantique des champs", Presses polytechnique et univ. romandes, 2001

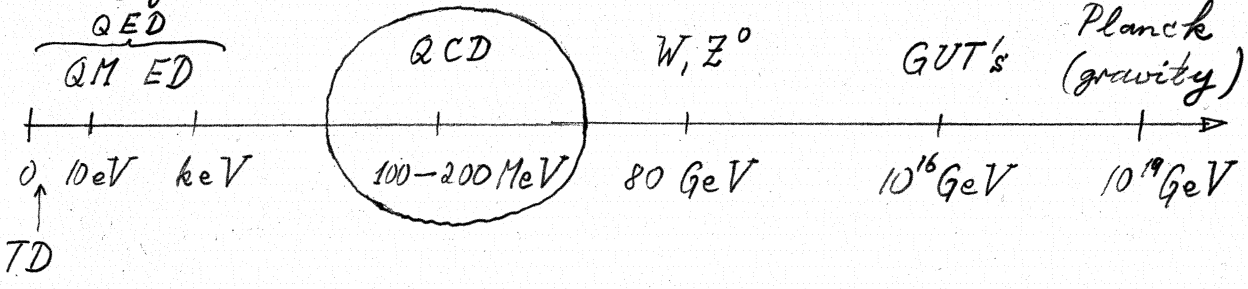
1. Introduction

The main purpose of this course is to introduce the audience to modern methods of quantum field theory, in particular non-abelian gauge (Yang-Mills) theories, and some of their applications to particle physics. The following topics will be discussed: Brief introduction to particle physics (e.g., quark model) and some of the key theoretical problems / classical gauge fields / path integrals in quantum mechanics (a brief review) / functional integral quantization of scalar and Fermi fields / spontaneous symmetry breaking and Goldstone's theorem / functional integral quantization of non-abelian gauge fields – the Faddeev-Popov method of gauge fixing / BRST symmetry (BV formalism?) / quantum effective action and -potential (generating functional for 1PI Green functions) / Slavnov-Taylor identities / perturbative renormalization of ultraviolet divergences /

Higgs mechanism / renormalization group methods and asymptotic freedom of QCD / applications to particle physics (e.g., Bjorken scaling)) / infrared slavery, the hypothesis of quark confinement, colour screening, introduction to lattice gauge theory / ...

1.1. Brief introduction to particle theory

Energy scales and masses



1 Rydberg = 13.605 eV, 1 keV = 10^3 eV, 1 MeV = 10^3 keV, 1 GeV = 10^3 MeV, ...

Masses of some elementary particles:

Leptons $e^{\pm}$: 0.511 MeV, $\mu^{\pm}$: 106 MeV, $\tau^{\pm}$: 1.777 GeV

neutrino masses very small, but > 0; (< 10 eV $\div$ < 19 MeV)

Mesons ($q\bar{q}$ bound states) π: ~ 135 MeV,

K: ~ 495 MeV, η: ~550 MeV

$\underbrace{\pi^+, \pi^0, \pi^-}_{\text{isospin triplet}}$; $\underbrace{K^+, K^-, \quad K^0, \overline{K^0}}_{\text{isospin doublets}}$; $\underbrace{\eta}_{\text{iso-singlet}}$: SU(3) octet

η': ~ 960 MeV (pseudoscalar SU(3) singlet)

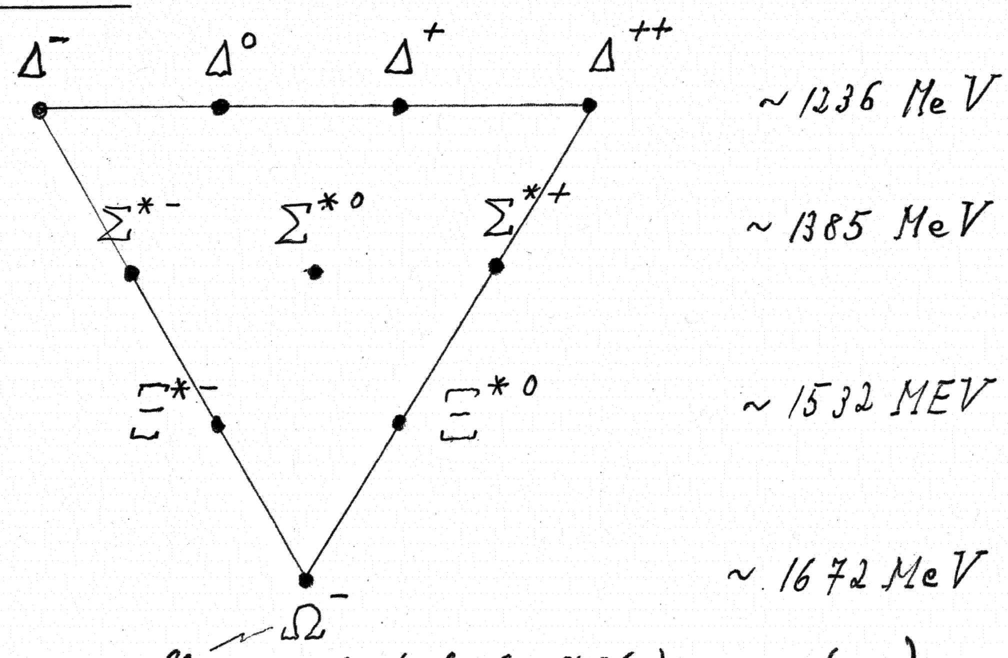
<u>Baryon octet</u> (bound states of three quarks), spin $\frac{1}{2}$:

p: 938 MeV, n: 939 MeV, Σ: ~ 1192 MeV, Λ: ~1115 MeV,

Ξ: ~ 1315 MeV;

$\underbrace{p, n}_{\text{isospin doublet}}$, Λ, Σ^+, Σ^0, Σ^-, Ξ^-, Ξ^0: SU(3) octet

<u>Baryon decuplet</u>, spin $\frac{3}{2}$:



(originally predicted by SU(3) symmetry)

Symmetries underlying all these multiplets of <u>hadrons</u> (mesons – spin 0, baryons – spin ½, 3/2) must be approximate, because the multiplets are not exactly degenerate. They are called <u>flavor</u>

symmetries. Earliest example: Isospin $SU(2)$ (Kemmer, Heisenberg). It was proposed by Gell-Mann and Neeman to interpret the octets of mesons and baryons and the decuplet of baryons as irreps. of $SU(3)$; ("eightfold way" – name inspired by Dyson's "threefold way" in th. of random matrices).

Why $SU(3)$?

Basis elements of complexified Lie algebra, $sl(2)$, of $SU(2)_{isospin}$: $E^1_\pm$ (raising- and lowering ops.), H^1 (with $I^z = \frac{1}{2} H^1$), satisfying

$$[H^1, E^1_\pm] = \pm 2 E^1_\pm, \quad [E^1_+, E^1_-] = H^1 \qquad (1)$$

To organize particles in higher multiplets not corresponding to irreps. of $SU(2)$, add further quantum number, = eigenvalue of generator H^2. Since H^1 and H^2 can be measured simultaneously,

$$[H^1, H^2] = 0. \qquad (2)$$

Could enlarge $SU(2)_{isospin}$ to $SU(2)\times U(1)$, with H^2 = generator of $U(1)$. $\longrightarrow$ Does <u>not</u> explain higher hadron multiplets in terms of irreps. of symmetry group! $\longrightarrow$ Add raising- and lowering ops. $E^2_{\pm}$, with

$$[H^2, E^2_{\pm}] = \pm 2 E^2_{\pm}, \quad [E^2_+, E^2_-] = H^2 \qquad (3)$$

<u>Requirement</u>: Action of any raising- or lowering op. on simultaneous eigenvector of (H^1, H^2) (with eigenvalue $(m^1, m^2) \in \mathbb{Z}\times\mathbb{Z}$) must again be an eigenvector of (H^1, H^2). $\Longrightarrow$

$$[H^i, E^j_{\pm}] = \pm A^{ij} E^j_{\pm}, \qquad (4)$$

with, of course, $A^{jj} = 2$. Because spec $(H^1, H^2) \in \mathbb{Z}\times\mathbb{Z}$, A^{ij} must be an integer.

<u>1^{st} possibility</u>: $A^{ij} = 0$, $i \neq j$. $\Rightarrow$ Symmetry group G would be $G = SU(2)\times SU(2)$, possibly enlarged by discrete symmetries $\longrightarrow$ does not lead to octets and decuplets.

2nd possibility: $G = SU(2) \times SU(2) \times SU(2)$, possibly enlarged by discrete symmetries; (i.e., add $H^\theta, E^\theta_\pm$ satisfying (1), + discrete symm. interchanging $\vec{I}^{\,2}$ and $\vec{I}^{\,\theta}$). This, too, is incompatible with observed props. of baryons; see ref. 2), Chapter 1.

3rd possibility: Continue from (1) - (4), with

$$A^{ij} \neq 0, \quad i \neq j. \qquad (5)$$

We define

$$E^\theta_\pm = [E^1_\pm, E^2_\pm]. \qquad (6)$$

Because of Jacobi identities, $E^\theta_\pm$ are again raising and lowering ops. for (H^1, H^2), and the same holds for $[E^1_\pm, E^2_\mp]$. Because of (5), either $E^\theta_\pm$ or $[E^1_\pm, E^2_\mp]$ must be $\neq 0$. By a simple re-definition ($E^2_\pm \to E^2_\mp$, $H^2 \to -H^2$), can always assume that $E^\theta_\pm$, as defined in (6) does not vanish.

We now attempt to complete (1) - (6) by requiring that

$$[E^i_\pm, E^j_\mp] = 0, \qquad (7)$$

for $i, j = 1, 2, \theta$. The Jacobi identity then implies that

$$A^{12} = A^{21} = -1. \tag{8}$$

Then (4) shows that

$$[H^1, E^2_\pm] = 2[I^2, E^2_\pm] = \mp E^2_\pm, \tag{9}$$

in accordance with the isospin assignments in the hadron octets and the baryon decuplet!

Setting $E^3_+ := E^\theta_-$; $E^3_- := E^\theta_+$, $H^3 := -H^1 - H^2$; (10)

one finds that

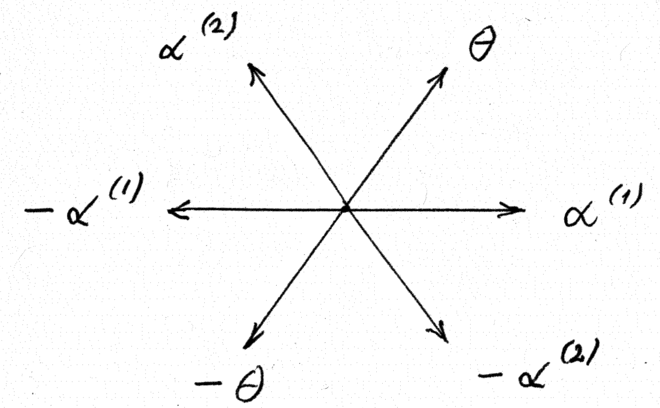
$$[H^3, E^3_\pm] = \pm 2 E^3_\pm \,, \quad [E^3_+, E^3_-] = H^3, \tag{11}$$

another $sl(2)$ subalgebra! The full set of commutation relations does not change under $(1,2,3) \to (2,3,1) \to (3,1,2)$.

(H^1, H^2) : generators of Cartan subalgebra.

Consider adjoint rep. of algebra $\mathfrak{g}$ generated by $(H^1, H^2, E^1_\pm, E^2_\pm, E^\theta_\pm)$ on $\mathfrak{g}$. Let $\alpha^{(1)} = (\alpha^{(1)}_1, \alpha^{(1)}_2)$, $\alpha^{(2)} = (\alpha^{(2)}_1, \alpha^{(2)}_2)$ and $\theta = (\theta_1, \theta_2)$ be the vectors of eigenvalues of (ad_{H^1}, ad_{H^2}) on the eigenvectors E^1_+, E^2_+ and E^θ_+. Then

$\alpha^{(1)} = (A^{11}, A^{12}) = (2,-1)$, $\alpha^{(2)} = (A^{21}, A^{22}) = (-1,2)$, $\theta = (1,1) = \alpha^{(1)} + \alpha^{(2)}$. We choose a scalar product in the plane such that $(\alpha^{(1)}, \alpha^{(2)})$, $(-\theta, \alpha^{(1)})$ and $(\alpha^{(2)}, -\theta)$ describe congruent pairs of vectors in the plane. $\Longrightarrow$



In conclusion, the algebra $\mathfrak{g}$ turns out to correspond to $sl(3)$. Replacing the generators $H^1, H^2, E^j_\pm$, $j = 1,2,\theta$, by H^1, $\frac{1}{\sqrt{3}}(H^1 + 2H^2)$, $E^j_+ + E^j_-$, $i(E^j_+ - E^j_-)$ (denoted by λ^a, $a = 1,2,\dots,8$ and called <u>Gell-Mann matrices</u>) and allowing only <u>real</u> linear combinations of them (hermiticity of generators!) we obtain the Lie algebra $su(3)$ of the compact Lie group, $SU(3)$, of unitary 3×3 matrices of determinant 1. The observed multiplets of mesons and baryons can be understood as irreps. of the (approximate flavor) symmetry group

$$G = SU(3), \tag{12}$$

(or of a larger group containing $SU(3)$). For details, see, in particular, ref 7), Chapter 3.

Representation theory of $SU(3)$, Clebsch-Gordan series

These things can be found in ref. 2), Chapter 1; and in ref. 7).

The "fundamental" irrep. of $SU(3)$ is

$$SU(3) \ni U \longmapsto U \in \mathcal{L}(\mathbb{C}^3), \tag{13}$$

where U is a 3×3 unitary matrix of determinant 1. It is 3-dim. and is denoted by "3" $\equiv (1,0)$.

A second, <u>inequivalent</u> irrep. of $SU(3)$ of dim. 3, denoted by $\overline{3} \equiv (0,1)$, is defined by

$$SU(3) \ni U \longmapsto \overline{U} = (U^*)^T \in \mathcal{L}(\mathbb{C}^3). \tag{14}$$

<u>Theorem.</u> (See, e.g., 2), Chapt. 1)

All finite-dim. irreps. of $SU(3)$ can be constructed by forming the tensor products

$$\underbrace{3 \otimes 3 \otimes \cdots \otimes 3}_{n} \otimes \underbrace{\bar{3} \otimes \cdots \otimes \bar{3}}_{m} , \qquad (15)$$

symmetrizing in the first n factors and in the second m factors and requiring that any trace ($\sim$ scalar product) between a factor 3 and a factor $\bar{3}$ vanishes. □

By symmetry, it is obviously enough to require that the trace between the first factor of 3 and the first factor of $\bar{3}$ in (15) vanish! Denoting

$$(\underbrace{3 \otimes_s \cdots \otimes_s 3}_{n}) \otimes (\underbrace{\bar{3} \otimes_s \cdots \otimes_s \bar{3}}_{m}) \qquad (16)$$

by $(n,0) \otimes (0,m)$ and the irrep. obtained by removing the trace between a 3 and a $\bar{3}$ by (n,m), we immediately find that

$$(n,0) \otimes (0,m) = (n,m) \oplus \left[(n-1,0) \otimes (0,m-1)\right] \qquad (17)$$

wherefrom the dimensions of all irreps. can be calculated: $\dim(n,m) = \frac{1}{2}(n+1)(m+1)(n+m+2)$.

<u>Examples.</u>

$(1,0) \simeq 3$, $(0,1) \simeq \bar{3}$, $(1,1) \simeq 8$ (adjoint representation of $SU(3)$ on $su(3)$), $(2,0) = 6$, $(0,2) = \bar{6}$, $(3,0) = 10$, $(0,3) = \overline{10}$, $(2,2) = 27$, $\dots$.

<u>Isospin- and hypercharge assignments</u>

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^3, \quad I^3 = \frac{1}{2} H^1 = \begin{pmatrix} 0 & & 0 \\ & \frac{1}{2} & \\ 0 & & -\frac{1}{2} \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -\frac{2}{3} & & 0 \\ & \frac{1}{3} & \\ 0 & & \frac{1}{3} \end{pmatrix} \text{ (hypercharge)}$$

Obviously,
$$Y = -\frac{2}{3} H^2 - \frac{1}{3} H^1. \tag{18}$$

The electric charge operator, Q, in a rep., ρ, of $SU(3)$ is then given by the <u>Gell-Mann–Nishijima formula</u>

$$Q = \rho(I^3) + \frac{1}{2} \rho(Y) \tag{19}$$

In the representation 3, Q is given by

$$Q = \begin{pmatrix} -\frac{1}{3} & & 0 \\ & \frac{2}{3} & \\ 0 & & -\frac{1}{3} \end{pmatrix}, \tag{20}$$

in units where some "elementary charge" $q = 1$.

One introduces one further quantum number, S, called "strangeness", corresponding to the generator

$$S = Y - B, \tag{21}$$

where B is the baryon number (op.). Knowing the eigenvalues of the commuting operators I^2, Y, Q and S in the representations $3, \bar{3}$ enables us to determine these eigenvalues in any irrep. (n,m), using that

$$[3 \equiv (1,0)]\Big|_{SU(2)_{isospin} \times U(1)_Y} = \left(\tfrac{1}{2}\right)^{1/3} \oplus (0)^{-2/3}, \tag{22}$$

where $(s)^y$ stands for the representation of $SU(2)$ with spin s and of $U(1)_Y$ with hypercharge $Y = y$.

All we need to know to do such calculations and many others is the Clebsch-Gordan series for $SU(3)$. The Clebsch-Gordan series is well explained in Ref. 2), Chapt. 1. Some important examples of decompositions of tensor product reps. are:

$$3 \otimes \bar{3} = 1 \oplus 8 ,$$

where 1 denotes the trivial representation,

$$3 \otimes 3 = \bar{3} \oplus 6 ,$$

where $\bar{3} \simeq$ antisymm. tensors of rank 2,

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 ,$$

or

$$27 \otimes 10 = 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 35 \oplus \overline{35} \oplus 64 \oplus 81 ;$$

etc.

As an exercise, I propose you verify the following table of quantum numbers for the octet of baryons:

	I^z	Y	Q	S
p	$1/2$	1	1	0
n	$-1/2$	1	0	0
Λ	0	0	0	-1
Σ^+	1	0	1	-1
Σ^0	0	0	0	-1
Σ^-	-1	0	-1	-1
Ξ^0	$1/2$	-1	0	-2
Ξ^-	$-1/2$	-1	-1	-2

The quantum number Q is in units of an "elementary charge" q. We know that the charge of the proton is $= e$, where e is the elementary electric charge first measured by Millikan. Thus,

$$q = e \tag{23}$$

Exercise: Determine these quantum numbers for the octet of mesons and for a particle transforming under the rep. 3 of $SU(3)$. The states of such a particle are described by vectors

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} =: \begin{pmatrix} s \\ u \\ d \end{pmatrix} \in \mathbb{C}^3$$

Then we have the table

	I^z	Y	Q	S
s	0	$-2/3$	$-1/3$	-1
u	$1/2$	$1/3$	$2/3$	0 (!)
d	$-1/2$	$1/3$	$-1/3$	0 (!)

The exclamation sign (!) indicates that we define baryon number B in such a way that the "strangeness" of u and d vanishes. We then find that, in the representation 3,

$$B = 1/3 . \tag{24}$$

Were there particles transforming under the representation 3 of SU(3), they would have <u>fractional electric charges</u> $-\frac{e}{3}, \frac{2e}{3}$ (with anti-particles in $\bar{3}$ having charges $\frac{e}{3}, -\frac{2e}{3}$). Such particles have not been observed <u>directly</u> in nature!

<u>The problem of the missing particles transforming under 3 or $\bar{3}$ — the quark model</u>

Apparently, one does not directly observe any elementary particles transforming under the representations 3 and $\bar{3}$ of $SU(3)_{flavor}$. This is a shame, because, thanks to

$$3 \otimes \bar{3} = 1 \oplus 8 \tag{25}$$

and (prove (26)!)

$$\begin{aligned} 3 \otimes 3 \otimes 3 &= (\bar{3} \oplus 6) \otimes 3 \\ &= (\bar{3} \otimes 3) \oplus (6 \otimes 3) \\ &= 1 \oplus 8 \oplus 8 \oplus 10 \end{aligned} \tag{26}$$

one could view the meson octet as consisting of boundstates of particles, q, transforming under 3 and their anti-particles, $\bar{q}$, transforming under $\bar{3}$. The η'-meson would then correspond to the trivial rep. 1 in the decomposition of $3 \otimes \bar{3}$.

Problem ("$U(1)$-problem") Why then is η' so much heavier than the meson octet?

Moreover, thanks to (26), one could interpret the baryon octets and the decuplet as bound states of three q's.

Such considerations gave rise to the idea that particles $q = \begin{pmatrix} s \\ u \\ d \end{pmatrix}$ and $\bar{q}$ do exist, but do not appear in asymptotic (scattering) states. This is the story of the quark model of Gell-Mann and Zweig.

For the observed particles to have the right spin $\left(s = 0, \text{ for mesons; } s \text{ half-integer, for baryons, with } s_p = s_n = \frac{1}{2},\ s_{\Delta^{++}} = \frac{3}{2}, \dots\right)$ the quarks must have spin $s_q = s_{\bar{q}} = \frac{1}{2}$.

Problem. What force binds the quarks and anti-quarks together in mesons and baryons?

Why do quarks and anti-quarks not appear in asymptotic states?

In order to get some glimpse of a solution to this problem, consider the Δ^{++} baryon, and try to view it as a bound state of three u-quarks. This state would then be described by a 3-quark wave function

$$\Psi_{\Delta^{++}} = \psi^{(1)}_{spin} \otimes \psi^{(2)}_{space} \otimes \psi^{(3)}_{SU(3)_{flavor}} \qquad (27)$$

Because $s_{\Delta^{++}} = \frac{3}{2}$, while $s_q = \frac{1}{2}$, $\psi^{(1)}_{spin}$ must be symmetric, and the relative orbital angular momentum must vanish, which is naturally accomplished if $\psi^{(2)}_{space}$ is symmetric. Finally, since Δ_{++} belongs to the decuplet of flavor-SU(3), i.e., tranforms under the 10 in the decomposition (26), $\psi^{(3)}_{SU(3)_{flavor}}$ must be symmetric, too. Thus $\Psi_{\Delta^{++}}$ is completely symmetric in the spin, position and flavor quantum numbers of the three quarks. This suggest that quarks are bosons. But since they have spin ½, this would imply a violation of the connection between spin and statistics for the quarks and the baryons, which would then be bosons — in violation of the observed Fermi statistics of baryons.

The way out is to assume that quarks carry a further quantum number that one calls color. The wave function of Δ^{++} then contains another factor, $\psi^{(4)}_{color}$, that must be <u>totally anti-symmetric</u> in the color quantum numbers of the three quarks. This is only possible if the color quantum number can take <u>at least three distinct values.</u> Simplicity suggests that it may take <u>exactly three distinct values</u>; (call them "red", "blue" and "yellow"). On color space, $\mathbb{C}^3$, the groups $U(3)$ and $SU(3)$ naturally act. The simplest choice (leading to a "simple" Lie group symmetry) is to choose

$$G_{color} = SU(3). \tag{28}$$

Then an anti-symmetric factor $\psi^{(4)}_{color}$ in the three-quark wave function, $\Psi_{\Delta^{++}}$, describing the Δ^{++} resonance carries the <u>trivial representation</u>, the "1" in the decomposition (26) (corresponding to the determinant: $U^{i_1}_{j_1} U^{i_2}_{j_2} U^{i_3}_{j_3} \cdot \varepsilon^{j_1 j_2 j_3} = \det(U)\, \varepsilon^{i_1 i_2 i_3} = \varepsilon^{i_1 i_2 i_3}$, for $U \in SU(3)$, where $\varepsilon^{i_1 i_2 i_3} = 0$ if 2 i's coincide, and $\varepsilon^{i_1 i_2 i_3} = \mathrm{sig}(i_1, i_2, i_3)$, otherwise). <u>One can reproduce the correct</u>

quantum numbers of all observed mesons and baryons by assuming that $SU(3)_{color}$ is represented trivially on all meson states (the "1" in Eq. (25)) and baryon states (the "1" in Eq. (26)).

Such ideas were proposed by Greenberg and justified, field-theoretically, by Doplicher, Haag and Roberts (with earlier unpubl. work due to Fierz and Glaser).

Another argument in favor of an $SU(3)_{color}$ symmetry results from an analysis of the $\pi^0 \to \gamma + \gamma$ decay (Steinberger), as already remarked in QFT I.

Now, one might speculate that $SU(3)_{color}$ is the symmetry of a force that binds q-$\bar{q}$ pairs into mesons and three quarks into baryons, and that this force is so strong that it binds the quarks (and anti-quarks) permanently together: Permanent quark confinement!

The best bet for constructing a theory of quarks and of the forces that bind them permanently together is to interpret $SU(3)_{color}$ as a local gauge group and introduce corresponding non-abelian gauge fields, generalizing what worked in QED, where the gauge group is $U(1)_{em}$, and the forces mediated by the corresponding gauge field, the electromagnetic vector potential A_μ, explain how electrons and nuclei are bound together in atoms and molecules (albeit not "permanently"). This theory of the strong interactions between quarks and anti-quarks is called

QCD

(for "quantum chromodynamics"). It was first fully formulated by Fritzsch, Gell-Mann and Leutwyler. It will be the main character of these lectures.

It is sort of strange that, while an approximate global symmetry, $SU(3)_{flavor}$, of the physics of hadrons, which really isn't a fundamental

symmetry, at all, gave rise to the idea of the quark model of hadrons, once this model had been introduced and people had recognized that it forces one to postulate the presence of a color quantum number and of $SU(3)_{color}$ and that it was a great idea to gauge $SU(3)_{color}$, the flavor symmetry became quite insignificant and, in many respects, faded away. Nevertheless, it is good to recall that $SU(3)_{flavor}$ had some considerable practical success. Much of this success relies on applications of the Wigner-Eckart theorem for $SU(3)_{flavor}$ to matrix elements of the electromagnetic interaction Hamiltonian, which breaks $SU(3)_{flavor}$ (see (19)!) and transforms under the adjoint rep. $(1,1) = 8$ of $SU(3)$.*)
Wigner-Eckart then leads to relations between the magnetic moments and masses of different

*) It has zero isospin and hypercharge $\rightarrow \propto \Lambda$.

baryons:

1. Magnetic moments

$$\mu(\Lambda) = \frac{1}{2}\mu(n), \quad \mu(\Sigma^+) = \mu(p),$$
$$\mu(\Xi^0) = \mu(n), \quad \mu(\Xi^-) = \mu(\Sigma^-) = -[\mu(p) + \mu(n)],$$
$$\mu(\Sigma^0) = -\frac{1}{2}\mu(n).$$

For the decuplet, all magnetic moments turn out to come out $\propto$ to electric charge of baryon.

2. Electromagnetic mass splittings

$$m_{\Xi^-} - m_{\Xi^0} = m_{\Sigma^-} - m_{\Sigma^+} + \underbrace{m_p - m_n}_{(\simeq -1\,MeV)} \quad (\simeq 6.5 \pm 1.0\,MeV)$$

Another related application is the once famous Gell-Mann-Okubo mass formula (based on assuming that $SU(2)_{isospin}$ and hypercharge are exact symmetries, while some interactions break $SU(3)_{flavor}$).

All this is well explained in ref. 2), Chapt. 1; see also 3).

The fading of $SU(3)_{flavor}$

With the prediction of "charm" (by Glashow, Iliopoulos and Maiani), and, later on, of "top" and "bottom", and with the discovery of the ψ/J and

Υ, ... mesons, it became clear that $SU(3)_{flavor}$ does <u>not have a fundamental meaning</u> in the theory of quarks and hadrons. But the quark model of hadrons and $SU(3)_{color}$ remained alife and well! If one includes the <u>weak interactions</u> of quarks and leptons ($\nearrow$ electroweak theory) then it becomes evident that quarks and leptons must be organized in "<u>generations</u>", of which <u>three</u> are known, at present:

$$[u, d], \; [c, s], \; [t, b] \qquad \text{(quarks)}$$

$$[(e_L, \nu_e), e_R], \; [(\mu_L, \nu_\mu), \mu_R], [(\tau_L, \nu_\tau), \tau_R] \quad \text{(leptons)}$$

Much more about this later in this course!

Let us recall, once again the various

<u>Fundamental Interactions</u>

(i) Gravitation ($\nearrow$ Einstein's GR)

(ii) Electromagnetism ($\nearrow$ QED; see QFT I)

(iii) Weak interactions (→ Fermi theory, GSW)

(iv) Strong interactions (→ QCD)

We will not consider problems related to a quantum theory of gravitation ("quantum gravity") in this course. We will mainly focus on (iv); (ii) has been the subject of QFT I. The "Standard Model" of (ii)–(iv) is the subject of specialized courses.

Some past and present key problems in particle theory:

(1) Renormalizable QFT of electroweak and strong interactions – Standard Model & beyond

(2) Structure of known particles (leptons, mesons, baryons, vector bosons, ...)

(3) Origin of masses and mass hierarchies; spontaneous symmetry breaking and Higgs

(4) Finiteness of electromagnetic mass differences

(5) Bjorken scaling in deep inelastic electron–nucleon scattering; asymptotic freedom

(6) Quark confinement and colour screening

(7) Why three generations of quarks and leptons?

(8) Nature and origin of dark matter and dark energy; cosmological-constant problem

(9) Unification: Supersymmetry, GUT's, string theory, "quantum gravity", ...

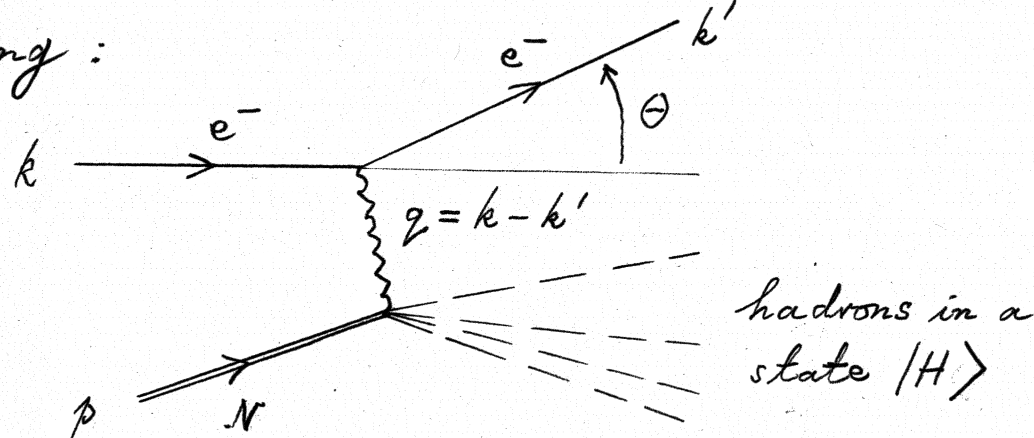
(10) How do our theories explain experiments?

etc.

In the 1960's and 70's, there has been enormous progress on problems (1), (2), (4), (5) and partial aspects of (3). Moreover, deep ideas on problem (9) were proposed. Since then, much progress on problems (3), (6) and (9) has been made, which is, however, not conclusive. A real lot is understood about problem (10), up to energy scales of order several hundred GeV. There is no deep understanding of problems (3) and (6)-(9), yet; and experiments at energies around the TeV scale have only recently (or will soon be) performed: Tevatron, LHC.

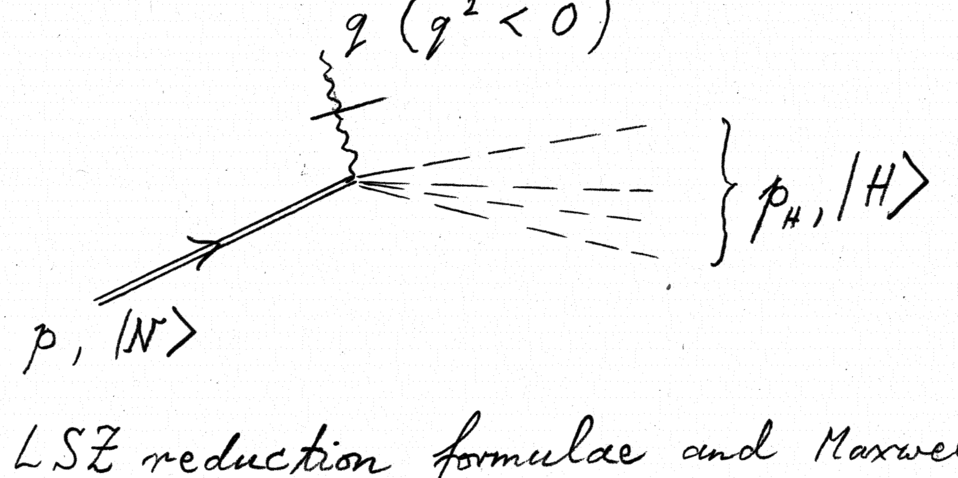
A comment on deep inelastic electron-nucleon scattering - Problem (5).

We consider high-energy electron-nucleon scattering :



(29)

We are interested in calculating the spin-averaged cross section for this process, where the unobserved, outgoing hadrons are summed over (i.e., we sum over the states $|H\rangle$), but the electron scattering angle θ is observed. This cross section can be calculated from



(30)

Using LSZ reduction formulae and Maxwell's equation (e.g. $\Box A_\mu = J_\mu$, in Lorentz gauge), we find that the inclusive cross section associated with (29) is proportional to the matrix

$$W^{\mu\nu}(p,q) := \frac{1}{2} \sum_{s_N} \sum_{|H\rangle} \delta^{(4)}(p_H - p - q) \times$$
$$\times \langle H|\hat{J}^\mu(q)|p, s_N\rangle_N \overline{\langle H|\hat{J}^\nu(q)|p, s_N\rangle_N} \quad (31)$$

for a proper normalization of the one-nucleon states $|p, s_N\rangle_N$. Because of the averaging over nucleon spin, s_N, and the summation over $|H\rangle$, $W^{\mu\nu}(p,q)$ is a Lorentz tensor depending on the two four-vectors p and q. It must therefore be a linear combination of $p^\mu p^\nu$, $p^\mu q^\nu$, $q^\mu p^\nu$, $q^\mu q^\nu$ and $g^{\mu\nu}(\equiv \eta^{\mu\nu})$, with coefficients that can only depend on the invariants q^2 and

$$\nu := -\frac{q \cdot p}{m_N}, \quad (32)$$

because $p^2 = m_N^2$; ($|p, s_N\rangle$ is on-shell).

Current conservation implies that

$$q_\mu W^{\mu\nu}(p,q) = q_\nu W^{\mu\nu}(p,q) = 0 \qquad (33)$$

Moreover, $\overline{W^{\mu\nu}} = W^{\nu\mu}$, and $(W^{\mu\nu})$ is a positive-definite matrix. Using also that parity is conserved in the process (29), we find that

$$W^{\mu\nu}(p,q) = -\left(\frac{q^\mu q^\nu}{q^2} - \eta^{\mu\nu}\right) W_1(\nu, q^2)$$
$$+ \frac{1}{m_N^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q^\nu\right) W_2(\nu, q^2), \qquad (33)$$

where W_1 and W_2 are <u>positive</u> functions. Plugging (33) into (31), (30) and then into (29), we find for the inclusive differential cross section in the nucleon rest frame

$$\frac{d^2\sigma}{d\Omega\, d\nu} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left(W_2 + 2W_1 \tan^2\left(\frac{\theta}{2}\right)\right), \qquad (34)$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is the solid angle into which the electron is scattered, and $\left(\frac{d\sigma}{d\Omega}\right)_{Mott}$ is the differential cross section for relativistic elastic electron scattering in a Coulomb field, with the spins of the incoming and outgoing electron not measured. By formula (13.63) of QFT I with $v \simeq c$, $Z = 1$ and $e^2 := \frac{e_R^2}{4\pi}$, we have that

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} \simeq \frac{e^4}{4\,E_{e^-}^2} \cdot \frac{\cos^2\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)},$$

where $E_{e^-} = k \cdot p / m_N$ is the energy of the incident electron in the nucleon rest frame.

Naiively, one would expect that the cross section (34) should fall off rapidly, as $q^2 \to -\infty$, for fixed p_H^2, because it should be proportional to an e.m. form factor for the transition from $|p,(\cdot)\rangle_N$ into $|H\rangle$. However, the SLAC-MIT collaboration headed by Friedman, Kendall and Taylor found, in an experiment at SLAC in 1968, that

$$\nu W_2(\nu, q^2) \simeq \text{const.} \qquad (35)$$

in q^2, for fixed values of

$$\omega := -\frac{2m_N \nu}{q^2} > 1.$$

The electron scattering angle θ was between $6°$ and $10°$, so that $\tan^2\left(\frac{\theta}{2}\right) < 7.5 \times 10^{-3}$, which explains why W_1 is unimportant in the analysis of those experimental data. In the limit $q^2 \to -\infty$, p_H^2 tends to ∞, which is why these processes are called "deep inelastic scattering".

Bjorken and Feynman came up with arguments that

$$\nu W_2(\nu, q^2) \to F_2(\omega), \quad W_1(\nu, q^2) \to F_1(\omega), \quad (36)$$

as $q^2 \to -\infty$, for fixed ω.

Feynman assumed that the photon in the process (30) interacts with constituents of the nucleon, which he called "partons", and that in the limit, where $q^2 \to -\infty$, these partons behave essentially like free charged particles.

We label the partons in the nucleon N by $i = 1, \dots, k$. Let $\mathcal{F}_i(x)\,dx$ be the probability that

that parton i carries a fraction in $[x, x+dx]$ of the momentum, $\vec{p}$, of the nucleon. Since $\mathcal{F}_i(x)\,dx$ is a probability measure, and by total momentum conservation, we have that

$$\int_0^1 dx\ \mathcal{F}_i(x) = 1 \text{ and } \sum_{i=1}^{k} \int_0^1 dx\ x\,\mathcal{F}_i(x) = 1. \qquad (37)$$

For elastic scattering of a highly relativistic electron off a relativistic parton with 4-momentum xp $(0 \leq x \leq 1)$, electric charge eQ_i (and spin $\frac{1}{2}$) we find

$$\frac{d^2\sigma}{d\Omega\, d\nu} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} Q_i^2 \left(1 - \frac{q^2}{2(m_N x)^2} \tan^2\left(\frac{\theta}{2}\right)\right) \delta\left(\nu + \frac{q^2}{2m_N x}\right),$$

with $(m_N x)^2 = (xp)^2 \overset{!}{=} (\underset{\substack{\uparrow \\ \text{4-momentum of} \\ \text{outgoing parton}}}{q + xp})^2 = q^2 + 2\nu m_N x + (m_N x)^2$;

i.e., $\nu = -\dfrac{q^2}{2m_N x}$.

A derivation of this formula may be found in:
J.-P. Derendinger, "Théorie quantique des champs", Presses polytechniques et universitaires romandes 2001.

The term $\propto \tan^2\left(\frac{\theta}{2}\right)$ is appropriate for a parton with spin $\frac{1}{2}$ and magnetic moment $\dfrac{eQ_i}{2m_N x}\vec{S}$.

Summing over all partons, we find that

$$\frac{d^2\sigma}{d\Omega\, d\nu} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left\{ \sum_{i=1}^{k} Q_i^2 \int_0^1 dx\, \mathcal{F}_i(x) \left(1 + \frac{q^2}{2(m_N x)^2} \tan^2\left(\frac{\theta}{2}\right)\right) \times \right.$$

$$\left. \delta\left(\nu + \frac{q^2}{2 m_N x}\right) \right\} \qquad (38)$$

Comparing this formula with expression (34), we find that

$$W_2(\nu, q^2) = \sum_{i=1}^{k} Q_i^2 \int_0^1 dx\, \mathcal{F}_i(x)\, \delta\left(\nu + \frac{q^2}{2 m_N x}\right)$$

$$= \frac{1}{\nu\omega} \sum_{i=1}^{k} Q_i^2\, \mathcal{F}_i\left(\frac{1}{\omega}\right), \qquad (39)$$

and

$$W_1(\nu, q^2) = \cdots = \frac{\omega\nu}{2 m_N} W_2(\nu, q^2), \qquad (40)$$

in accordance with (35), (36)! (The relation (40) is originally due to Callan and Gross. It agrees with experiment to within 10% - 15%.)

Eqs. (37) and (39) then yield the sum rule

$$\int_1^\infty \nu(\omega)\, W_2(\nu(\omega), q^2) \frac{d\omega}{\omega} = \sum_{i=1}^{k} Q_i^2,$$

etc.

It is tempting to identify the charged partons with the quarks. Comparison of sum rules, like the one above, with experimental data then suggests that a non-zero fraction of the nucleon 4-momentum p must be carried by electrically neutral partons. These "particles" can be interpreted as the field quanta of the force field that binds the quarks into baryons or mesons: the "gluons".

Apparently, in scattering processes at very large values of $-q^2$, the partons behave approximately like freely moving point particles; in particular, the quarks behave like fractionally charged Dirac particles: "Asymptotic Freedom."

In view of Problems (1), (5) and (10), the question to be answered is whether one can construct a renormalizable QFT of quarks and gluons "explaining" Asymptotic Freedom. The answer is: QCD!

Combining the material discussed in QFT I with the one sketched in this Introduction, we arrive at the following picture:

There are __leptons__ $[(e_L, \nu_e), e_R]$, $[(\mu_L, \nu_\mu), \mu_R]$ and $[(\tau_L, \nu_\tau), \tau_R]$. They have electromagnetic (QFT I) and weak (later in this course) interactions, but apparently no strong interactions. From the experimental data on baryons and mesons one is led to conclude that there are __quarks__ $[u, d]$, $[c, s]$ and $[t, b]$ (with electric charges $[\frac{2}{3}, -\frac{1}{3}]$, respectively). They have strong interactions (somehow related to the symmetry $SU(3)_{color}$), electromagnetic and weak interactions.

In QFT I, we found that it is a very successful idea to describe the electromagnetic interactions between charged particles in the form of a gauge theory: QED. The corresponding gauge field is the electromagnetic vector potential, A_μ. The idea may then be plausible that the weak and strong interactions might be mediated by gauge fields, too. Since the strong interactions are associated with a non-abelian symmetry group, $SU(3)_{color}$, the gauge theory describing them may be expected to be a <u>non-abelian gauge theory</u> with gauge group $SU(3)$.

We will see that the weak interactions can be thought to be mediated by $SU(2)$ gauge fields only coupling to <u>left-handed</u> particles.

With these ingredients one can formulate the "jewel" of present-day particle physics: the so-called <u>Standard Model</u> of Particle Physics.

1.2 The Lagrangian of the Standard Model

The standard model emerges from the work of Glashow, Salam and Weinberg (electro-weak theory) and Fritsch, Gell-Mann and Leutwyler (QCD). It is a gauge theory with gauge group

$$G = SU(3)_c \times SU(2)_L \times U(1)_y \qquad (41)$$

The corresponding gauge fields are denoted as follows.

$SU(3)$: A^a_μ, $a = 1, \dots, 8$ (gluons)

$SU(2)_L$: W^A_μ, $A = 1, 2, 3$

$U(1)_y$: B_μ

$\Big\}$ $(W^{\pm}, Z^0, \gamma)$

A gauge field configuration, (A, W, B), determines a connection on a principal bundle with fibres $\simeq G$, and base space = space-time M^4, and on any associated vector bundle. A connection gives rise to a notion of parallel transport and of covariant derivative:

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} - i A^a_\mu \lambda^a - i W^A_\mu S^A - i B_\mu Y, \qquad (42)$$

where λ^a, $a = 1, \cdots, 8$, are the Gell-Mann matrices and $S^A = \frac{1}{2}\sigma^A$, $A = 1, 2, 3$, are the usual generators of $su(2)$. These notions from differential geometry will be clarified later.

The components of the curvature 2-form of the connection corresponding to ∇_μ in (42) are given by

$$[\nabla_\mu, \nabla_\nu] = F^a_{\mu\nu}\lambda^a + W^A_{\mu\nu} S^A + B_{\mu\nu} Y, \qquad (43)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, \qquad (44)$$

with f^{abc} the structure constants of $su(3)_{color}$,

$$W^A_{\mu\nu} = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu + \varepsilon^{ABC} W^B_\mu W^C_\nu, \qquad (45)$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad (46)$$

The quark- and lepton fields can be interpreted as sections of associated vector bundles transforming under the following representations of the gauge group $G = SU(3)_c \times SU(2)_L \times U(1)_Y$; (we decompose all the quark- and lepton fields, which are spin $\frac{1}{2}$ Dirac fields, into chiral, i.e., left-handed and right-handed components):

	Notation	Representations
Left handed quarks	$q_L^{(n)}$	$(3, 2, 1/6)$
Right handed quarks	$u_R^{(n)}$	$(\bar{3}, 1, 2/3)$
	$d_R^{(n)}$	$(\bar{3}, 1, -1/3)$
Left handed leptons	$l_L^{(n)}$	$(1, 2, 1/2)$
Right handed leptons	$\psi_R^{(n)}$	$(1, 1, -1)$

Here $n = 1, 2, 3$ is the generation index.

Instead of introducing right-handed fermion fields, we could introduce the charge conjugates of the right-handed fermion fields:

transforming under the following representations of the gauge group $G = SU(3)_c \times SU(2)_L \times U(1)_Y$; (we decompose all the quark- and lepton fields, which are spin $\frac{1}{2}$ Dirac fields, into chiral, i.e., left-handed and right-handed components):

	Notation	Representations
Left handed quarks	$q^{(n)}$	$(3, 2, 1/6)$
Right handed quarks	$u^{(n)}$ $d^{(n)}$	$(3, 1, 2/3)$ $(3, 1, -1/3)$
Left handed leptons	$\ell^{(n)}$	$(1, 2, -1/2)$
Right handed leptons	$e^{(n)}$	$(1, 1, 1)$

Here $n = 1, 2, 3$ is the <u>generation index</u>.

Next, we introduce scalar fields H^α, $\alpha = 1, 2$, the <u>Higgs fields</u>.

$$\psi_R \longrightarrow (\psi_R)^C = C\gamma^0(\psi_R^*)^T = (\psi^C)_L, \quad (47)$$

and formulate everything in terms of only left-handed fields. The passage from ψ to ψ^C amounts to replacing particles by their anti-particles. In (47), one chooses C to be given by $i\gamma^2\gamma^0$. (Exercise: Verify that this choice of C really amounts to charge conjugation and that $(\psi_R)^C$, as defined in (47), is indeed left-handed.)

Scalar fields. In the standard model of particle physics, all masses of quarks, leptons and the weak gauge bosons, $W^\pm$ and Z^0, are generated by interactions of the corresponding Dirac- and gauge fields with scalar fields, called Higgs fields, via the so-called Higgs mechanism. Thus, we must introduce scalar fields H^α, $\alpha = 1, 2$; and we choose them to transform under the following representations of $G = SU(3)_c \times SU(2)_L \times U(1)_y$:

$$(1, 2, -1/2) \tag{48}$$

We use the following notations:

$$H := \begin{pmatrix} H^1 \\ H^2 \end{pmatrix}, \quad \bar{H} := -i\left(H^* \sigma^2\right)^T = \begin{pmatrix} H^{2*} \\ -H^{1*} \end{pmatrix} \tag{49}$$

Then $\bar{H}$ transforms under

$$(1, 2, 1/2).$$

The quantity

$$\varepsilon_{\alpha\beta} \bar{H}^\alpha H^\beta = H^{1*} H^1 + H^{2*} H^2 = H^* H \tag{50}$$

is invariant under transformations by elements of G.

We are now prepared to introduce the <u>Lagrangian density of the standard model</u>.

$$\mathcal{L} = \mathcal{L}_{kin.} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}, \tag{51}$$

where

$$\mathcal{L}_{kin.} = \mathcal{L}_{YM} + \mathcal{L}_F, \tag{52}$$

$$\mathcal{L}_{YM} = -\frac{1}{4g_s^2} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4g^2} W^A_{\mu\nu} W^{A\mu\nu} - \frac{1}{4(g')^2} B_{\mu\nu} B^{\mu\nu}; \tag{53}$$

$$\mathcal{L}_F = i\sum_{n=1}^{3}\Big[\bar{q}^{(n)}\not{\nabla}^{q} q^{(n)} + \bar{u}^{(n)}\not{\nabla}^{u} u^{(n)} + \bar{d}^{(n)}\not{\nabla}^{d} d^{(n)} + \bar{l}^{(n)}\not{\nabla}^{l} l^{(n)} + \bar{\psi}^{(n)}\not{\nabla}^{\psi}\psi^{(n)}\Big], \tag{54}$$

where $\not{\nabla}^{\rho} = \gamma^{\mu}\nabla^{\rho}_{\mu}$ and the superscript ρ on ∇^{ρ}_{μ} indicates the representation of $G = SU(3)_c \times SU(2)_L \times U(1)_y$ under which the corresponding matter field transforms;

$$\mathcal{L}_{Yukawa} = -\sum_{n,m=1}^{3}\Big[\lambda^{nm}_{d}\,\varepsilon_{\alpha\beta}H^{\alpha}\bar{d}^{(n)}q^{(m)\beta} + \lambda^{nm}_{u}\,\varepsilon_{\alpha\beta}\bar{H}^{\alpha}\bar{u}^{(n)}q^{(m)\beta} + \lambda^{nm}_{l}\,\varepsilon_{\alpha\beta}H^{\alpha}\bar{\psi}^{(n)}l^{(m)\beta}\Big] + h.c.; \tag{55}$$

and

$$\mathcal{L}_{Higgs} = (\nabla^{H}_{\mu}H)^{*}(\nabla^{H\mu}H) - V(H,H^{*}), \tag{56}$$

with

$$V(H,H^{*}) = \frac{\lambda}{2}(H^{*}H)^{2} - \mu^{2}(H^{*}H) \tag{57}$$

The matrices λ^{nm}_{x}, $x = d, u, l$, of Yukawa coupling constants, together with an appropriate choice of V, give rise to the masses of quarks and leptons; except that the neutrinos remain massless. The kinetic energy term in (56), together with an appropriate choice of V, render the intermediate vector bosons of the weak interactions, $W^{\pm}, Z^{0}$, massive via the Higgs mechanism.

If Nature teaches us that neutrinos are massive — as it appears to do in recent experiments — we are led to also introduce right-handed neutrinos transforming trivially under G. These particles can therefore not be detected in strong or electro-weak processes. They are described by right-handed Dirac fields, $\nu_R^{(n)}$, with Lagrangian density

$$\mathcal{L}_{\nu_R} = i \sum_{n=1}^{3} \overline{\nu}_R^{(n)} \not{\partial} \nu_R^{(n)} - \frac{1}{2} \sum_{n,m=1}^{3} M_n (\nu_R^{(n)})^T C \nu_R^{(n)} - h.c. - \sum_{n,m=1}^{3} \lambda_\nu^{nm} \varepsilon_{\alpha\beta} \overline{H^\alpha \nu_R^{(n)}} \ell^{(m)\beta} - h.c., \tag{58}$$

where the M_n's are <u>Majorana masses</u>; (see QFT I, Chapt. 5), and the λ_ν^{nm}'s are further Yukawa couplings giving rise to masses of left-handed neutrinos.

The Lagrangian of the standard model contains the following numerous parameters:

g_s, g, g' : gauge coupling constants;

$\lambda_d^{nm}, \lambda_u^{nm}, \lambda_\ell^{nm}$ (λ_ν^{nm}) : Yukawa couplings; they are complex but are determined by observation only up to certain phases; (redefinition of spinor fields!);

(M_n : Majorana masses of right handed neutrinos)

λ, μ : Parameters of the Higgs potential.

From $\lambda_d^{n,m}$ and $\lambda_u^{n,m}$ one can infer the <u>Cabibbo-Kobayashi-Maskawa matrix</u> describing the weak couplings between quarks of different generations and a complex phase parametrizing CP violation in the charged-current weak interactions (which, among other things, has important implications for cosmology).

The task of a modern particle theorist is to understand the relativistic quantum theory determined by the Lagrangian $\mathcal{L}$ in (51)-(58). In particular, we have to learn how to do concrete calculations of observational significance starting from $\mathcal{L}$, including calculations of radiative corrections (renormalization theory, renormalization group calculations, etc.). We will mostly focus on the theory of the strong interactions among quarks (QCD), neglecting their electro-weak interactions.

It has become customary to quantize gauge field theories, such as the standard model, formally by using the formalism of functional (or path) integrals, which, as first recognized by P.A.M. Dirac, is well adapted to the Lagrangian formulation of the classical field theory; a theme later worked out more fully by R.P. Feynman. Functional integrals are well adapted to calculating vacuum expectation values of time-ordered products of field operators (from which one can derive scattering amplitudes via LSZ reduction formulae). They are convenient to derive the Feynman rules of the theory.

We thus proceed to develop (formally) a theory of functional integration, starting with quantum mechanics and then proceeding to QFT.

2. The Dirac–Feynman Path Integral in Quantum Mechanics.

To start with, I recommend to the reader to study the following two papers:

P. A. M. Dirac, "The Lagrangian in Quantum Mechanics", Phys. Zeitschrift der Sowjetunion 3, 64–72 (1933).

R. P. Feynman, "Space–Time Approach to Non-Relativistic Quantum Mechanics", Rev. Mod. Phys. 20, 367 (1948).

We owe our understanding of QFT, in particular of non-abelian gauge theories – perturbatively and non-perturbatively – largely to a formulation of QFT in terms of functional (or path) integrals. Path-integral quantization of classical field theories yields a manifestly Lorentz-covariant treatment of the corresponding QFT's; (similarly to Schwinger's action formalism). At the formal level, it is very flexible. It also allows for a natural passage to "Euclidian Field Theory", which is based on an analytic continuation of Green functions in the time variables from the real to the imaginary axis. In Euclidian Field Theory, functional integrals often aquire a mathematically precise probabilistic interpretation. This is the starting point for numerous non-perturbative studies of QFT.

In this chapter, we study the path-integral quantization of classical Lagrangian systems with finitely many degrees of freedom. For simplicity, we consider systems with a configuration space $\mathbb{R}^n$; (to be generalized later). Their classical phase space is then $\mathbb{R}^{2n}$. We choose coordinates q^j, p_j, $j = 1, \dots, n$ on $\mathbb{R}^{2n}$, where the q^j's are configuration space coordinates, and the p_j's are the corresponding canonical momenta, with

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \qquad (2.1)$$

where $\{\cdot,\cdot\}$ is the Poisson bracket.

According to Heisenberg and Dirac, canonical quantization of a classical Hamiltonian system with phase space $\mathbb{R}^{2n}$ relies on a replacement of the canonical coordinates q^j, p_j by operators Q^j, P_j, $j = 1, \dots, n$, and of the Poisson bracket $\{\cdot,\cdot\}$ by $-\frac{i}{\hbar}[\cdot,\cdot]$, where $\hbar$ is Planck's constant and $[A,B] = AB - BA$ is the commutator of A with B. Thus

$$[Q^i, Q^j] = [P_i, P_j] = 0, \quad [Q^i, P_j] = i\hbar\,\delta^i_j. \qquad (2.2)$$

Let $h(p,q)$ denote the Hamilton function of a classical Hamiltonian system with phase space $\mathbb{R}^{2n}$. The Hamiltonian, $H(P,Q)$, of the corresponding quantum system is obtained by replacing p_j by P_j and q^j by Q^j, $j=1,\dots,n$, in $h(p,q)$ and adopting an ordering prescription of the Q's and the P's such that

$$H(P,Q)^* = H(P,Q) \tag{2.3}$$

(with $P_j^* = P_j = \frac{\hbar}{i}\frac{\partial}{\partial q^j}$ and $(Q^j)^* = Q^j = \text{mult.}\, q^j$, $j=1,\dots,n$, in the Schrödinger representation).

Obviously, $H(P,Q)$ is determined by $h(p,q)$ only up to terms of $O(\hbar)$.

Let $|q\rangle = \delta(q-q')$ be the generalized eigenvector of Q^i corresponding to the eigenvalue q^i, and let $|p\rangle = (2\pi\hbar)^{-n/2} \exp(ip\cdot q'/\hbar)$, $p\cdot q' := \sum_{j=1}^{n} p_j \cdot q'^j$, be the generalized eigenvector of P_i corresponding to p_i, $i=1,\dots,n$. We will work in the Schrödinger representation, throughout this chapter. Clearly,

$$\langle q|p\rangle = \int d^n q' \,\delta(q-q')(2\pi\hbar)^{-\frac{n}{2}} e^{ip\cdot q'/\hbar} = (2\pi\hbar)^{-\frac{n}{2}} e^{ip\cdot q/\hbar} \tag{2.4}$$

We have the completeness relations

$$\int d^n q \, |q\rangle\langle q| = \mathbb{1}, \quad \int d^n p \, |p\rangle\langle p| = \mathbb{1}. \qquad (2.5)$$

With the rule for how to determine $H(P,Q)$ described above we find that

$$\langle p | H(P,Q) | q\rangle = h(p,q)\langle p|q\rangle + O(\hbar), \qquad (2.6)$$

with $\langle p|q\rangle = \overline{\langle q|p\rangle}$.

We now attempt to calculate the propagator $\exp(-itH/\hbar)$ from $h(p,q)$. We first choose $t = \varepsilon$, with $|\varepsilon|$ "very small". Then

$$\langle q'|e^{-i\varepsilon H/\hbar}|q\rangle \overset{(2.5)}{=} \int \frac{d^n p}{(2\pi\hbar)^{n/2}} e^{ip\cdot q'/\hbar} \langle p|e^{-i\varepsilon H/\hbar}|q\rangle$$

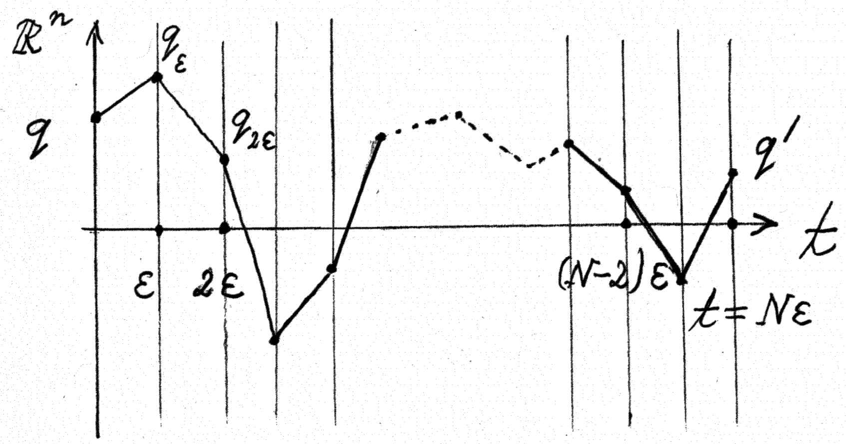
$$\approx \int \frac{d^n p}{(2\pi\hbar)^{n/2}} e^{ip\cdot q'/\hbar} \langle p|\mathbb{1} - \frac{i\varepsilon}{\hbar} H|q\rangle$$

$$\overset{(2.6)}{=} \int \frac{d^n p}{(2\pi\hbar)^{n/2}} e^{ip\cdot q'/\hbar} \left(1 - \frac{i\varepsilon}{\hbar} h(p,q) + O(\varepsilon)\right)\langle p|q\rangle$$

$$\overset{(2.4)}{\approx} \int \frac{d^n p}{(2\pi\hbar)^n} e^{i[p\cdot(q'-q) - \varepsilon h(p,q)]/\hbar + O(\varepsilon)} \qquad (2.7)$$

For finite time t (>0), we decompose the time interval $[0,t]$ in $N \to \infty$ very short subintervals, setting

$\varepsilon = \dfrac{t}{N}$. In every such subinterval we then use (2.7).



This yields the formula

$$\langle q' | e^{-itH/\hbar} | q \rangle = \langle q' | \left(e^{-i\varepsilon H/\hbar} \right)^N | q \rangle$$

$$= \int d^n q_1 \cdots d^n q_{N-1} \langle q' | e^{-i\varepsilon H/\hbar} | q_{N-1} \rangle \times$$

$$\times \langle q_{N-1} | e^{-i\varepsilon H/\hbar} | q_{N-2} \rangle \cdots \langle q_1 | e^{-i\varepsilon H/\hbar} | q \rangle$$

$$\overset{(2.7)}{=} \lim_{N\to\infty} \int_{\substack{q_0 = q \\ q_N = q'}} \prod_{i=1}^{N-1} \frac{d^n q_i \, d^n p_i}{(2\pi\hbar)^n} \frac{d^n p_0}{(2\pi\hbar)^n} \times$$

$$\times \exp \frac{i}{\hbar} \cdot \frac{t}{N} \left[\sum_{j=0}^{N-1} p_j \cdot \frac{q_{j+1} - q_j}{(t/N)} - h(p_j, q_j) + O(\hbar) \right] \tag{2.8}$$

We formally exchange the limit $N \to \infty$ with integration, with the prescription that the p-integrals be evaluated before the q-integrals. This yields the

Feynman path integral over phase space:

$$\langle q' | e^{-itH/\hbar} | q \rangle = \int\limits_{\substack{q(0)=q \\ q(t)=q'}} \mathcal{D}q \int \mathcal{D}p \, \exp\Big[\frac{i}{\hbar}\int_0^t ds \, \{p(s)\cdot\dot{q}(s) - h(p(s),q(s)) + O(\hbar)\}\Big], \quad (2.9)$$

with

$$\text{"}\mathcal{D}q\,\mathcal{D}p = \prod_s \underbrace{\frac{d^nq(s)\,d^np(s)}{(2\pi\hbar)^n}}_{\text{Liouville measure on } \mathbb{R}^{2n}}\text{"} \quad (2.10)$$

Mathematically, the R.S. of (2.10) does not make much sense; it is <u>defined</u> as being given by the R.S. of (2.8).

Next, we consider the following Hamilton function:

$$\left.\begin{aligned} h(p,q) &= \sum_{j=1}^n \frac{p_j^2}{2m_j} + V(q), \\ &\text{and we choose} \\ H(P,Q) &= \sum_{j=1}^n \frac{P_j^2}{2m_j} + V(Q) \end{aligned}\right\} \quad (2.11)$$

and set the terms of $O(\hbar)$ in (2.6), (2.8), (2.9) to zero.

The $\mathcal{D}p$-integral in (2.9) is then a <u>Gaussian integral</u>, and we can evaluate it explicitly:

$$\int \mathcal{D}p \, \exp\Big[\frac{i}{\hbar}\int_0^t ds \, \{p(s)\cdot\dot{q}(s) - \sum_{j=1}^n \frac{p_j(s)^2}{2m_j}\}\Big] =$$

$$= \mathcal{N} \exp\left[\frac{i}{\hbar}\int_0^t ds \sum_{j=1}^{n} \frac{m_j\, \dot{q}^j(s)^2}{2}\right], \qquad (2.12)$$

where $\mathcal{N}$ is a q- independent (divergent) normalization factor. Plugging (2.12) into (2.9), we find that

$$\langle q'|e^{-itH/\hbar}|q\rangle = \mathcal{N} \int\limits_{\substack{q(0)=q\\ q(t)=q'}} \mathcal{D}q \; \exp\left[\frac{i}{\hbar}\int_0^t \left(\sum_{j=1}^{n} \frac{m_j\, \dot{q}^j(s)^2}{2}\right) - V(q(s))\right]$$

$$= \mathcal{N}\int\limits_{\substack{q(0)=q\\ q(t)=q'}} \mathcal{D}q \; \exp\Big[\frac{i}{\hbar} \underbrace{\int_0^t ds\, L(q(s),\dot{q}(s),s)}_{\text{action } S(q(\cdot))!}\Big] \qquad (2.13)$$

A mathematically precise definition of the R.S. of (2.12) starts from (2.8): Plugging (2.11) into (2.8), we formally get

$$\langle q'|e^{-itH/\hbar}|q\rangle = \lim_{N\to\infty} Z_N \int\limits_{\substack{q_0=q\\ q_N=q'}} \prod_{j=1}^{N-1} d^n q_j \;\times \qquad (2.14)$$

$$\times \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1} \frac{t}{N}\left\{\left(\sum_{i=1}^{n} \frac{m_i}{2}\cdot \frac{(q^i_{j+1}-q^i_j)^2}{(t/N)^2}\right) - V(q_j)\right\}\right],$$

where Z_N is a normalization factor. For appropriate choices of V, (2.13) is a consequence of the Trotter product formula. Formulae (2.13), (2.14) represent the

Dirac – Feynman path integral over configuration space in the Lagrangian formalism.

Before we generalize our findings we present a short

Digression on Gaussian Integrals.

1. Let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\prod_{i=1}^{N} dx_i$ the Lebesgue measure on $\mathbb{R}^N$, A an $N \times N$ matrix with $A + \bar{A} > 0$, and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{C}^N$. Then

$$\int_{\mathbb{R}^N} \exp\Big[-\frac{1}{2}\sum_{j,k=1}^{N} x_j A_{jk} x_k + i\sum_{j=1}^{N}\xi_j x_j\Big]\prod_{j=1}^{N} dx_j$$

$$= (2\pi)^{N/2}(\det A)^{-1/2}\exp\Big[-\frac{1}{2}\sum_{j,k=1}^{N}\xi_j (A^{-1})_{jk}\xi_k\Big] \quad (2.15)$$

Proof. W.l.o.g., we may suppose that $A > 0$ can be diagonalized. Then (2.15) follows from

$$\int_{\mathbb{R}} e^{-\frac{\lambda}{2}x^2} e^{i\xi x}\, dx = \int_{\mathbb{R}} e^{-\frac{\lambda}{2}\left(x - \frac{i\xi}{\lambda}\right)^2 - \frac{\xi^2}{2\lambda}}\, dx$$

$$= e^{-\frac{\xi^2}{2\lambda}}\frac{1}{\sqrt{\lambda}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dy = \sqrt{\frac{2\pi}{\lambda}}\, e^{-\frac{\xi^2}{\lambda}}.$$

Note that the result (2.15) can be obtained by finding the critical point of the quadratic form

$$Q(x) = \frac{1}{2}(x, Ax) - i(\xi, x)$$

given by $\bar{x} = iA^{-1}\xi$. Then $Q(\bar{x}) = \frac{1}{2}(\xi, A^{-1}\xi)$, (real scalar product), and we can write

$$\int_{\mathbb{R}^N} \exp\left[-\frac{1}{2}(x, Ax) + i(\xi, x)\right] \mathcal{D}x$$
$$= (\det A)^{-1/2} e^{-Q(\bar{x})}$$
$$= (\det A)^{-1/2} e^{-\frac{1}{2}(\xi, A^{-1}\xi)}, \qquad (2.16)$$

where $\mathcal{D}x := (2\pi)^{-N/2} \prod_{i=1}^{N} dx_i$. As a corollary, we find that

$$\int \mathcal{D}x \, P(x) e^{-Q(x)} = P\left(-i\frac{\delta}{\delta\xi}\right) \int \mathcal{D}x \, e^{-Q(x)}, \qquad (2.17)$$

where $\frac{\delta}{\delta\xi}$ is the usual gradient ("nabla") in the variables $\xi_1, \dots, \xi_N$, and P is a polynomial in $x_1, \dots, x_N$. Eq. (2.17) follows by exchanging $\frac{\delta}{\delta\xi}$ with $\int(\cdot)$.

2. Let $z = (z_1, \dots, z_N) \in \mathbb{C}^N$,

$$\mathcal{D}z := \prod_{i=1}^{N} i \frac{dz_i \wedge d\bar{z}_i}{2\pi} = \prod_{i=1}^{N} \frac{d\,\mathrm{Re}\,z_i \wedge d\,\mathrm{Im}\,z_i}{\pi}. \qquad (2.18)$$

Let A be an $N \times N$ matrix with $A + A^* > 0$, and let ξ and η be vectors in $\mathbb{C}^N$, $\langle \cdot, \cdot \rangle$ the scalar product on $\mathbb{C}^N$ taken to be anti-linear in the first argument. Then

$$\int_{\mathbb{C}^N} \mathcal{D}z \; \exp\left[-\langle z, Az\rangle + i\langle \xi, z\rangle + i\langle z, \eta\rangle\right]$$

$$= (\det A)^{-1} \exp\left[-\langle \xi, A^{-1}\eta\rangle\right], \qquad (2.19)$$

as follows by quadratic completion; (the fact that there are no terms in the exponent on the R. S. of (2.19) quadratic in ξ or η can be understood from the "gauge-invariance" of $\mathcal{D}z \exp[-\langle z, Az\rangle]$ under $z \to e^{i\alpha} z$, $\bar{z} \to e^{-i\alpha}\bar{z}$!).

We now apply the simple formulae in digression 1 to formula (2.14). In order to satisfy the condition that $A + \bar{A} > 0$ (in 1, above), we give time t a small imaginary part:

$$t \to t e^{-i\theta}, \quad 0 < \theta < \pi \qquad (2.20)$$

Then

$$\operatorname{Re}\left[\frac{i}{\hbar}\sum_{j=0}^{N}\left(\frac{te^{-i\theta}}{N}\right)^{-1}\left(\sum_{i=1}^{n}\frac{m_i}{2}\cdot\left(q^i_{j+1} - q^i_j\right)^2\right)\right] < 0$$

and

$$\operatorname{Re}\left[-\frac{i}{\hbar}\sum_{j=0}^{N}\frac{te^{-i\theta}}{N}V(q_j)\right] < 0,$$

provided $V(q) \geq 0$. Then the integrals on the R.S. of (2.14) are absolutely convergent. In particular, for

$$V(q) = Q(q),$$

where Q is a positive quadratic form in $q = (q_1, \cdots, q_n)$, the calculations in 1 can be applied to (2.14) and yield a path integral expression for the propagator of the n-diml. harmonic oscillator. Formally,

$$\langle q' | e^{-ite^{-i\theta} H/\hbar} | q \rangle = \int_{\substack{q(0)=q \\ q(t)=q'}} \mathcal{D}q \, e^{\frac{i}{\hbar} \int_0^t L_\theta(q(s), \dot{q}(s))\, ds}, \quad (2.21)$$

where

$$L_\theta(q, \dot{q}) = \sum_{i=1}^{n} \frac{m_i (\dot{q}^i)^2}{2} e^{i\theta} - V(q) e^{-i\theta} \quad (2.22)$$

Henceforth (2.13), (2.14) are <u>defined</u> as the limit

$$\theta \searrow 0 \quad (2.23)$$

of expressions like (2.21)!

It is of interest to also consider the limit

$$\theta \nearrow \frac{\pi}{2}, \quad (2.24)$$

which is called <u>"Wick rotation"</u>. In this limit, (2.21) yields the so-called <u>"Feynman-Kac formula"</u>:

$$\langle q' | e^{-tH} | q \rangle = \int dW^t_{qq'}(q(\cdot)) \exp\left[-\frac{1}{\hbar} \int_0^t V(q(s))\, ds\right], \quad (2.25)$$

where

$$\text{"} dW^t_{qq'} = N e^{-\int_0^t ds \left[\sum_{i=1}^n \frac{m_i \dot{q}^i(s)^2}{2\hbar} \right]} \delta(q(0)-q)\,\delta(q(t)-q')\,\mathcal{D}q \text{"} \tag{2.26}$$

is the Wiener measure on the space of continuous paths $\{q(\cdot) \mid q : [0,t] \to \mathbb{R}^n,\ q(0) = q,\ q(t) = q'\}$. The Wiener measure is normalized such that

$$\int dW^t_{qq'}(q(\cdot)) = \exp(-tH_0/\hbar)_{q'q} \tag{2.27}$$

where $H_0 = -\sum_{i=1}^n \frac{\hbar^2}{2m_i} \frac{\partial^2}{(\partial q^i)^2}$. It describes Brownian motion with a diffusion tensor $\left(\frac{\hbar}{m_i}\delta_{ik}\right)$.

If V is bounded from below and polynomially bounded then the R.S. of (2.25) is well defined, mathematically, and determines a semigroup, $\{e^{-tH}\}_{t\geq 0}$ of bounded, selfadjoint operators that has an analytic extension to all complex times t in the domain $\{z \mid \operatorname{Re} z > 0\}$. By passing to the boundary values $\operatorname{Re} t \searrow 0$, $\operatorname{Im} t \in \mathbb{R}$ arbitrary, one can recover the propagator $\exp(-itH/\hbar)$.

All this will turn out to have a natural generalization to QFT.

Groundstate ("vacuum") of H and time-ordered Green functions in the path-integral formalism.

Consider a Hamiltonian H of the form (2.11), and assume that

$$E_0 = \inf \operatorname{spec} H \tag{2.28}$$

is an eigenvalue of H. Then (by Perron-Frobenius) E_0 is simple, and the corresponding eigenvector, $\Omega(q)$, the "groundstate" of H, is unique up to a phase, which can be chosen such that $\Omega(q) > 0$. We also use the notation $\Omega(q) = \langle q | \Omega \rangle$. To simplify our notation we set $\hbar = 1$, and we define $t^\theta = t e^{-i\theta}$. By the spectral theorem, we have that, for $t > 0$, $0 < \theta < \pi$,

$$e^{-it^\theta H} = e^{-it^\theta E_0} \left\{ |\Omega\rangle\langle\Omega| + O\left(e^{-t \sin\theta \Delta E}\right) \right\}, \tag{2.29}$$

where $\Delta E = \operatorname{dist}(E_0, \operatorname{spec} H \setminus \{E_0\})$ is the energy gap above the groundstate energy E_0. Since $\langle \Omega | q \rangle > 0$, $\langle \Omega | q' \rangle > 0$, (2.29) implies that

$$\begin{aligned} E_0 &= \lim_{t \to \infty} - \frac{1}{it^\theta} \ln \langle q' | e^{-it^\theta H} | q \rangle \\ &= \lim_{t \to \infty} - \frac{1}{it^\theta} \ln \int \mathcal{D}q \; e^{i \int_{-t/2}^{t/2} ds \, L_\theta(q(s), \dot{q}(s))}, \end{aligned} \tag{2.30}$$

with $q(-\frac{t}{2}) = q$, $q(\frac{t}{2}) = q'$ arbitrary.

Let $O_1, \dots, O_k$ be polynomially bounded functions of q and $t_1 < \dots < t_k$ an ordered n-tuple of times. In the Heisenberg picture, we associate with O_α the operator

$$\hat{O}^\theta_\alpha(t) = e^{it^\theta H} \hat{O}_\alpha e^{-it^\theta H}.$$

The <u>time-ordered Green function</u> of the operators $O_1, \dots, O_n$ is defined by $(q_i := q(-T),\ q_f := q(T))$

$$\langle \Omega | \hat{O}_k(t_k) \cdots \hat{O}_1(t_1) | \Omega \rangle$$
$$= \lim_{\theta \searrow 0} \langle \Omega | \hat{O}_k e^{-i(t_k^\theta - t_{k-1}^\theta)(H-E_0)} \hat{O}_{n-1} \cdots e^{-i(t_2^\theta - t_1^\theta)(H-E_0)} \hat{O}_1 | \Omega \rangle$$

$$= \lim_{\theta \searrow 0} \lim_{T \to \infty} \frac{1}{Z_T^\theta} \int_{q_{i,f}=0} \mathcal{D}q\, O_k(q(t_k)) \cdots O_1(q(t_1)) \times e^{i\int_{-T}^{T} ds\, L_\theta(q(s), \dot{q}(s))}, \tag{2.31}$$

where

$$Z_T^\theta := \int_{q_{i,f}=0} \mathcal{D}q\, e^{i\int_{-T}^{T} ds\, L_\theta(q(s), \dot{q}(s))} \tag{2.32}$$

Since $H - E_0 \geq 0$ and $\mathrm{Im}(t_j^\theta - t_{j-1}^\theta) < 0$, for $\theta > 0$, the L.S. of (2.31) makes sense!

A sketch of a proof of (2.31), (2.32) goes as follows.

(1) We use that

$$e^{-it^\theta(H-E_0)}|0\rangle = \langle\Omega|0\rangle|\Omega\rangle + \mathcal{O}\left(e^{-t\sin\theta\,\Delta E}\right),$$

see (2.29). Hence

$$\langle 0|e^{-iT^\theta(H-E_0)}\hat{\mathcal{O}}_k^\theta(t_k)\cdots\hat{\mathcal{O}}_1^\theta(t_1)e^{-iT^\theta(H-E_0)}|0\rangle$$

$$\xrightarrow[T\to\infty]{} \langle 0|\Omega\rangle\langle\Omega|0\rangle\,\langle\Omega|\hat{\mathcal{O}}_k^\theta(t_k)\cdots\hat{\mathcal{O}}_1^\theta(t_1)|\Omega\rangle;$$

in particular

$$\langle 0|e^{-2iT^\theta(H-E_0)}|0\rangle \xrightarrow[T\to\infty]{} \langle 0|\Omega\rangle\langle\Omega|0\rangle > 0.$$

Hence

(2) $$\langle 0|e^{-iT^\theta(H-E_0)}\hat{\mathcal{O}}_k^\theta(t_k)\cdots\hat{\mathcal{O}}_1^\theta(t_1)e^{-iT^\theta(H-E_0)}|0\rangle$$

$$= e^{-2iT^\theta E_0}\int\prod_{\alpha=1}^{k+1}d^nq_\alpha\,\mathcal{O}_\alpha(q_\alpha)\langle q_\alpha|e^{-i(t_\alpha^\theta-t_{\alpha-1}^\theta)H}|q_{\alpha-1}\rangle,$$

where $q_0=0,\ t_0=-T,\ t_{k+1}=T$ and $\mathcal{O}_{k+1}(q)=\delta_0(q)$.

$$\cdots = e^{-2iT^\theta E_0}\int\prod_{\alpha=1}^{k}d^nq_\alpha\,\mathcal{O}_\alpha(q_\alpha)\int\mathcal{D}q\prod_{\alpha=0}^{k+1}\delta_{q_\alpha}(q(t_\alpha))\times$$

$$\times\exp\left[i\int_{-T}^{T}ds\,L_\theta(q(s),\dot q(s))\right]$$

$$= e^{-2iT^\theta E_0}\int_{q(\pm T)=0}\mathcal{D}q\,\mathcal{O}_k(q(t_k))\cdots\mathcal{O}_1(q(t_1))e^{i\int_{-T}^{T}ds\,L_\theta(q(s))} \qquad (2.33)$$

Formulae (2.31) and (2.32) readily follow from (1) and (2).

Note that the R.S. of (2.31), (2.33) is symmetric under permuting the operators $O_1(q(t_1)) \cdots O_k(q(t_k))$. Thus

$$\langle \Omega | T[\hat{O}_k(t_k) \cdots \hat{O}_1(t_1)] | \Omega \rangle$$

$$= \lim_{\theta \searrow 0} \lim_{T \to \infty} \frac{1}{Z_T^\theta} \int_{q_{i,f}=0} \mathcal{D}q \; O_k(q(t_k)) \cdots O_1(q(t_1)) \times \exp\left[i \int_{-T}^{T} ds \, L_\theta(q(s), \dot{q}(s)) \right], \quad (2.34)$$

where $T[\cdots]$ denotes time-ordering.

If the functions $O_j(q)$ are linear functions of q expression (2.34) can be obtained from the functional

$$e^{iW(J)} := \lim_{\theta \searrow 0} \lim_{T \to \infty} \frac{1}{Z_T^\theta} \int_{q_{i,f}=0} \mathcal{D}q \exp i \int_{-T}^{T} ds \{ L_\theta(q(s), \dot{q}(s)) + J(s) \cdot q(s) \} \quad (2.35)$$

by calculating the functional derivative

$$(-i)^k \left(\delta^k / \delta J_{i_k}(t_k) \cdots \delta J_{i_1}(t_1) \right) e^{iW(J)} \Big|_{J \equiv 0} \quad (2.36)$$

The functional $W(J)$ is then the generating functional of the connected time-ordered Green functions of (functions of) the q_i's.

After these generalities, we consider the simple example of a one-dimensional harmonic oscillator, with

$$H(P,Q) = \frac{1}{2}\left(P^2 + \mu^2 Q^2\right), \qquad (2.37)$$

where μ is the frequency of the oscillator, $Q = \text{mult.}\, q$, $q \in \mathbb{R}$, and $P = i\, d/dq$. The generalization to $n = 2, 3, \cdots$ coupled harmonic oscillators is immediate.

For H as in (2.37), the functional integral in (2.35) is Gaussian, and we can apply the formulae of digression 1 in the limit where $N \to \infty$:

$$e^{iW_0(J)} = \lim_{\theta \searrow 0} \lim_{T\to\infty} \frac{1}{Z_T^\theta} \int_{q_{i,f}=0} \mathcal{D}q \; \exp i \int_{-T}^{T} ds \left[\frac{1}{2}\dot{q}(s)^2 e^{i\theta} - \frac{\mu^2}{2} q(s)^2 e^{-i\theta} + J(s)q(s)\right] \qquad (2.38)$$

It is convenient to use Fourier transformation in time s to evaluate the exponent of the integrand on the R.S. of (2.38):

$$\lim_{T\to\infty} \int_{-T}^{T} ds \left[\frac{1}{2}\dot{q}(s)^2 e^{i\theta} - \frac{\mu^2}{2} q(s)^2 e^{-i\theta} + J(s)q(s)\right]$$

$$= \int d\omega \left[\frac{1}{2}\left[e^{i\theta}\omega^2 - e^{-i\theta}\mu^2\right] |\hat{q}(\omega)|^2 + \hat{J}(\omega)\,\overline{\hat{q}(\omega)} \right], \qquad (2.39)$$

where $\overline{\hat{q}(\omega)} = \hat{q}(-\omega)$, because $q(s)$ is real-valued. Apart from that, the variables $\hat{q}(\omega) \in \mathbb{C}$, $\omega \in \mathbb{R}$, are <u>independent</u>. Furthermore,

$$\mathcal{D}q = \text{Jac}\ \mathcal{D}\hat{q}, \qquad (2.40)$$

where Jac is the Jacobian of the change of variables $q(s) \longleftrightarrow \hat{q}(\omega)$. Since Fourier transformation is unitary, Jac = 1 (formally). Hence

$$e^{iW_0(J)} = \lim_{\theta \searrow 0} \frac{1}{Z_\infty^\theta} \int \mathcal{D}\hat{q}\ \exp i \int d\omega \left\{ \frac{1}{2}\left[e^{i\theta}\omega^2 - e^{-i\theta}\mu^2\right] |\hat{q}(\omega)|^2 + \hat{J}(\omega)\,\overline{\hat{q}(\omega)} \right\}. \qquad (2.41)$$

The normalization factor Z_∞^θ is chosen such that $W_0(J \equiv 0) = 0$. By formula (2.16),

$$W_0(J) = \lim_{\theta \searrow 0} \frac{1}{2} \int \hat{J}(-\omega) \left[e^{i\theta}\omega^2 - e^{-i\theta}\mu^2\right]^{-1} \hat{J}(\omega)\, d\omega$$

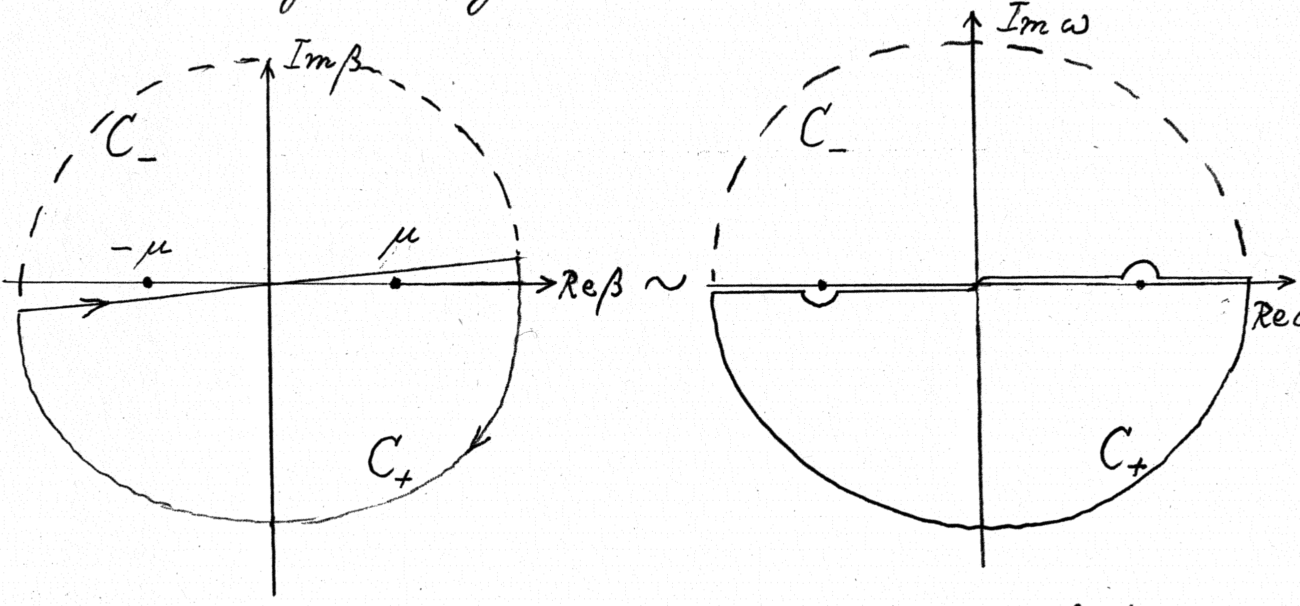
$$= \lim_{\theta \searrow 0} \frac{-i}{2} \int dt \int ds\ J(t)\, \Delta_F(t-s)\, J(s), \qquad (2.42)$$

where

$$\Delta_F(t) = \lim_{\theta \searrow 0} \frac{i}{2\pi} \int d\omega \frac{e^{-i\omega t}}{\omega^2 e^{i\theta} - \mu^2 e^{-i\theta}}$$

$$= \lim_{\theta \searrow 0} \frac{e^{i\theta}}{2\pi} \int d\omega \frac{e^{-i\omega t}}{(\omega e^{i\theta} + \mu)(\omega e^{i\theta} - \mu)} \qquad (2.43)$$

Let $\beta = \omega e^{i\theta}$, $\omega \in \mathbb{R}$. Then the β-integration contour is given by



where C_+ must be chosen if $t>0$ and C_- if $t<0$.

Thus,
$$\Delta_F(t) = \lim_{\varepsilon \searrow 0} \frac{i}{2\pi} \int d\omega \frac{e^{-i\omega t}}{\omega^2 - \mu^2 + i\varepsilon}$$

is the <u>Feynman propagator</u> of a "(0+1)-dimensional scalar field theory" with "mass" μ.

By (2.36), (2.42),

$$\langle \Omega | T[q(t) q(s)] | \Omega \rangle = \frac{\delta^2}{\delta J(t) \delta J(s)} W_0(J)$$

$$= i \Delta_F (t-s). \qquad (2.44)$$

Higher connected Green functions vanish; (<u>Wick's</u>

theorem).

Next, we consider a Hamiltonian

$$H(P,Q) = \frac{P^2}{2} + V(Q),$$

where

$$V(q) = \frac{\mu^2}{2} q^2 + U(q),$$

and $U(q) \geqq 0$ is a polynomial. Then, by (2.35), (2.36) and (2.38),

$$\begin{aligned} e^{iW(J)} &= \lim_{\theta \downarrow 0} N \int \mathcal{D}q \exp i \int ds \left\{ \frac{1}{2} \dot{q}(s)^2 e^{i\theta} - \frac{\mu^2}{2} q(s)^2 e^{-i\theta} \right. \\ &\qquad \left. - U(q(s)) e^{-i\theta} - J(s) q(s) \right\} \\ &= \exp i \int ds\, U\!\left(-i \frac{\delta}{\delta J(s)}\right) e^{iW_0(J)}, \end{aligned} \tag{2.45}$$

which is a convenient starting point for perturbation theory in U. Eq. (2.45) can be rewritten as follows. We first prove the identity

$$F(-i\nabla_x)\, G(x) = G(-i\nabla_y)\, F(y) e^{ix\cdot y}\Big|_{y=0}, \tag{2.46}$$

where $x = (x_1, \cdots, x_N)$, $y = (y_1, \cdots, y_N)$, and F and G are entire functions. It is enough to prove (2.46) for plane waves: $F(x) = e^{ia\cdot x}$, $G(x) = e^{ib\cdot x}$, $a, b \in \mathbb{R}^N$.

The

$$e^{ia\cdot(-i\nabla_x)}\, e^{ib\cdot x} = e^{ib\cdot(x+a)},$$

while

$$e^{ib\cdot(-i\nabla_y)}\, e^{i(x+a)\cdot y} = e^{i(x+a)\cdot(y+b)}$$

Setting $y=0$ in the second identity yields the desired result.

Applying (2.46) to (2.45) we find that

$$e^{iW(J)} = N \exp\left[\frac{1}{2}\int dt \int ds\, \Delta_F(t-s)\frac{\delta^2}{\delta q(t)\delta q(s)}\right]$$

$$\exp i\int ds\left\{U(q(s)) + J(s)q(s)\right\}\Big|_{q\equiv 0}, \qquad (2.47)$$

as convenient a starting point for perturbation theory as (2.45); (N is a normalization factor chosen such that $W(J\equiv 0)=0$).

Let us briefly return to (2.45): On the R.S. of this equation we let θ approach the value $\frac{\pi}{2}$, instead of $\theta \searrow 0$. Then

$$e^{W^E(J)} = N^E \int \mathcal{D}q \exp\left[-\int ds\left\{\frac{1}{2}\dot{q}(s)^2 + \frac{\mu^2}{2}q(s)^2 + U(q(s)) + iJ(s)q(s)\right\}\right]$$

$$= \exp\left[-\int ds\, U\left(-i\frac{\delta}{\delta J(s)}\right)\right] e^{W_0^E(J)}, \qquad (2.48)$$

where

$$W_0^E(J) = -\frac{1}{2}\int dt \int ds \; J(t)\, S_0(t-s)\, J(s),$$

and

$$S_0(t) = \frac{1}{2\pi}\int d\omega \,\frac{e^{-i\omega t}}{\omega^2+\mu^2} = \frac{e^{-\mu|t|}}{2\mu} \qquad (2.49)$$

is the so called Euclidian two-point Green-, or Schwinger function. Formulae (2.48) and (2.45) are connected to each other by analytic continuation in time from the real to the imaginary axis. While (2.35), (2.45) are formal objects that do not have a precise mathematical meaning, the huge advantage of (2.48) is that it has a rigorous mathematical interpretation; (Ornstein-Uhlenbeck process, Feynman-Kac formula). One should really interpret (2.35), (2.45) as an analytic continuation of (2.48) in the time variables from the imaginary- back to the real axis.

Our last topic in this chapter is to sketch how the results found above can be extended to mechanical systems with general configuration spaces (M, g), where M is a smooth n-dimensional Riemannian manifold equipped with a Riemannian metric $g = (g_{ij}(q))$. The phase space of such systems is then given by

$$\Gamma = T^*M, \tag{2.50}$$

where T^*M is the $2n$-dimensional cotangent bundle with base space M and fibres $\simeq \mathbb{R}^n$ (momentum space). Locally, one can always choose coordinates $(q^i, p_i)_{i=1,\cdots,n}$ on Γ such that the natural symplectic 2-form, ω, on Γ takes the form

$$\omega = \sum_i dp_i \wedge dq^i \tag{2.51}$$

(Darboux's theorem). The Liouville measure, $d\lambda(q,p)$, on Γ is given by

$$d\lambda(q,p) = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \tag{2.52}$$

We consider Hamiltonian systems where the metric g on M is determined by the kinetic energy, T:

$$T = \frac{1}{2} \sum_{i,j=1}^{n} \dot{q}^i g_{ij}(q) \dot{q}^j . \qquad (2.53)$$

Let $(g^{ij}(q))$ denote the inverse metric. Then the Hamilton functions of such systems have the form

$$h(p,q) = \frac{1}{2} \sum_{i,j=1}^{n} \left(p_i - A_i(q)\right) g^{ij}(q) \left(p_j - A_j(q)\right) + V(q), \qquad (2.54)$$

where $V(q) \geq 0$ is a function on M, and $A = \sum_{i=1}^{n} A_i \, dq^i$ is a 1-form describing an external "electromagnetic vector potential".

Schrödinger knew already in 1926 how to set up the wave mechanics of such systems! To quantize them, using formal path integrals, we start from expression (2.9). We define

$$\text{"}\mathcal{D}\lambda = \prod_s \frac{d\lambda(q(s), p(s))}{(2\pi\hbar)^n}$$

$$= \prod_s \frac{d^n q(s) \, d^n p(s)}{(2\pi\hbar)^n}\text{"} \qquad (2.55)$$

Then, formally (closing both eyes and hoping for the best)

$$\langle q' | e^{-itH/\hbar} | q \rangle = \int_{\substack{q(0)=q \\ q(t)=q'}} \mathcal{D}\lambda \exp\Big[\frac{i}{\hbar}\int_0^t ds \,\{p(s)\cdot\dot{q}(s) - h(p(s),q(s)) + O(\hbar)\}\Big], \tag{2.56}$$

where $H = H(P,Q)$ is the quantum Hamiltonian of the system already constructed by Schrödinger, and $Q^i|q\rangle = q^i|q\rangle$. Note that the integrand on the R.S. of (2.56) is a <u>Gaussian</u> in $p(\cdot)$. So we can hope to do the $p(\cdot)$-integrals using the formulae in digression 1. We start by changing variables:

$$\pi_i(s) := p_i(s) - A_i(q(s)).$$

Then $d\lambda(q,p) = d\lambda(q,\pi)$, $d^n p(s) = d^n \pi(s)$.

Carrying out the $p(\cdot)$-integrals in (2.56) then amounts to calculating

$$\int \mathcal{D}\pi \exp\Big[\frac{i}{\hbar}\int_0^t ds\,\Big\{(\pi(s)+A(q(s)))\cdot\dot{q}(s) - \frac{1}{2}\sum_{i,j=1}^{n}\pi_i(s)g^{ij}(q(s))\pi_j(s)\Big\}\Big] \tag{2.57}$$

This integral factorizes over s; so we may use formula (2.15) to evaluate it: Let $g(q) = \det(g_{ij}(q)) =$

$\det(g^{ij}(q))^{-1}$. Then, by eq. (2.15), (2.57) is given by

$$\mathcal{N}\left[\prod_s \sqrt{g(q(s))}\right] \exp\left[\frac{i}{\hbar}\int_0^t ds \left\{A(q(s))\cdot\dot{q}(s) + \frac{1}{2}\sum_{i,j=1}^n \dot{q}^i(s) g_{ij}(q(s))\dot{q}^j(s)\right\}\right] \tag{2.58}$$

We now set

$$\text{"}\ \mathcal{D}q = \prod_s \sqrt{g(q(s))}\, d^n q(s) \text{"}. \tag{2.59}$$

Note that $\sqrt{g(q)}\, d^n q$ is the Riemannian volume form on M! We also set

$$L(q(s), \dot{q}(s)) := \frac{1}{2}\sum_{i,j=1}^n \dot{q}^i(s) g_{ij}(q(s))\dot{q}^j(s) + \sum_{i=1}^n A_i(q(s))\dot{q}^i(s) - V(q(s)). \tag{2.60}$$

Then (2.56) – (2.60) yield our "master formula":

$$\langle q' | e^{-itH/\hbar} | q\rangle = \mathcal{N} \int_{\substack{q(0)=q \\ q(t)=q'}} \mathcal{D}q \, \exp\Big[\frac{i}{\hbar}\underbrace{\int_0^t ds\, L(q(s), \dot{q}(s))}_{=\text{ action}}\Big], \tag{2.61}$$

as in (2.13), but with $\mathcal{D}q$ the Riemannian volume form on path space $\times_s M_s$.

From here on, one may repeat the yoga from (2.25) through (2.36). We will not present the details.

3. Functional-integral quantization of scalar fields; perturbation theory; spontaneous symmetry breaking and Goldstone's theorem

In this chapter, we extend the path-integral approach to quantum theory developed in the last chapter to scalar quantum field theories. We introduce the notion of "effective actions", and we recall some basic facts about perturbation theory and renormalization theory. We conclude with a discussion of spontaneous symmetry breaking and of Goldstone's theorem.

3.1. Functional-integral quantization of self-interacting scalar field theories.

Prerequisite for this section is Chapt. 5 of QFT I, on Lagrangian field theory. For simplicity, we start by considering theories of a real scalar field, $\varphi(\vec{x},t)$, $(\vec{x},t) \in M^d$, $d = (2,3)\,4$. Generalizations to N-component fields $\vec{\phi} = (\varphi_1, \varphi_2, \cdots, \varphi_N)$, $N \geq 2$, will turn out to be immediate.

Our notations are as follows: M^d denotes d-dimensional Minkowski space-time; $x = (\vec{x}, t) \equiv (x^0, \vec{x})$ are the Cartesian coordinates of a space-time point in M^d; in these coordinates the Lorentz metric on M^d is given by $(\eta_{\mu\nu})$, with $\eta_{00} = 1$, $\eta_{ii} = -1$, $i = 1, \cdots, d-1$, and $\eta_{\mu\nu} = 0$, for $\mu \neq \nu$. We set $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, $(\eta^{\mu\nu} = \eta_{\mu\nu})$.

From Chapt. 5 we recall that a typical Lagrangian density for a theory of a real scalar field has the form

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - U(\varphi), \tag{3.1}$$

where
$$U(\varphi) \overset{e.g.}{=} \frac{\lambda}{4!}\varphi^4 - \frac{\mu^2}{2}\varphi^2, \tag{3.2}$$

$\lambda > 0$, $\mu^2 \in \mathbb{R}$. The "momentum" canonically conjugate to φ is defined by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi(x))} = \partial_0 \varphi(x) \equiv \dot{\varphi}(x) \tag{3.3}$$

The Hamiltonian density is then given by

$$\mathcal{H}(\varphi, \pi) = \left[\pi \dot{\varphi} - \mathcal{L}(\varphi, \partial_\mu \varphi)\right]_{\dot{\varphi} = \pi}$$

$$= \frac{1}{2}\left[\pi^2 + (\vec{\nabla}\varphi)^2\right] + U(\varphi). \qquad (3.4)$$

The classical Hamilton functional is

$$H = \int\limits_{t=\text{const.}} d^{d-1}x \ \mathcal{H}(\varphi(x), \pi(x)) \qquad (3.5)$$

and the action is

$$S = \int dt \int d^{d-1}x \left[\pi(x)\dot{\varphi}(x) - \mathcal{H}(\varphi(x), \pi(x))\right] \quad (3.6)$$

For $U(\varphi) = \frac{m^2}{2}\varphi^2$ $(\lambda = 0,\ \mu^2 = -m^2 < 0)$, $\mathcal{L}, \mathcal{H}$ and S are quadratic in the field quantities. The corresponding classical field theory describes free fields satisfying the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0 \qquad (3.7)$$

The quantization of free field theory in the operator formalism on Fock space has been taught in QFT I. This theory is a canonical quantum field theory, with equal-time canonical commutation relations

$$[\pi(\vec{x},0), \varphi(\vec{y},0)] = -i\,\delta^{(d-1)}(\vec{x}-\vec{y}), \quad (3.8)$$

in units where $\hbar = c = 1$; (Heisenberg commutation relations). One arrives at (3.8) by replacing the Poisson brackets of the classical field quantities by "$i \times$ commutators" of the corresponding field operators. The analogy between mechanics and field theory is as follows:

$$\left.\begin{aligned} q_j(t) &\longleftrightarrow \varphi(\vec{x}, t) \\ p_j(t) &\longleftrightarrow \pi(\vec{x}, t) \end{aligned}\right\} \qquad (3.9)$$

Apparently, field theory is nothing but a mechanics of systems with ∞ many degrees of freedom; $(j = 1, \dots, n \rightarrow \vec{x} \in \mathbb{R}^{d-1})$. Formally, we may introduce generalized eigenvectors, $|\varphi_c\rangle$, of all the commuting operators $\{\varphi(\vec{x}, 0)\}_{\vec{x} \in \mathbb{R}^{d-1}}$,

$$\varphi(\vec{x}, 0)|\varphi_c\rangle = \varphi_c(\vec{x})|\varphi_c\rangle,$$

and generalized eigenvectors, $|\pi_c\rangle$, of the commuting momenta $\{\pi(\vec{x}, 0)\}_{\vec{x} \in \mathbb{R}^{d-1}}$,

$$\pi(\vec{x}, 0)|\pi_c\rangle = \pi_c(\vec{x})|\pi_c\rangle,$$

with
$$\langle \varphi_c | \pi_c \rangle \propto \exp i \int \varphi_c(\vec{x}) \cdot \pi_c(\vec{x})\, d^{d-1}x.$$

The subscript "c" stands for "classical (field configuration)". It will be dropped from now on.

Formally, we can carry over the methods from Chapt. 2 to the present context. For example, the propagator of a scalar field theory is given by the phase-space functional integral

$$\langle \varphi' | e^{-itH} | \varphi \rangle$$

$$= \mathcal{N} \int_{\substack{\varphi(\vec{x},0)=\varphi(\vec{x}) \\ \varphi(\vec{x},t)=\varphi'(\vec{x})}} \mathcal{D}\varphi \, \mathcal{D}\pi \, \exp i \int_0^t ds \int d^{d-1}x \left[\pi(\vec{x},s)\dot{\varphi}(\vec{x},s) - \mathcal{H}(\varphi,\pi)(\vec{x},s) \right], \qquad (3.10)$$

where

$$\text{"} \mathcal{D}\varphi \, \mathcal{D}\pi = \prod_{\vec{x},s} d\varphi(\vec{x},s) \, d\pi(\vec{x},s) \text{"}, \qquad (3.11)$$

$\prod_{\vec{x}} d\varphi(\vec{x},\cdot) \, d\pi(\vec{x},\cdot)$ is the formal Liouville measure on classical phase space, and $\mathcal{N}$ is a (φ- and φ'- independent) divergent normalization factor.

Formula (3.10) is analogous to (2.9). It is mathematically entirely ill-defined! It is based on the prejudice that the Schrödinger picture makes sense in QFT, which, for $d \geq 4$, is certainly not the case.

One could improve the mathematical standing of (3.10) by first replacing space $\mathbb{R}^{d-1}$ by a compact box, Λ, and then approximating Λ by a finite lattice, Λ_ε, of lattice spacing $\varepsilon > 0$. One would then consider the limits $\Lambda \nearrow \mathbb{R}^{d-1}$, $\varepsilon \searrow 0$ after having carried out the path integration; ("lattice approximation to QFT"). This could be the subject of a special course.

At a formal level, eqs. (3.10) and (3.11) represent an excellent starting point for studies of QFT.

For $\mathcal{H}$ as in (3.4), the integrand in (3.10) is a Gaussian in $\pi(\cdot)$. Using formula (2.15), we may formally carry out the $\pi(\cdot)$-integral: We replace s by $s\,e^{-i\theta}$ and $\frac{\partial}{\partial s}$ by $e^{i\theta}\frac{\partial}{\partial s}$, $0 < \theta < \pi$.

Then the exponent on the R.S. of (3.10) is given by

$$i\int_0^t ds \int d^{d-1}x \Big[\pi(\vec{x},s)\dot{\varphi}(\vec{x},s) - \frac{1}{2}e^{-i\theta}\Big\{\pi^2(\vec{x},s) + (\vec{\nabla}\varphi)^2(\vec{x},s) + U(\varphi(\vec{x},s))\Big\}\Big].$$

It has a strictly negative real part (for any constant configuration $\varphi(\cdot)$), provided $0 < \theta < \pi$.

Thus, formally, we can apply (2.15) (with $N \to \infty$) to carry out the $\pi(\cdot)$-integral and find:

$$\langle \varphi' | e^{-itH} | \varphi \rangle$$

$$= \lim_{\theta \searrow 0} Z_{\theta,t}^{-1} \int_{\substack{\varphi(\vec{x},0)=\varphi(\vec{x}) \\ \varphi(\vec{x},t)=\varphi'(\vec{x})}} \mathcal{D}\varphi \, \exp i \int_0^t ds \int d^{d-1}x \, \mathcal{L}_\theta(\varphi(x), \partial_\mu \varphi(x)), \tag{3.12}$$

where

$$\mathcal{L}_\theta(\varphi(x), \partial_\mu \varphi(x)) = \frac{1}{2}\Big[\dot{\varphi}(x)^2 e^{i\theta} - (\vec{\nabla}\varphi)^2(x) e^{-i\theta} - U(\varphi(x)) e^{-i\theta}\Big], \tag{3.13}$$

and Z_θ is a φ- and φ'- independent normalization factor. Expressions (3.12), (3.13) represent "Feynman's path integral" over field-configuration space.

Next, we formally repeat the arguments between (2.28) and (2.34). This yields expressions for the time-ordered vacuum expectation values of field operators of the quantum field theory. Let Ω denote the groundstate, or vacuum, of the quantum field theory with Lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi)$ as in (3.1). Then we formally find that

$$\langle\Omega|T[\varphi(\vec{x}_1, t_1 e^{-i\theta}) \cdots \varphi(\vec{x}_n, t_n e^{-i\theta})]|\Omega\rangle$$

$$= \lim_{T\to\infty} Z_{\theta,T}^{-1} \int \mathcal{D}\varphi \;\; \varphi(\vec{x}_1, t_1) \cdots \varphi(\vec{x}_n, t_n) \times$$

$$\times \exp i \int_{-T}^{T} dt \int d^{d-1}x \; \mathcal{L}_\theta(\varphi(x), \partial_\mu \varphi(x)) \qquad (3.14)$$

$$= Z_\theta^{-1} \int \mathcal{D}\varphi \; \varphi(\vec{x}_1, t_1) \cdots \varphi(\vec{x}_n, t_n) \exp i \int d^d x \; \mathcal{L}_\theta(\cdots),$$

where time ordering is defined by

$$T[\varphi(\cdot, t_1 e^{-i\theta}) \cdots \varphi(\cdot, t_n e^{-i\theta})]$$

$$= \varphi(\cdot, t_{\pi 1} e^{-i\theta}) \cdots \varphi(\cdot, t_{\pi n} e^{-i\theta}), \qquad (3.15)$$

with $t_{\pi(j-1)} > t_{\pi j}$, for all j; $0 < \theta < \pi$,

and $\mathcal{L}_\theta$ is as in (3.13); finally

$$\text{"}\; Z_{\theta,T} = \int \mathcal{D}\varphi \exp i \int_{-T}^{T} dt \int d^3x \; \mathcal{L}_\theta(\varphi(x), \partial_\mu \varphi(x)). \text{"}$$

The Green functions of direct physical interest are obtained as the limits of those in (3.14), as $\theta \searrow 0$. It is also of interest to consider the limit $\theta \nearrow \frac{\pi}{2}$, ("<u>Wick rotation</u>"). Then

$$S_n(\vec{x}_1, t_1, \ldots, \vec{x}_n, t_n) :=$$

$$\langle\Omega|T[\varphi(\vec{x}_1, -it_1) \cdots \varphi(\vec{x}_n, -it_n)]|\Omega\rangle$$

$$= Z^{-1} \int \varphi(\vec{x}_1, t_1) \cdots \varphi(\vec{x}_n, t_n) \times$$

$$\times \exp - \int d^d x \left[\frac{1}{2} (\nabla\varphi)(x) \cdot (\nabla\varphi)(x) + U(\varphi(x)) \right] \mathcal{D}\varphi, \quad (3.16)$$

where ∇ is the d-dimensional gradient. Note that, formally,

$$Z^{-1} \exp - \int d^d x \left[\frac{1}{2} (\nabla\varphi)(x) \cdot (\nabla\varphi)(x) + U(\varphi(x)) \right] \mathcal{D}\varphi \quad (3.17)$$

is a <u>positive</u> <u>probability</u> <u>measure</u> on function space. In the special case where $U(\varphi) = \frac{m^2}{2} \varphi^2$ is quadratic in φ, with $m^2 \geq 0$, the measure (3.17) has a precise mathematical meaning as a <u>Gaussian measure</u> on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions with <u>mean</u> 0 and <u>covariance</u> $(-\Delta + m^2)^{-1}$ (with $m^2 > 0$, for $d = 1, 2$), where $\Delta = \nabla \cdot \nabla$ is the d-dimensional Laplacian. It describes the free scalar field of mass m at imaginary times.

The distributions $S_n(x_1, \ldots, x_n)$, $x_i = (\vec{x}_i, t_i)$, $i = 1, \ldots, n$, are called <u>Euclidian Green-</u> or <u>Schwinger functions</u>. The existence of an analytic continuation of Green functions in the time variables, $t_i \to t_i e^{-i\theta}$, from the real $(\theta = 0)$ to the imaginary axis $(\theta = \frac{\pi}{2})$, and back, has been proven (essentially) by Osterwalder and Schrader to be a consequence of (a suitably strong form of) the axioms of local relativistic QFT; (see QFT I).

Expression (3.16) shows that the Schwinger functions are symmetric in their arguments, which is related to the fact that the field quanta of the theory obey Bose statistics.

Note that, for $U(\varphi) = \frac{m^2}{2}\varphi^2$,

$$
\begin{aligned}
S_2(x,y) &\underset{t>s}{=} \langle 0 \,|\, \varphi(\vec{x}, -it)\, \varphi(\vec{y}, -is) \,|\, 0 \rangle \\
&= \left(-\Delta + m^2\right)^{-1}(x,y) \\
&= \int \frac{d^d k}{(2\pi)^d} \, \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \\
&= \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \, \frac{e^{-\omega(\vec{k})|t-s|}}{2\omega(\vec{k})} \, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \qquad (3.18)
\end{aligned}
$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ is the energy of a relativistic particle of mass m with momentum $\vec{k}$, and $|0\rangle$ is the vacuum of the free field theory.

Let $P(\varphi)$ and $Q(\varphi)$ be (polynomial) functions of the time-0-field operators $\{\varphi(\vec{x},0)\}_{\vec{x}\in\mathbb{R}^{d-1}}$.

Then we have the identity

$$
\begin{aligned}
&\langle\Omega|\,[P(\varphi),Q(\varphi)]\,|\Omega\rangle \\
&\quad = \lim_{\varepsilon\searrow 0}\langle\Omega|\{P(\varphi)e^{-\varepsilon H}Q(\varphi)-Q(\varphi)e^{-\varepsilon H}P(\varphi)\}|\Omega\rangle \\
&\quad = \lim_{\varepsilon\searrow 0} Z^{-1}\int\{P(\varphi(\cdot,\varepsilon))-P(\varphi(\cdot,-\varepsilon))\}\,Q(\varphi(\cdot,0))\,\times \\
&\qquad \times\exp-\int d^dx\left[\tfrac{1}{2}(\nabla\varphi)(x)\cdot(\nabla\varphi)(x)+U(\varphi(x))\right]\mathcal{D}\varphi,
\end{aligned}
\tag{3.19}
$$

where H is the Hamiltonian of the theory, with $H\Omega=0$. The (formal) proof is similar to that of (3.16). (Identity (3.19) – extended to N-component scalar fields, with $N>2$ – will be useful for a proof of Goldstone's theorem.)

In this course, we will not be interested in the precise mathematical meaning of formulae like (3.14) and (3.16), but use them as a starting point for a <u>perturbative analysis:</u> We expand the R.S. of (3.14), (3.16) in a power series in powers of $U(\varphi)$. The resulting terms can be calculated quite explicitly by carrying out the appropriate Gaussian integrals.

Actually, for a Lagrangian density $\mathcal{L}$ as in eqs. (3.1), (3.2), it is more convenient to rescale the fields

$$\varphi =: \frac{\phi}{\sqrt{\lambda}} \tag{3.20}$$

Then

$$\mathcal{L}_\theta(\varphi(x), \partial_\mu \varphi(x)) = \lambda^{-1}\Big[\frac{1}{2}\dot{\phi}(x)^2 e^{i\theta} - \frac{1}{2}(\vec{\nabla}\phi)^2(x) e^{-i\theta}$$
$$- \frac{1}{4!}\phi(x)^4 e^{-i\theta} + \frac{\mu^2}{2}\phi(x)^2 e^{-i\theta}\Big].$$

An action functional is then defined as

$$S_\theta(\phi) := \lambda \int d^d x \, \mathcal{L}_\theta\left(\frac{\phi(x)}{\sqrt{\lambda}}, \partial_\mu \frac{\phi(x)}{\sqrt{\lambda}}\right). \tag{3.21}$$

Green functions can be calculated by functional integration with the measure

$$Z_\theta^{-1} e^{iS_\theta(\phi)/\lambda} \mathcal{D}\phi. \tag{3.22}$$

Note that the coupling constant λ plays the rôle of Planck's constant $\hbar$. The limit $\lambda \searrow 0$ thus corresponds to the <u>classical limit</u>. An expansion of Green functions in powers of λ around $\lambda = 0$

yields the <u>loop expansion</u>: The contribution in order λ^n consists of summing all Feynman amplitudes corresponding to Feynman diagrams with precisely n <u>loops</u>, as the reader may recall from QFT I.

To understand these matters more precisely, we first consider a <u>free field theory</u>, with

$$U(\varphi) = \frac{m^2}{2}\varphi^2, \quad m^2 \geq 0. \tag{3.23}$$

Our first result is

<u>Wick's Theorem.</u> For U as in (3.23), we have that

$$\langle 0 | T[\varphi(\vec{x}_1, t_1 e^{-i\theta}) \cdots \varphi(\vec{x}_n, t_n e^{-i\theta})] | 0 \rangle$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{1}{2^k k!} \displaystyle\sum_{\pi \in S_{2k}} \prod_{j=1}^{k} \Delta_F^\theta (x_{\pi(2j-1)} - x_{\pi(2j)}), & n = 2k, \end{cases} \tag{3.24}$$

$k = 0, 1, 2, \cdots$, where S_{2k} is the group of permutations of $2k$ elements, $|0\rangle$ is the vacuum of the free theory, and

$$\Delta_F^\theta(x) = -i \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{(k_0)^2 e^{i\theta} - (\vec{k}^2 + m^2) e^{-i\theta}} \tag{3.25}$$

is the Feynman propagator.

Wick's theorem, eq. (3.24), follows from (3.14) and (3.23) by explicitly carrying out the Gaussian functional integral; (exercise!).

From (3.25) we find that

$$\lim_{\theta \searrow 0} \Delta_F^\theta(x) = \lim_{\theta \searrow 0} -i e^{i\theta} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{-i\vec{k}\cdot\vec{x}} \int \frac{dk_0}{2\pi} \frac{e^{ik_0 t}}{(k_0 e^{i\theta}+\omega)(k_0 e^{i\theta}-\omega)}$$

$$= \lim_{\varepsilon \searrow 0} -i \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{k^2 - m^2 + i\varepsilon}, \tag{3.26}$$

with $\omega = \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$; see (2.43).

It is useful to consider the <u>generating functional</u> of time-ordered Green functions of free fields, with $U(\varphi)$ as in (3.23):

$$e^{iW_0^\theta(J)} := Z_\theta^{-1} \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\langle \varphi, A^\theta \varphi\rangle + i\langle J, \varphi\rangle}, \tag{3.27}$$

where $\langle \varphi, \psi \rangle := \int d^d x \, \varphi(x)\psi(x)$, and

$$(A^\theta \varphi)(x) := -i\left(e^{i\theta}\frac{\partial^2}{\partial t^2} - e^{-i\theta}(\Delta_{d-1} - m^2)\right)\varphi(x) \tag{3.28}$$

Note that

$$(A^\theta)^{-1}(x,y) = -\Delta_F^\theta(x-y). \tag{3.29}$$

In (3.27)

$$Z_\theta := \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\langle \varphi, A^\theta \varphi \rangle} = \text{const.} \det(A^\theta)^{-1/2},$$

where const. is a (divergent) constant, and $\det(A^\theta)$ requires a proper definition (e.g. via Sealey's ζ-function method); see (2.15). By (2.15), (2.16), the R.S. of (3.27) is found to be

$$e^{iW_0^\theta(J)} = \exp\left[-\frac{1}{2}\langle J, (A^\theta)^{-1} J\rangle\right]$$

$$= \exp\left[\frac{1}{2}\langle J, \Delta_F^\theta J\rangle\right] \qquad (3.30)$$

Wick's theorem follows by noting that (3.27) implies that

$$\langle 0 | T[\varphi(\vec{x}_1, t_1 e^{-i\theta}) \cdots \varphi(\vec{x}_n, t_n e^{-i\theta})] | 0 \rangle$$

$$= \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} e^{iW_0^\theta(J)} \Big|_{J \equiv 0} \qquad (3.31)$$

and using (3.30) on the R.S. of (3.31).

Next, we attempt to evaluate the time-ordered Green functions of an interacting scalar QFT by starting from (3.14) and doing perturbation theory in $U(\varphi)$, using Wick's theorem and the linked cluster theorem;

see QFT I!

3.2. Perturbation theory in $\mathcal{L}_I$.

In the following, we do not explicitly display the parameter θ. We will be interested in the following two regimes:

(i) real-time Green functions: $\theta \searrow 0$;

(ii) Euclidian Green (Schwinger) functions: $\theta = \frac{\pi}{2}$ (Wick rotation to purely imaginary times).

The vector $|0\rangle$ denotes the Fock vacuum of a free field theory with Lagrangian density

$$\mathcal{L}_0(\varphi, \partial_\mu \varphi) = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2, \tag{3.32}$$

where $m \geqslant 0$ is the mass of the free field. Furthermore, $\mathcal{L}_I(\varphi)$ is the <u>interaction Lagrangian</u>; e.g.

$$\mathcal{L}_I(\varphi) = \frac{\lambda}{4!} \varphi^4 - \frac{m^2 + \mu^2}{2} \varphi^2, \tag{3.33}$$

and

$$\mathcal{L} := \mathcal{L}_0 + \mathcal{L}_I . \tag{3.34}$$

By $|\Omega\rangle$ we denote the (formal) vacuum vector of the interacting theory. We use the notations

$$\langle (\cdot) \rangle_0 := Z_0^{-1} \int \mathcal{D}\varphi \, (\cdot) \, e^{i\int \mathcal{L}_0 \, d^dx} , \tag{3.35}$$

an expectation value in a Gaussian "measure", and

$$\langle\langle (\cdot) \rangle\rangle := Z^{-1} \int \mathcal{D}\varphi \, (\cdot) \, e^{i\int \mathcal{L} \, d^dx} \tag{3.36}$$

Formally, we have that

$$\begin{aligned} &\langle \Omega | T[\varphi(x_1) \cdots \varphi(x_n)] | \Omega \rangle \\ &= \langle\langle \varphi(x_1) \cdots \varphi(x_n) \rangle\rangle, \quad \text{by (3.14) and (3.36)} \\ &= \frac{\langle \varphi(x_1) \cdots \varphi(x_n) \exp i \int \mathcal{L}_I(\varphi(y)) \, d^dy \rangle_0}{\langle \exp i \int \mathcal{L}_I(\varphi(y)) \, d^dy \rangle_0} , \end{aligned} \tag{3.37}$$

by (3.34) – (3.36). We may now expand numerator and denominator on the R.S. of (3.37) in powers of $\mathcal{L}_I$:

$$\begin{aligned} &\langle \varphi(x_1) \cdots \varphi(x_n) \exp i \int \mathcal{L}_I(\varphi(y)) \, d^dy \rangle_0 \\ &= \sum_{k=0}^{\infty} \frac{i^k}{k!} \int \cdots \int \prod_{j=1}^{k} d^dy_j \, \langle \varphi(x_1) \cdots \varphi(x_n) \, \times \\ &\qquad \times \mathcal{L}_I(\varphi(y_1)) \cdots \mathcal{L}_I(\varphi(y_k)) \rangle_0 ; \end{aligned} \tag{3.38}$$

the expansion of the denominator being obtained by setting $n = 0$ in (3.38).

By (3.31), (3.30) and (3.27),

$$\langle \varphi(x_1)\cdots\varphi(x_n)\,\mathcal{L}_I(\varphi(y_1))\cdots\mathcal{L}_I(\varphi(y_k))\rangle_0$$

$$= \langle 0|T[\varphi(x_1)\cdots\varphi(x_n)\mathcal{L}_I(\varphi(y_1))\cdots\mathcal{L}_I(\varphi(y_k))]|0\rangle \tag{3.39}$$

This expression is ill-defined when we attempt to evaluate it by applying Wick's theorem, eq. (3.24), <u>even if all arguments</u> $x_1,\dots,x_n,y_1,\dots,y_k$ <u>are different from each other.</u> This is because we have not Wick-ordered the interaction Lagrangians $\mathcal{L}_I(\varphi(y_j))$. Of course, this can be done! It suffices to know how to Wick-order an exponential,

$$e^{i\langle J,\varphi\rangle} = e^{i\int J(x)\varphi(x)\,d^dx}$$

of the free field φ: We define the <u>Wick-ordered exponential</u> of φ by

$$:e^{i\langle J,\varphi\rangle}: = \frac{e^{i\langle J,\varphi\rangle}}{\langle e^{i\langle J,\varphi\rangle}\rangle_0} = e^{i\langle J,\varphi\rangle}\,e^{-\frac{1}{2}\langle J,\Delta_F J\rangle}, \tag{3.40}$$

see (3.30). Then

$$:\varphi(y_1)\cdots\varphi(y_\ell): = \frac{\delta^\ell}{\delta J(y_1)\cdots\delta J(y_\ell)} :e^{i\langle J,\varphi\rangle}:\Big|_{J\equiv 0} \tag{3.41}$$

If we Wick order the interaction Lagrangians, $\mathcal{L}_I(\varphi(y_j)) \to :\mathcal{L}_I(\varphi(y_j)):$, in (3.39) then expression (3.39) is well-defined as long as all the $n+k$ arguments, $x_1, \dots, x_n, y_1, \dots, y_k$, are different from each other. But they are in general ($d \geq 3$) <u>not</u> tempered distributions and hence <u>cannot</u> be integrated over $y_1, \dots, y_k$, due to non-integrable short-distance singularities $\left((y_i - y_j)^2 \to 0\right)$. These are a manifestation of the <u>ultraviolet divergences</u> we have been fighting with in QFT I; ($\nearrow$ <u>renormalization theory</u>).

From now on, we may interpret $\mathcal{L}_I(\varphi(y))$ as the Wick-ordered interaction Lagrangian – if we like – and we continue with formal calculations: We plug (3.39) into (3.38) and find

$$\left\langle \varphi(x_1) \cdots \varphi(x_n) \exp i \int \mathcal{L}_I(\varphi(y)) d^d y \right\rangle_0$$

$$= \sum_{k=0}^{\infty} \frac{i^k}{k!} \int \cdots \int \prod_{j=1}^{k} d^d y_j \, \langle 0 | T[\varphi(x_1) \cdots \varphi(x_n) \times$$

$$\times \mathcal{L}_I(\varphi(y_1)) \cdots \mathcal{L}_I(\varphi(y_k))] | 0 \rangle$$

$$= \langle 0 | T[\varphi(x_1)\cdots\varphi(x_n) \exp i \int \mathcal{L}_I(\varphi(y))\, d^d y] | 0 \rangle .$$

Thus, with (3.37),

$$\langle \Omega | T[\varphi(x_1)\cdots\varphi(x_n)] | \Omega \rangle$$

$$= \frac{\langle 0 | T[\varphi(x_1)\cdots\varphi(x_n) e^{i\int \mathcal{L}_I(\varphi(y))\, d^d y}] | 0 \rangle}{\langle 0 | e^{i\int \mathcal{L}_I(\varphi(y))\, d^d y} | 0 \rangle} , \qquad (3.42)$$

which is the famous Gell-Mann-Low formula. For the time being, it only makes sense if we regularize the theory at short distances:

$$\mathcal{L}_I(\varphi(y)) \longrightarrow \mathcal{L}_I(\varphi_\kappa(y)), \qquad (3.43)$$

where $\varphi_\kappa(y) := \int d^d z \, \delta_\kappa(y-z)\, \varphi(z)$,

and $\delta_\kappa > 0$ is a smooth approximation to the δ-function, with $\delta_\kappa \to \delta$, as $\kappa \to \infty$. In this case, we can appeal to the linked cluster theorem to make sense of the R.S. of (3.39).

Using (3.30) and (3.31), we can rewrite (3.39) as

$$\langle \varphi(x_1)\cdots\varphi(x_n)\,\mathcal{L}_I(\varphi(y_1))\cdots\mathcal{L}_I(\varphi(y_k))\rangle_0$$

$$= (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(y_1)}\right)\cdots\mathcal{L}_I\left(-i\frac{\delta}{\delta J(y_k)}\right)e^{iW_0(J)}, \tag{3.44}$$

with $W_0(J)$ as in (3.30). We define

$$e^{i\tilde{W}(J)} := \exp i\int d^d y\,\mathcal{L}_I\left(-i\frac{\delta}{\delta J(y)}\right)e^{iW_0(J)} \tag{3.45}$$

and $W(J) := \tilde{W}(J) - \tilde{W}(0)$. Then, by (3.44), (3.45) and (3.39), (3.42), we have that

$$\langle\Omega|\,T[\varphi(x_1)\cdots\varphi(x_n)]\,|\Omega\rangle$$

$$= (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\, e^{iW(J)}\Big|_{J\equiv 0} \tag{3.46}$$

Expressions (3.45) and (3.46) are a convenient starting point for perturbation theory.

<u>Theorem.</u> $W(J)$ is the generating function for the <u>connected</u> time-ordered Green functions of the theory. In perturbation theory, only Feynman amplitudes corresponding to <u>connected</u> Feynman graphs contribute to <u>connected</u> Green functions. (See QFT I!)

3.3. A lightening review of renormalization theory

In this section, we choose the space-time dimension d to be $= 4$. Furthermore, for fun, we study Euclidian field theory obtained by setting the parameter $\theta = \frac{\pi}{2}$.

The Euclidian propagator is given by

$$S_2(x,y) = \left(-\Delta + m^2\right)^{-1}(x,y) = -\Delta_F^{\theta=\frac{\pi}{2}}(x-y)$$

$$\propto |x-y|^{-2}, \quad \text{as} \quad x \to y. \tag{3.47}$$

In any non-trivial QFT in $d=4$ dimensions, divergent integrals appear in the evaluation of Feynman amplitudes contributing to Green functions, due to the short-distance singularities in (3.47). Renormalization is a procedure to remove these divergences (order by order in perturbation theory) by adding counter terms to the Lagrangian of the theory; (see QFT I). In a renormalizable QFT, the addition of counterterms must amount to a redefinition of the parameters (field strength, masses and coupling constants) appearing in the Lagrangian.

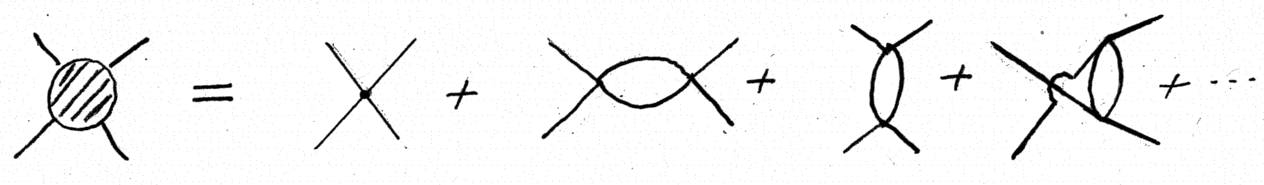
As an example, we consider φ^4 – theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - V(\varphi),$$

with

$$V(\varphi) = \frac{\lambda}{4!}\varphi^4 - \frac{m^2+\mu^2}{2}\varphi^2 \qquad (3.48)$$

The first few terms in the expansion of the connected 4–point Green function correspond to the graphs



Cutting off momentum-space integrals at some large momentum Λ, or, equivalently, x-space integrals at a distance $\frac{1}{\Lambda}$, we find, after amputation of external lines (see QFT I):

$$\Gamma^{(4)}_\Lambda = -\lambda + a\lambda^2 \ln \Lambda + b\lambda^2 + \cdots, \qquad (3.49)$$

where a is Λ-independent constant, and b is a well-defined function of the external momenta in $\Gamma^{(4)}_\Lambda$. Formula (3.49) suggests to change the Lagrangian of the theory to

$$\mathcal{L} \to \mathcal{L} - a \ln \Lambda \frac{\lambda^2}{4!} \varphi^4, \qquad (3.50)$$

(renormalization of coupling constant to order λ^2).

The counterterm in (3.50) cancels the divergencies in (3.49) to order λ^2.

One may formulate a general rule for how to choose the counterterms in any order of perturbation theory.

First we note that

$$S_2(\beta x, \beta y) \sim \frac{1}{\beta^2 |x-y|}, \quad \text{as } \beta \to 0.$$

Thus, setting $\varphi_\beta(x) = \beta \varphi(\beta x)$, we have that

$$\langle 0 | \varphi_\beta(x) \varphi_\beta(y) | 0 \rangle = \beta^2 S_2(\beta x, \beta y) \to \frac{1}{|x-y|^2},$$

as $\beta \to 0$. We thus say that the (ultraviolet) dimension of a free scalar field is 1, namely the exponent of β in the definition of φ_β.

The Lagrangian density $\mathcal{L}$ of a field theory can be written as

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \mathcal{L}_i \equiv \mathcal{L}_0 + \mathcal{L}_I, \qquad (3.50)$$

with $\mathcal{L}_0$ as in (3.32), where every $\mathcal{L}_i$ is a monomial in φ and derivatives of φ. The dimension of $\mathcal{L}_i$,

$[\mathcal{L}_i]$, is defined to be the order of $\mathcal{L}_i$ in φ and in first-derivative operators. Thus, for example,

$$[\varphi^4] = 4 \ , \ [\varphi^2] = 2 \ , \ [(\nabla\varphi)^2] = 4 .$$

We imagine that we calculate a contribution to a connected Green function, such as $\Gamma^{(4)}$, corresponding to a Feynman graph with n_i vertices of type $\mathcal{L}_i$. Then, to arbitrary order in perturbation theory, all divergences in this contribution can be cancelled by counterterms, which are local polynomials in φ and derivatives of φ, whose dimension, d_{ct}, satisfies the bound

$$d_{ct} - 4 \leq \sum n_i ([\mathcal{L}_i] - 4) \qquad (3.51)$$

Of course only counterterms satisfying (3.51) that are compatible with the symmetries of the original field theory will appear, provided our cutoff procedure respects these symmetries; (use, e.g., dimensional regularization!).

Let us check (3.51) in the example of φ^4-theory.

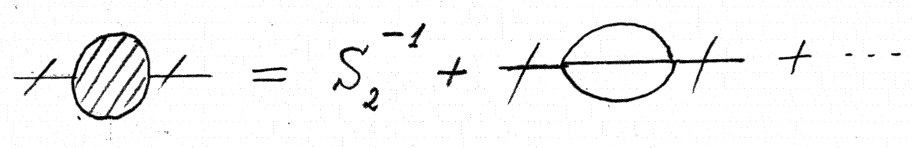
In this example, $\mathcal{L}_1 \propto \varphi^4$, hence $[\mathcal{L}_1] = 4$, and $\mathcal{L}_2 \propto \varphi^2$, i.e., $[\mathcal{L}_2] = 2$. Then the R. S. of (3.51) is ≤ 0. Hence, only counterterms of dimension $d_{ct} \leq 4$ are needed to render the theory finite. Preserving the symmetries of the theory, only <u>three</u> types of counterterms may appear:

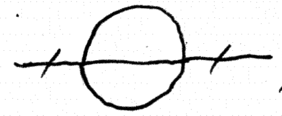
$$\mathcal{L} \rightarrow \mathcal{L} + \frac{A}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{B}{2}\varphi^2 - \frac{C}{4!}\varphi^4, \quad (3.52)$$

where A, B and C are cutoff (Λ-) dependent constants, (expected to diverge, as $\Lambda \to \infty$). The calculation (3.49) gave

$$C = a\lambda^2 \ln \Lambda + O(\lambda^3),$$

while A and B remained undetermined. The constants A and B can be calculated by considering the 2-point function, $\Gamma^{(2)}$,



Then the counterterms $\propto A, B$ cancel the infinities in the Feynman amplitudes corresponding to the graphs , ... Note that the counterterms in

(3.52), which render the theory finite to order λ^2, all have dimension $d_{ct} \leq 4$. Incorporating them as new vertices of the theory leads to the conclusion that the R.S. of (3.51) is still ≤ 0. Thus, at the next order in λ, and hence at any order in λ, all counterterms needed to render the theory finite are still of the form (3.52). These counterterms are of the form of the three terms in the original Lagrangian (3.48) and hence can be interpreted as readjustments of the parameters in the original Lagrangian:

$A \leftrightarrow$ rescaling of φ, i.e., field-strength renormalization

$B \leftrightarrow$ mass renormalization

$\lambda \leftrightarrow$ coupling constant renormalization.

We conclude that, apparently, φ^4-theory is a renormalizable QFT (see QFT I), while φ^6-theory is non-renormalizable in $d=4$, but renormalizable in $d=3$.

The finite ambiguities in choosing A, B and C can be eliminated by imposing suitable renormalization conditions on the 2-point ($\Gamma^{(2)}$) and 4-point ($\Gamma^{(4)}$) Green functions.

3.4. The effective action of a QFT

Recall the definition of the generating function, $W(J)$, of connected Green functions:

$$W(J) = \tilde{W}(J) - \tilde{W}(0), \tag{3.53}$$

where

$$e^{i\tilde{W}(J)} = \int \mathcal{D}\varphi \exp i \int d^dx \left[\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) + J(x)\varphi(x)\right]$$

$$= \exp i \int d^dy\, \mathcal{L}_I\left(-i\frac{\delta}{\delta J(y)}\right) e^{iW_0(J)}, \tag{3.54}$$

where $W_0(J) = -\frac{i}{2}\langle J, \Delta_F J\rangle$. In the Euclidian region $(\theta = \frac{\pi}{2})$, we drop all the "i's" and set

$$e^{\tilde{W}(J)} = \exp \int d^dy\, \mathcal{L}_I\left(\frac{\delta}{\delta J(y)}\right) e^{-W_0(J)} \tag{3.55}$$

with $W_0(J) = \frac{1}{2}\langle J, (-\Delta_d + m^2)^{-1} J\rangle$; (signs are an invention of the devil and depend on definitions and conventions; I cannot guarantee that I choose them correctly — but eq. (3.55) is perfect!).

Eq. (3.55) is equivalent to

$$e^{W(J)} = Z^{-1}\int \mathcal{D}\varphi \exp\Big[-\frac{1}{2}\langle \varphi, (-\Delta_d + m^2)\varphi\rangle - \int d^d x \,\{U(\varphi(x)) + J(x)\varphi(x)\}\Big], \quad (3.56)$$

with

$$Z = \int \mathcal{D}\varphi \exp\Big[-\frac{1}{2}\langle \varphi, (-\Delta_d + m^2)\varphi\rangle - \int d^d x \; U(\varphi(x))\Big].$$

Note that $W(J)$ is <u>convex</u> as a functional of J; (exercise).

Let $\varphi_c(x)$ be an arbitrary classical field configuration. We define the <u>effective action</u> of the theory by

$$\Gamma(\varphi_c) := \sup_J \{\langle J, \varphi_c\rangle - W(J)\} \qquad (3.57)$$

$$= -\inf_J \{W(J) - \langle J, \varphi_c\rangle\}.$$

Eq. (3.57) defines the convex <u>Legendre transform</u> of $W(J)$. Since W is convex, Γ is a <u>convex functional</u> of φ_c (exercise!); see TdW. If, in (3.57), the "sup" is a maximum then

$$\varphi_c(x) = \frac{\delta W(J)}{\delta J(x)} \qquad (3.58)$$

Using the inverse Legendre transform,

$$W(J) = \sup_{\varphi_c} \{ \langle \varphi_c, J \rangle - \Gamma(\varphi_c) \},$$

and assuming that the "sup" is a "max", we also have that

$$J(x) = \frac{\delta \Gamma(\varphi_c)}{\delta \varphi_c(x)} \tag{3.59}$$

Eq. (3.58) is an equation for J, given φ_c; while (3.59) is an equation for φ_c, given J. By (3.55), we have that

$$\frac{\delta W(J)}{\delta J(x)} e^{W(J)} = \frac{\delta}{\delta J(x)} e^{W(J)}$$

$$= \text{const.} \exp \int d^d y \, \mathcal{L}_I\left(\frac{\delta}{\delta J(y)}\right) \frac{\delta W_0(J)}{\delta J(x)} e^{W_0(J)}$$

$$= \exp L \left(-\Delta_d + m^2\right)^{-1} J \exp(-L) \, e^{W(J)}, \tag{3.60}$$

where $L := \int d^d y \, \mathcal{L}_I\left(\frac{\delta}{\delta J(y)}\right)$. Hence

$$\left[\left(-\Delta_d + m^2\right) \frac{\delta W(J)}{\delta J(x)}\right] e^{W(J)} = \left(e^L J(x) e^{-L}\right) e^{W(J)}.$$

Next,

$$e^L J(x) e^{-L} = J(x) + [L, J(x)] + \frac{1}{2}[L,[L,J(x)]] + \cdots$$

From the definition of L we arrive at

$$[L, J(x)] = \mathcal{L}_I'\left(\frac{\delta}{\delta J(x)}\right),$$

which is a differential operator in J and hence commutes with J, so that $[L, \cdots, [L, J]] = 0$. Thus

$$\left[(-\Delta+m^2)\frac{\delta W(J)}{\delta J(x)}\right]e^{W(J)} = \left[J(x) + \mathcal{L}_I'\left(\frac{\delta}{\delta J(x)}\right)\right]e^{W(J)},$$

or, with (3.58),

$$(-\Delta+m^2)\varphi_c(x) = J(x) + e^{-W(J)}\mathcal{L}_I'\left(\frac{\delta}{\delta J(x)}\right)e^{W(J)},$$

which yields

$$\begin{aligned}(-\Delta+m^2)\varphi_c(x) - \mathcal{L}_I'(\varphi_c(x)) - J(x) &= \text{terms involving second or higher derivatives of } W(J) \text{ w.r. to } J \qquad (3.61)\\ &= \text{"quantum corrections".}\end{aligned}$$

For $\mathcal{L}$ as in (3.48), we find that

$$\begin{aligned}(-\Delta+m^2)\varphi_c(x) - \mathcal{L}_I'(\varphi_c(x)) - J(x)\\ = -\frac{\lambda}{3!}\frac{\delta^2\varphi_c}{\delta J(x)^2} - \frac{\lambda}{4}\frac{\delta\varphi_c^2}{\delta J(x)} + (m^2+\mu^2)\frac{\delta\varphi_c}{\delta J(x)} \qquad (3.62)\end{aligned}$$

<u>Theorem.</u> The effective action, $\Gamma(\varphi_c)$, is the generating function of the <u>"vertex functions"</u>, which are given by the sum of all <u>one-particle irreducible Feynman amplitudes</u> with external lines amputated:

$$\Gamma(\varphi_c) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \Gamma^{(n)}(x_1, \dots, x_n) \prod_{j=1}^{n} \varphi(x_j)\, d^d x_j, \quad (3.63)$$

where $\Gamma^{(n)}(x_1, \dots, x_n)$ is the proper vertex n-point function in position space with external lines amputated.

The physical meaning of the effective potential

By eq. (3.56),

$$\varphi_c(x) := \frac{\delta W(J)}{\delta J(x)} = \langle \varphi(x) \rangle_J = \langle \Omega_J | \varphi(x) | \Omega_J \rangle, \quad (3.64)$$

where the subscript J indicates that we are considering a theory with a Lagrangian density

$$\mathcal{L}(\varphi, \partial_\mu \varphi) \to \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) + J(x)\varphi(x).$$

Its Hamiltonian is time-dependent,

$$H_t = H + \int d^{d-1}x\, J(\vec{x}, t)\, \varphi(\vec{x}).$$

Differentiating (3.64) with respect to $J(y)$, we find that

$$\frac{\delta \varphi_c(x)}{\delta J(y)} = \frac{\delta \langle \varphi(x) \rangle_J}{\delta J(y)} = \langle \varphi(x) \varphi(y) \rangle_J - \langle \varphi(x) \rangle_J \langle \varphi(y) \rangle_J,$$

etc. Setting $J(x) \equiv 0$, at the end of our calculations, we find, using (3.59), that

$$0 = \frac{\partial \Gamma(\varphi_c)}{\partial \varphi_c(x)}, \text{ for } J \to 0, \quad (3.65)$$

i.e., φ_c must be a critical point of $\Gamma(\varphi_c)$ when $J \to 0$. This fact will play an important rôle in our analysis of spontaneous symmetry breaking.

In order to understand the meaning of $\Gamma(\varphi_c)$, we consider the special case where

$$J(x) = J_0(\vec{x})\, \chi_{[0,T]}(t). \tag{3.66}$$

Returning to formula (3.12), but with t replaced by $-it$ $\left(\theta = \frac{\pi}{2}\right)$, and combining it with (3.56), we find that, for J as in (3.66),

$$e^{W(J)} = Z^{-1} \int \mathcal{D}\varphi \, \exp\Big[-\frac{1}{2}\langle \varphi, (-\Delta + m^2)\varphi \rangle - \int d^d x \, U(\varphi(x))$$
$$+ \int_0^T dt \int d^{d-1}x \, J_0(\vec{x})\varphi(\vec{x},t)\Big]$$

$$= \langle \Omega | e^{-TH_{J_0}} | \Omega \rangle, \tag{3.67}$$

where

$$H_{J_0} = H - \int d^{d-1}x \, J_0(\vec{x})\varphi(\vec{x}), \tag{3.68}$$

and H is the Hamiltonian of the theory with $J \equiv 0$. In (3.67), $|\Omega\rangle$ is the vacuum of the theory with $J \equiv 0$; i.e., $H|\Omega\rangle = 0$.

(By Perron – Frobenius,) the groundstate, $|\Omega_{J_0}\rangle$, of H_{J_0} has a non-vanishing overlap with the vacuum $|\Omega\rangle$; i.e.,

$$\left|\langle \Omega_{J_0} | \Omega \rangle\right| > 0.$$

Hence

$$E(J_0) = \lim_{T\to\infty} -\frac{1}{T} \ln \langle \Omega | e^{-TH_{J_0}} | \Omega \rangle$$

$$= \lim_{T\to\infty} -\frac{1}{T} W\left(J = J_0 \otimes \chi_{[0,T]}\right), \qquad (3.69)$$

where $E(J_0)$ is the groundstate energy of H_{J_0}.

The Feynman – Hellman theorem (1st order perturbation theory) tells us that

$$\frac{\delta E(J_0)}{\delta J_0(\vec{x})} = -\langle \Omega_{J_0} | \varphi(\vec{x}) | \Omega_{J_0} \rangle =: -\varphi_c(\vec{x}, 0) \qquad (3.70)$$

Hence

$$\langle \Omega_{J_0} | H | \Omega_{J_0} \rangle \overset{(3.68)}{=} \langle \Omega_{J_0} | H_{J_0} | \Omega_{J_0} \rangle + \int d^{d-1}x \, J_0(\vec{x}) \langle \Omega_{J_0} | \varphi(\vec{x}) | \Omega_{J_0} \rangle$$

$$\overset{(3.70)}{=} E(J_0) - \int d^{d-1}x \, J_0(x) \frac{\delta E(J_0)}{\delta J_0(x)}$$

$$= E(J_0) + \frac{1}{T} \langle J_0 \otimes \chi_{[0,T]}, \varphi_c \rangle \qquad (3.71)$$

Expressed in terms of W,

$$\langle \Omega_{J_0} | H | \Omega_{J_0} \rangle = \lim_{T\to\infty} -\frac{1}{T}\left\{ W(J) - \langle J, \varphi_c \rangle \right\}\Big|_{J = J_0 \otimes \chi_{[0,T]}}$$

$$= \lim_{T\to\infty} -\frac{1}{T}\left\{ W(J) - \langle J, \frac{\delta W}{\delta J} \rangle \right\}\Big|_{J = J_0 \otimes \chi_{[0,T]}}$$

Fixing $\varphi_c(x) = \varphi_c(\vec{x}) \otimes \chi_{[0,T]}(T)$ and choosing J such that $\varphi_c(x) = \delta W(J)/\delta J(x)$, we find that

$$\lim_{T\to\infty} \frac{1}{T}\Gamma(\varphi_c) = \langle \Omega_{J_0} | H | \Omega_{J_0} \rangle \Big|_{\varphi_c = \frac{\delta W(J)}{\delta J}}. \quad (3.72)$$

Let us choose space to be a cube, Λ, with sides of length L, with periodic boundary conditions imposed at $\partial\Lambda$, and $J_0(\vec{x}) \equiv J_0 = \text{const.}$ on Λ. Then $\varphi_c(\vec{x}) \equiv \phi_c$ is constant on Λ, as well. We then define a function

$$V(\phi_c) := \lim_{T,L\to\infty} \frac{1}{TL^{d-1}} \Gamma(\varphi_c)$$

$$(3.73)$$

$$= \lim_{L\to\infty} \frac{1}{L^{d-1}} \langle \Omega_{J_0} | H | \Omega_{J_0} \rangle \Big|_{\varphi_c = \frac{\delta W(J)}{\delta J}}.$$

The function $V(\phi_c)$ is called effective potential.

Because it is the Legendre transform of

$$-\mathcal{E}(J_0) := \lim_{T,L\to\infty} \frac{1}{TL^{d-1}} W\left(J = J_0 \chi_{\Lambda\times[0,T]}\right), \qquad (3.74)$$

which is convex in J_0, it is convex in ϕ_c. It is equal to the energy density, $\langle \Omega_{J_0} | \mathcal{H} | \Omega_{J_0} \rangle$, in the groundstate $|\Omega_{J_0}\rangle$ of the Hamiltonian H_{J_0}, where J_0 is chosen such that

$$\phi_c = \left.\frac{\delta W(J)}{\delta J(x)}\right|_{J\equiv J_0 = \text{const.}} = \langle \Omega_{J_0} | \varphi(\vec{x}) | \Omega_{J_0} \rangle$$

By inverting the Legendre transformation, we see that

$$J_0 = \frac{dV(\phi_c)}{d\phi_c} \qquad (3.75)$$

Apparently, $J_0 = 0$ corresponds to critical points of V.

If we choose $J_0(\vec{x}) = J_0 \cos(\vec{k}\cdot\vec{x} + \alpha)$ to be periodic on Λ then $\varphi_c(\vec{x}) = \phi_c \cos(\vec{k}\cdot\vec{x} + \beta)$ (ϕ_c constant) is periodic on Λ, as well. We find that

$$\lim_{T,L\to\infty} \frac{1}{TL^{d-1}} \Gamma(\varphi_c) = \frac{1}{2^d}\left[V(\phi_c) + \frac{Z(\phi_c)}{2}\phi_c^2 |\vec{k}|^2 + O(|\vec{k}|^4)\right]. \quad (3.76)$$

We may then impose as renormalization conditions for a perturbative treatment of a φ^4 - theory:

$$\lambda_R = \frac{d^4 V}{d\phi_c^4}\Big|_{\phi_c=0}, \quad m_R^2 = \frac{d^2 V}{d\phi_c^2}\Big|_{\phi_c=0}, \tag{3.77}$$

and

$$Z(\phi_c = 0) = 1. \tag{3.78}$$

3.5. One-loop calculation of the effective potential in φ^4 - theory, $d=4$.

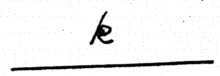
We use the parametrization of φ^4 - theory introduced in (3.20) – (3.22):

$$S(\phi) = \int d^d x \left[\frac{1}{2} (\nabla\phi)^2(x) + \frac{1}{4!}\phi(x)^4 - \frac{\mu^2}{2}\phi(x)^2 \right] \tag{3.79}$$

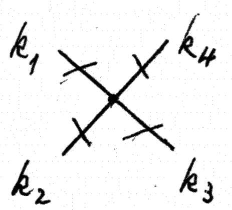
giving rise to the functional measure

$$Z^{-1} e^{-\frac{1}{\lambda}S(\phi)} \mathcal{D}\phi. \tag{3.80}$$

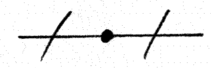
The imaginary-time Feynman rules in momentum space are:



$$\longleftrightarrow \quad \frac{\lambda}{k^2}$$



$$\longleftrightarrow - \frac{1}{4!\,\lambda}\, \delta(k_1 + k_2 + k_3 + k_4)$$

k_1  $k_2 \quad \longleftrightarrow \quad \frac{\mu^2}{2\lambda}\,\delta(k_1+k_2).$

The original physical field is given by

$$\varphi = \frac{\phi}{\sqrt{\lambda}}\,;$$

but it is convenient to work with the fields ϕ.

Consider a Feynman amplitude contributing to a proper vertex function $\hat{\Gamma}^{(n)}(k_1,\cdots,k_n)$ (see (3.63)) with n external legs amputated with the inverse propagators $\lambda^{-1}k_i^2$, $i=1,\cdots,n$; ($\hat{\Gamma}^{(n)}$ is the Fourier transform of $\Gamma^{(n)}$ in (3.63)). Such a Feynman amplitude corresponds to a 1PI Feynman graph with I internal propagators proportional to λ and V vertices proportional to $\frac{1}{\lambda}$. At every vertex, there is a δ-function imposing 4-momentum conservation. By translation invariance, the sum of all 4-momenta flowing into the graph must be $=0$. Thus, to calculate the corresponding amplitude, we must carry out $I-(V-1)$ integrations over momentum space $\mathbb{R}^4$. We remember from QFT I that the number of such integrations is equal to the number, L, of <u>loops</u> in the diagram. Thus the amplitude is proportional to

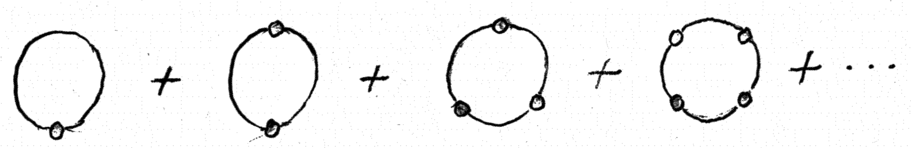
$$\lambda^{I-V} = \lambda^{-1} \cdot \lambda^{L} \tag{3.81}$$

Perturbation theory in powers of λ thus amounts to an expansion in the number, L, of loops in the Feynman graphs corresponding to amplitudes contributing to some $\Gamma^{(n)}$.

From (3.73) we infer that the effective potential, $V(\phi_c)$, is given by

$$V(\phi_c) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\Gamma}^{(n)}(0, \dots, 0)\, \phi_c^n \tag{3.82}$$

All <u>one-loop contributions</u> to $V(\phi_c)$ then correspond to the following Feynman graphs



(3.83)

where the dots, $\circ$, stand for a sum of terms with $n = 0, 1, 2, \cdots$ external lines corresponding to terms in U of order $n+2$ in ϕ, of which two ϕ's are contracted and the remaining ones are set $= \phi_c$. Thus, the dot stands for

$$-\!\!\circ\!\!- \quad \longleftrightarrow \quad U''(\phi_c) = \frac{1}{2}\phi_c^2 - \mu^2.$$

The sum of the amplitudes corresponding to the graphs in (3.83) is therefore given by

$$V_{1loop}(\phi_c) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{U''(\phi_c)}{k^2}\right)^n, \qquad (3.84)$$

where $\frac{1}{2n}$ is the appropriate symmetry factor.

The series on the R.S. of (3.84) can be summed, and we get

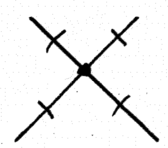
$$V_{1loop}(\phi_c) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln\left(k^2 + U''(\phi_c)\right), \qquad (3.85)$$

up to a (divergent) constant. Of course, the integral on the R.S. is divergent. We cut it off by requiring that $|k| \leq \Lambda < \infty$. Introducing polar coordinates in momentum space $\mathbb{R}^4$ and changing variables to $\tau := k^2$, we may carry out the integral on the R.S. of (3.85), cutoff at $\tau = \Lambda^2$, explicitly and find

$$V_{1\,loop}(\phi_c) = \frac{\Lambda^2}{32\pi^2} U''(\phi_c) + \frac{U''(\phi_c)^2}{64\pi^2}\left(\ln\left(\frac{U''(\phi_c)}{\Lambda^2}\right) - \frac{1}{2}\right) + C_\Lambda, \qquad (3.86)$$

where $C_\Lambda = O(\Lambda^4 \ln \Lambda)$ is a (divergent) constant.

The only tree diagrams contributing to $V(\phi_c)$ are

 and 

and, taking into account the quadratic and quartic counterterms, yields

$$V_{0\,loop}(\phi_c) = \frac{1}{4!\lambda}\phi_c^4 + \frac{\mu^2}{2\lambda}\phi_c^2 + \frac{B_\Lambda}{2}\phi_c^2 + \frac{C_\Lambda}{4!}\phi_c^4; \qquad (3.87)$$

see (3.52). Note that we can choose the coefficients, B_Λ and C_Λ, of the counterterms to cancel the divergent terms, $\propto \phi_c^2$ and $\propto \phi_c^4$, in (3.86). Having done this, we find that, to order λ^0, $V(\phi_c)$ is given by

$$V(\phi_c) = \frac{1}{\lambda} U(\phi_c) + \frac{U''(\phi_c)^2}{64\pi^2} \ln U''(\phi_c) + \frac{b}{2}\phi_c^2 + \frac{c}{4!}\phi_c^4, \qquad (3.88)$$

up to a constant, with

$$U(\phi_c) = \frac{1}{4!}\phi_c^4 + \frac{\mu^2}{2}\phi_c^2 , \qquad (3.89)$$

where the last two terms in (3.88) are an expression of the ambiguity in choosing the counterterms and are fixed by imposing renormalization conditions.

We observe that if we had chosen U to be a polynomial of order ≥ 6 then $U''(\phi_c)^2$ would be of order $2 \deg U - 4 > \deg U$, and we would have to introduce counterterms of degree $> \deg U$. This shows that theories with $\deg U \geq 6$ are <u>non-renormalizable.</u>

We also note that, for $\mu^2 < 0$,

$$U''(\phi_c) = \frac{1}{2}\phi_c^2 + \mu^2 \qquad (3.90)$$

becomes negative, as soon as $|\phi_c| < \sqrt{-2\mu^2}$, and hence $\ln U''(\phi_c)$ has an <u>imaginary part</u> when $|\phi_c| < \sqrt{-2\mu^2}$. In this region, formula (3.88) becomes <u>incorrect</u>! The correct definition is

$$V(\phi_c) \equiv V(\phi_c^*), \qquad (3.91)$$

for $|\phi_c| < \phi_c^*$, where $\phi_c^* > 0$ is the largest critical point of V. Thus $V(\phi_c)$ is flat, for $\phi_c \in [-\phi_c^*, \phi_c^*]$. This is a signal of spontaneous symmetry breaking.

There is another (and better!) way of arriving at the results (3.86), (3.88): We set

$$\phi(x) = \phi_c + \chi(x) \quad (\phi_c = \text{const.}).$$

Then the action functional (3.79) is given by

$$S(\phi) = \int d^d x \Big[\frac{1}{2} (\nabla \chi)^2(x) + U(\phi_c) + U'(\phi_c)\chi(x) + \frac{1}{2} U''(\phi_c)\chi^2(x) + \mathcal{O}(\chi^3) \Big] \tag{3.92}$$

We choose a "source term" $J(x) \equiv J_0 = \text{const.}$ such that

$$\langle \phi(x) \rangle_J = \phi_c, \text{ i.e., } \langle \chi \rangle_J = 0, \tag{3.93}$$

where $\langle (\cdot) \rangle_J$ is the expectation value in the measure

$$e^{-\tilde{W}(J)} e^{-S(\phi) + J_0 \int d^d x\, \phi(x)} \mathcal{D}\phi,$$

see (3.64). Formally

$$\mathcal{D}\phi = \mathcal{D}\chi \tag{3.94}$$

If we neglect terms $\propto \mathcal{O}(\chi^3)$ in $S(\phi)$, i.e., if we do calculations only to one-loop order, we may choose $J_0 = U'(\phi_c)$, so that there isn't any term linear in χ in $S(\phi) - J_0 \int d^d x\, \phi(x)$. Then

$$e^{-\Gamma(\phi_c)} \underset{\text{to 1 loop}}{\simeq} \int \mathcal{D}\chi \exp - \frac{1}{\lambda}\int d^d x \left[\frac{1}{2}(\nabla\chi)^2(x) + U(\phi_c) + \frac{1}{2} U''(\phi_c)\chi(x)^2\right]$$

$$= e^{-\frac{1}{\lambda}\int U(\phi_c)}\left(\det\left(-\Delta + U''(\phi_c)\right)\right)^{-1/2}, \qquad (3.95)$$

as long as $U''(\phi_c) \geq 0$; see (2.16). But

$$\det\left(-\Delta + U''(\phi_c)\right) = \exp \operatorname{Tr} \ln\left(-\Delta + U''(\phi_c)\right) \qquad (3.96)$$

Taking logarithms and dividing by the volume $\int d^d x\, 1$, we find that

$$V(\phi_c) \underset{\text{to 1 loop}}{=} \frac{1}{\lambda} U(\phi_c) + \frac{1/2}{\left(\int d^d x\, 1\right)} \times \operatorname{Tr} \ln\left(-\Delta + U''(\phi_c)\right) + c.t.$$

$$= \frac{1}{\lambda} U(\phi_c) + \frac{1}{2}\int \frac{d^4 k}{(2\pi)^4} \ln\left(k^2 + U''(\phi_c)\right) + c.t.,$$

where c.t. stands for "counter terms (to order λ^0)". Thus we have reproduced (3.85), (3.88)!

We proceed to discussing spontaneous symmetry breaking.

3.6. Spontaneous symmetry breaking and Goldstone's theorem.

Whatever we have studied in Chapt. 3, so far, has an obvious generalization to field theories of N-component, real scalar fields

$$\vec{\varphi} = (\varphi_1, \cdots, \varphi_N), \tag{3.97}$$

$N \geq 2$. For $N = 2$, we may think of $\vec{\varphi}$ as representing a complex field

$$H := \varphi_1 + i\varphi_2 . \tag{3.98}$$

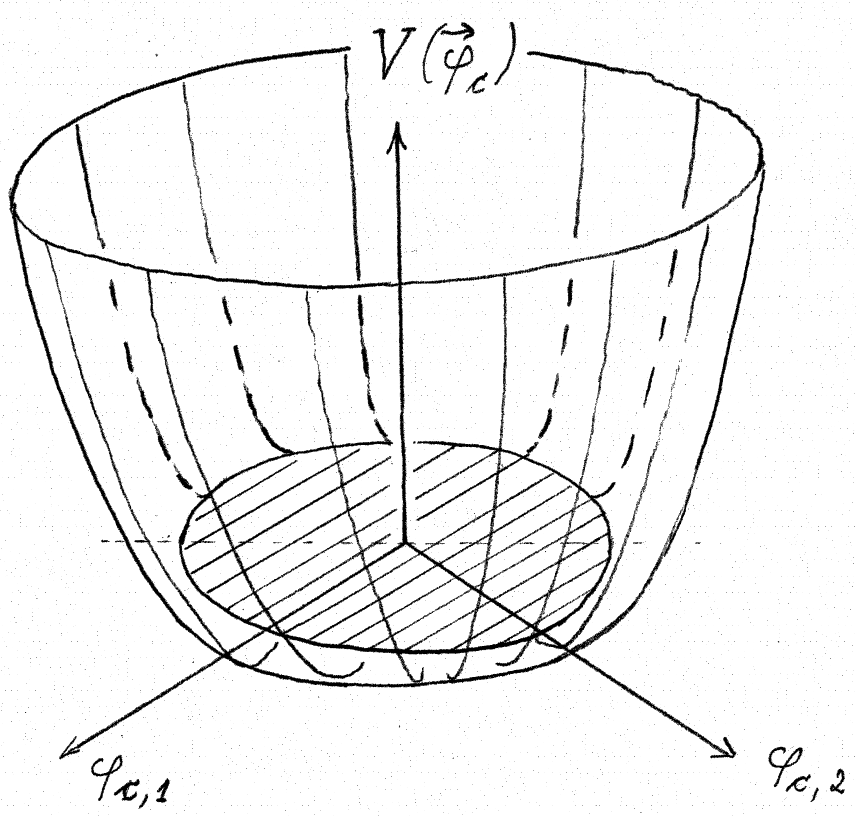
We consider a Lagrangian of the form

$$\mathcal{L}(\vec{\varphi}) = \frac{1}{2}(\partial_\mu \vec{\varphi})(\partial^\mu \vec{\varphi}) - U(|\vec{\varphi}|)$$

with

$$U(|\vec{\varphi}|) = \frac{\lambda}{4!}\left(|\vec{\varphi}|^2 - \rho^2\right)^2 \tag{3.99}$$

We let $V(\vec{\varphi}_c)$ denote the effective potential of this theory, as studied in the last two sections. The leading contribution to $V(\vec{\varphi}_c)$ is given by the convex envelope of $U(\vec{\varphi}_c)$, whose graph is sketched in the following figure:



The shaded disk lies in the (φ_1, φ_2)-plane and has radius ρ. $V(\vec{\varphi}_c)$ is flat on this disk and strictly convex for $|\vec{\varphi}_c| > \rho$. Note that, by (3.59)

$$\vec{J}(x) = \frac{\delta \Gamma(\varphi_c)}{\delta \vec{\varphi}_c(x)} = \frac{d\, V(\vec{\varphi}_c)}{d\, \vec{\varphi}_c}, \qquad (3.100)$$

if $\vec{\varphi}_c(x) \equiv \vec{\varphi}_c$, and hence $\vec{J}$, are constant. We recall that

$$\vec{\varphi}_c(x) = \langle \vec{\varphi}(x) \rangle_J = \langle \Omega_J | \vec{\varphi}(x) | \Omega_J \rangle \qquad (3.101)$$

By (3.100) and (3.101), any $\vec{\varphi}_c$ for which $dV(\vec{\varphi}_c)/d\vec{\varphi}_c$ vanishes corresponds to a vacuum state $|\Omega_0\rangle$ of the theory with $\vec{J} \equiv 0$ for which $\langle \Omega_0 | \vec{\varphi}(x) | \Omega_0 \rangle = \vec{\varphi}_c$.

If $\vec{\varphi}_c \neq 0$ this means that the vacuum state $|\Omega_0\rangle$ spontaneously breaks a symmetry of the theory! To understand what this means, we start by observing that the Lagrangian $\mathcal{L}$ in (3.99) is invariant under a global $O(N)$-symmetry (with $O(1) = \mathbb{Z}_2$): If

$$\vec{\varphi}(x) \longrightarrow R\vec{\varphi}(x), \quad R \in O(N), \qquad (3.102)$$

$\mathcal{L}$ does <u>not</u> change (for $\vec{J} \equiv 0$!). Obviously, this symmetry is spontaneously broken by any vacuum $|\Omega_0\rangle$ for which $\vec{\varphi}_c = \langle \Omega_0 | \vec{\varphi}(x) | \Omega_0 \rangle \neq 0$. The calculations leading to formula (3.88) suggest that, for $\rho^2 > 0$ large enough, spontaneous symmetry breaking occurs.

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We consider a Lagrangian of the form

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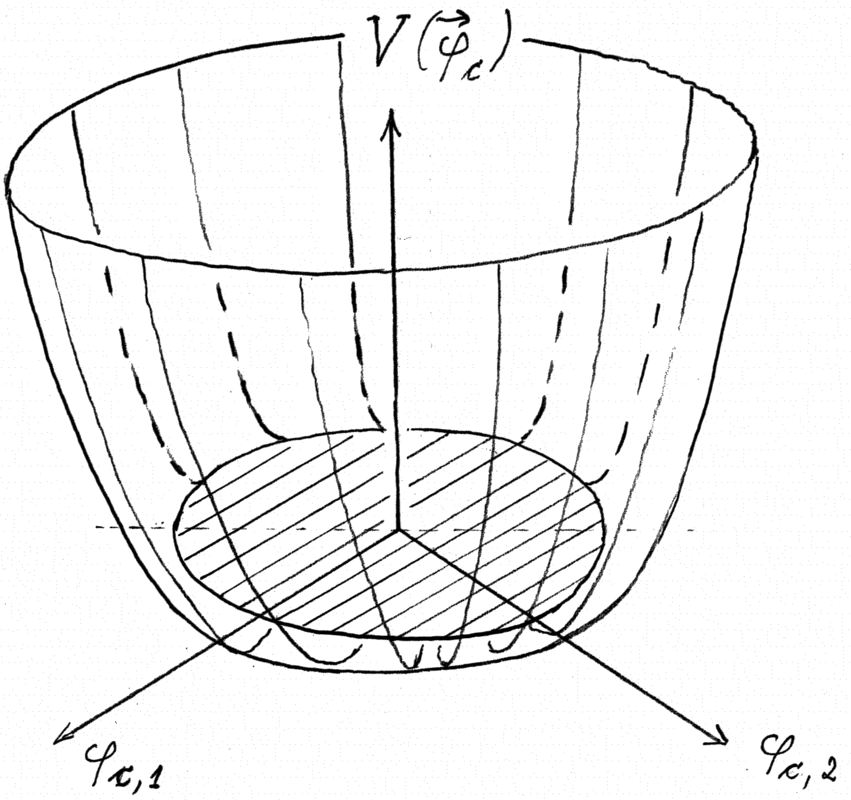

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if $\vec{\varphi}_c(x) \equiv \vec{\varphi}_c$, and hence $\vec{J}$, are constant. We recall that

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$\mathcal{L}$ does not change (for $\vec{J} \equiv 0$!). Obviously, this symmetry is spontaneously broken by any vacuum $|\Omega_{\vec{\varphi}_c}\rangle$ for which $\vec{\varphi}_c = \langle \Omega_{\vec{\varphi}_c} | \vec{\varphi}(x) | \Omega_{\vec{\varphi}_c} \rangle \neq 0$. The calculations leading to formula (3.88) suggest that, for $\rho^2 > 0$ large enough, spontaneous symmetry breaking occurs.

The physical meaning of the ball of zeros of $\dfrac{dV(\vec{\varphi}_c)}{d\vec{\varphi}_c}$

We have verified that, at least to 1-loop order, $O(\lambda^0)$, and for a choice of counter terms such that the constants b and c in (3.88) vanish,

$$\frac{dV(\vec{\varphi}_c)}{d\vec{\varphi}_c} \equiv \operatorname{grad}_{\vec{\varphi}_c} V(\vec{\varphi}_c)$$

vanishes on a ball $B_{\rho^*} := \{\vec{\xi} \in \mathbb{R}^N \,|\, |\vec{\xi}| \leq \rho^*\}$, where $0 < \rho^* \simeq \rho$, for λ small enough and ρ large enough. Any $\vec{\varphi}_c \in B_{\rho^*}$ with $\vec{\varphi}_c \neq 0$ corresponds to a vacuum state $|\Omega_{\vec{\varphi}_c}\rangle$ that <u>breaks</u> the $O(N)$-<u>symmetry</u> of the Lagrangian of the theory <u>spontaneously</u>.

Let $\vec{\varphi}_c$ be a point in the interior of B_{ρ^*}. Then there is a probability measure, $dP_{\vec{\varphi}_c}(\vec{\xi})$, with support on the $(N-1)$-dimensional sphere ∂B_{ρ^*} such that, for any polynomial, F, in the field operators,

$$\langle \Omega_{\vec{\varphi}_c} | F | \Omega_{\vec{\varphi}_c} \rangle = \int_{\partial B_{\rho^*}} dP(\vec{\xi}) \langle \Omega_{\vec{\xi}} | F | \Omega_{\vec{\xi}} \rangle ; \qquad (3.103)$$

in particular, for $F = \vec{\varphi}(x)$,

$$\vec{\varphi}_c = \int_{\partial B_{\rho^*}} dP(\vec{\xi}) \, \vec{\xi} ,$$

i.e. $\vec{\varphi}_c$ is the barycenter of dP. Eq. (3.103) is interpreted as saying that the vacuum state

$|\Omega_{\vec{\varphi}_c}\rangle$ describes "phase coexistence" of "pure phases" of the theory corresponding to vacuum states $|\Omega_{\vec{\xi}}\rangle$, with $\vec{\xi} \in \partial B_{\rho^*}$.

It is of interest to note that the theory with Lagrangian (3.99), with $\lambda = \frac{g}{N}$, $g > 0$, can be solved exactly in the limit where $N \to \infty$; ("spherical model limit"). We may return to this in the exercises. For $N = 1$, the symmetry, $O(1) = \mathbb{Z}_2$, is discrete; the symmetry operation consists of $\varphi(x) \mapsto -\varphi(x)$. The field theory then describes the continuum limit of the Ising model.

Goldstone bosons

Goldstone's theorem says that if the continuous symmetry $O(N)$, $N \geq 2$, is spontaneously broken the field theory describes $N-1$ massless particles, so called Goldstone bosons.

In order to develop some intuition for this phenomenon, we consider the example of an $N=2$ component $\lambda|\vec{\varphi}|^4$-theory.

Here $\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, $\phi := \varphi_1 + i\varphi_2$. The tree-level Lagrangian is given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \bar{\phi})(\partial^\mu \phi) - \frac{\lambda}{4!}\left(|\phi|^2 - \rho^2\right)^2.$$

It admits an $O(2)$-symmetry. In particular,

$$\begin{aligned} \varphi_1 &\mapsto \varphi_1 \cos\alpha + \varphi_2 \sin\alpha, \\ \varphi_2 &\mapsto -\varphi_1 \sin\alpha + \varphi_2 \cos\alpha, \end{aligned} \tag{3.104}$$

or $\phi \mapsto e^{i\alpha}\phi$, $0 \leqslant \alpha < 2\pi$, is a <u>global</u> $SO(2)$-symmetry. The minima of the potential $U(|\phi|) = \frac{\lambda}{4!}\left(|\phi|^2 - \rho^2\right)^2$ lie on the circle

$$|\phi|^2 = \varphi_1^2 + \varphi_2^2 = \rho^2. \tag{3.105}$$

We quantize the theory by expanding around one of these minima and applying renormalized perturbation theory. For example, we choose

$$\langle \phi \rangle = \rho \quad \left(\langle \varphi_2 \rangle = 0\right).$$

Setting $\tilde{\phi} := \phi - \langle \phi \rangle = \phi - \rho$, we find that

$$U(|\phi|) = \frac{\lambda}{4!}\left[|\tilde{\phi}|^4 + 4\rho\, \tilde{\varphi}_1 \tilde{\varphi}_2^2 + 4\rho\, \tilde{\varphi}_1^3\right] + \frac{\lambda}{6}\rho^2 \tilde{\varphi}_1^2 \tag{3.106}$$

Note that $\text{grad}\, U\Big|_{\tilde{\phi}} = 0$ (because $\phi = \rho$ is a minimum) and

$$\text{Hessian}\; U\Big|_{\tilde{\phi}=0} = \begin{pmatrix} \frac{\lambda}{3}\rho^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It appears that $\tilde{\varphi}_1$ describes a boson of mass $\simeq \frac{\lambda}{3}\rho^2$, while $\tilde{\varphi}_2$ describes a massless boson, which is called Goldstone boson. This is a general phenomenon whenever a continuous symmetry is spontaneously broken! We note that perturbation theory in the term $4\rho\,\tilde{\varphi}_1\,\tilde{\varphi}_2^{\,2}$ on the R.S. of (3.106) suggests that the $\tilde{\varphi}_1$-particle with tree-level mass $= \frac{\lambda}{3}\rho^2$ is actually unstable: It can decay into two Goldstone bosons; i.e., it is a resonance. (In the exercises, we attempt to estimate its life time.)

It may be useful to recast these findings in a slightly different form: We introduce polar coordinates in field space,

$$\varphi_1 = R\cos\theta\,, \quad \varphi_2 = R\sin\theta.$$

The group $SO(2)$ acts by $\theta \mapsto \theta + \alpha$. The Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu R)(\partial^\mu R) + \frac{1}{2}R^2(\partial_\mu\theta)(\partial^\mu\theta) - U(R).$$

Setting $\tilde{R} := R - \rho$, we find that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\tilde{R})(\partial^\mu\tilde{R}) + \frac{1}{2}(\tilde{R}+\rho)^2(\partial_\mu\theta)(\partial^\mu\theta) - U(\tilde{R}+\rho)$$

$$\simeq \frac{1}{2}(\partial_\mu\tilde{R})(\partial^\mu\tilde{R}) - \frac{1}{2}U''(\rho)\,\tilde{R}^2 + \frac{\rho^2}{2}(\partial_\mu\theta)(\partial^\mu\theta)$$

$$+ \text{ higher-order terms.} \tag{3.107}$$

Note that the particle described by the field θ is massless, because the θ-field enters $\mathcal{L}$ only through <u>first derivatives</u>.

More generally, let $\vec{\phi}$ be a field with N real components, and assume that the Lagrangian of $\vec{\phi}$ is invariant under

$$\vec{\phi} \longmapsto \exp\Big(\sum_{A=1}^{n} \xi_A T^A\Big)\cdot\vec{\phi}\,, \tag{3.108}$$

where $T_1, \cdots, T_n$ are the generators of the Lie algebra, $\mathfrak{g}$, of an n-dimensional Lie group G, and $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. We consider a Lagrangian $\mathcal{L}(\vec{\phi}, \partial_\mu \vec{\phi})$ <u>invariant</u> under (3.108). Let

$$\vec{\varphi}_c = \langle \Omega | \vec{\phi}(x) | \Omega \rangle \tag{3.109}$$

be the vacuum expectation value of $\vec{\phi}(x)$ in a vacuum, $|\Omega\rangle$, that breaks the symmetry G of $\mathcal{L}$ spontaneously. Let $H \subseteq G$ be the subgroup of G with the property that

$$h \vec{\varphi}_c = \vec{\varphi}_c, \text{ for all } h \in H.$$

We may choose a basis, $T^1, \cdots, T^n$, of $\mathfrak{g}$ such that $T_1, \cdots, T_m$, $m \leq n$, generate the Lie algebra, $\mathfrak{h}$, of H; i.e.,

$$T_A \vec{\varphi}_c = 0, \quad A = 1, \cdots, m.$$

Then

$$T_A \vec{\varphi}_c \neq 0, \quad \text{for } A = m+1, \cdots, n. \tag{3.110}$$

We conclude that the (effective) potential of the theory must admit an $(n-m)$-dimensional surface of minima passing through $\vec{\varphi}_c$. Locally, the parameters $\xi_{m+1}, \dots, \xi_n$ can be used as coordinates of this surface. Enlarging them to a complete local coordinate system in field space and rewriting $\mathcal{L}$ in the new coordinates, we observe that $\mathcal{L}$ depends on $\xi_{m+1}, \dots, \xi_n$ only through their first derivatives. Thus, the particles described by the fields $\xi_{m+1}, \dots, \xi_n$ apparently remain massless, i.e., there are

$$n - m = \dim G - \dim H = \dim \left(G/H\right)$$ massless Goldstone bosons.

If $\mathcal{L}$ contains small terms that break the symmetry G explicitly then the Goldstone bosons tend to aquire small masses. With this in mind, the pions can be interpreted as the Goldstone bosons of an approximate $SU(2)_L \times SU(2)_R$ symmetry of QCD spontaneously broken to a $SU(2)_{isospin}$ symmetry. The small mass terms for the u- and the d-quarks in the Lagrangian of QCD break the chiral $SU(2)_L \times SU(2)_R$ symmetry explicitly. This explains why the pions are not exactly massless. (More details will be studied in the exercises.)

"Proof" of spontaneous symmetry breaking for $\lambda|\vec{\phi}|^4$-theory in $d=3$ space-time dimensions

We consider a theory with Lagrangian

$$\mathcal{L}(\vec{\varphi}, \partial_\mu \vec{\varphi}) = \frac{1}{2}(\partial_\mu \vec{\varphi})\cdot(\partial^\mu \vec{\varphi}) - U(|\vec{\varphi}|),$$

where

$$U(|\vec{\varphi}|) = \frac{\lambda}{4!}\left(|\vec{\varphi}|^2 - \rho^2\right)^2. \qquad (3.111)$$

In three space-time dimensions, this theory makes perfect mathematical sense, non-perturbatively. We study it in the Euclidian region; (Wick rotation to imaginary time, $\theta = \frac{\pi}{2}$).

It is convenient to Wick-order the potential $U(|\vec{\varphi}|)$ with respect to the <u>massless</u> free field (which is only possible if $d \geq 3$!):

$$:|\vec{\varphi}|^2:(x) = \lim_{y \to x} \left\{ \vec{\varphi}(x) \cdot \vec{\varphi}(y) - \langle 0 | \vec{\varphi}(x) \cdot \vec{\varphi}(y) | 0 \rangle \right\}, \quad (3.112)$$

where

$$\langle 0 | \vec{\varphi}(x) \cdot \vec{\varphi}(y) | 0 \rangle = (-\Delta)^{-1}_{xy} = \frac{1}{4\pi |x-y|} ; \quad (3.113)$$

and similarly for $:|\vec{\varphi}|^4:$. We set

$$U_\Lambda(\rho^2; |\vec{\varphi}|) := \int_\Lambda d^3x \left\{ \frac{\lambda}{4!} :|\vec{\varphi}|^4:(x) + \delta m^2(\lambda) :|\vec{\varphi}|^2:(x) - \frac{\lambda}{12} \rho^2 :|\vec{\varphi}|^2:(x) \right\}$$

$$= U_\Lambda(0; |\vec{\varphi}|) - \frac{\lambda \rho^2}{12} \int_\Lambda d^3x :|\vec{\varphi}|^2:(x), \quad (3.114)$$

where Λ is a space-time cube with sides of length L, $\delta m^2(\lambda) = \text{const}\, \lambda^2$ is the logarithmically divergent coefficient of a mass counterterm (independent of ρ^2), and we omit the constant $\frac{\lambda}{4!} \rho^4$.

In $d=3$ dimensions, there are no field-strength and coupling-constant renormalizations!

We now define

$$Z_\Lambda(\rho^2) := \frac{1}{Z_\Lambda(0)} \int \mathcal{D}\vec{\varphi}\, e^{-\frac{1}{2}\int d^3x\, (\partial_\mu \vec{\varphi})(x)\cdot(\partial^\mu \vec{\varphi})(x)} \times$$

$$\times\, e^{-U_\Lambda(0;|\vec{\varphi}|)}\, e^{\frac{\lambda \rho^2}{12}\int_\Lambda d^3x\, :|\vec{\varphi}|^2:(x)}, \qquad (3.115)$$

and

$$F(\rho^2) := \lim_{L\to\infty} \frac{1}{L^3} \ln Z_\Lambda(\rho^2).$$

It is fairly easy to show that $F(\rho^2)$ is a <u>convex</u> function of ρ^2, with $F(\rho^2) \nearrow \infty$, as $\rho^2 \to \infty$.

Thus,
$$\frac{\partial F(\rho^2)}{\partial \rho^2} \overset{(3.115)}{=} \frac{\lambda}{12} \langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} \nearrow \infty,$$

as $\rho^2 \to \infty$. We conclude that, given any finite constant $M^2 > 0$, there exists some $\rho_M^2 < \infty$ such that

$$\langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} \geq M^2 > 0, \quad \text{for } \rho^2 > \rho_M^2 \qquad (3.116)$$

Next, we note that

$$\langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} = \lim_{y\to 0} \langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle_{\rho^2} - \frac{1}{4\pi|y|}$$

$$= \lim_{y\to 0} \langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle^c_{\rho^2} - \langle \vec{\varphi}(0)\rangle^2_{\rho^2} - \frac{1}{4\pi|y|} \qquad (3.117)$$

The Källen – Lehmann representation of the two-point function (see QFT I) says that

$$\langle \vec{\varphi}(y) \cdot \vec{\varphi}(0) \rangle^c_{\rho^2} = \int_0^\infty d\mu(a^2) \left(-\Delta + a^2\right)^{-1}_{y,0}$$

$$= \int_0^\infty d\mu(a^2) \, \frac{e^{-a|y|}}{4\pi |y|} \tag{3.118}$$

Since this theory is a <u>canonical</u> field theory (there is no field strength renormalization!), i.e.,

$$\left[\varphi_A(\vec{y}, 0), \pi_A(0,0)\right] = i\,\delta^{(2)}(\vec{y}),$$

it follows (exercises!) that

$$\int_0^\infty d\mu(a^2) = 1. \tag{3.119}$$

Thus

$$\langle \vec{\varphi}(y) \cdot \vec{\varphi}(0) \rangle^c_{\rho^2} - \frac{1}{4\pi|y|} = \int_0^\infty d\mu(a^2) \, \frac{e^{-a|y|} - 1}{4\pi|y|}$$

$$\leq 0. \tag{3.120}$$

Combining (3.116), (3.117) and (3.120), we conclude that

$$0 < M^2 \leq \langle \vec{\varphi}(0) \rangle^2_{\rho^2}, \text{ for } \rho^2 > \rho_M^2, \tag{3.121}$$

i.e., $\left|\langle \vec{\varphi}(0) \rangle_{\rho^2}\right| > 0$. This proves that the O(N)-

symmetry $\left(\vec{\varphi} = (\varphi_1, \cdots, \varphi_N), N \geq 1\right)$ is spontaneously broken for ρ^2 large enough. (Details can be found in Fröhlich, Simon and Spencer, Commun. Math. Phys. 50, (1976).)

These arguments fail in $d=2$, because Wick-ordering with respect to a massless free field is meaningless, and in $d \geq 4$, because the $\lambda |\vec{\varphi}|^4$-theory without ultraviolet cutoffs does not make sense. (If a suitable ultraviolet cutoff is introduced the same conclusions — spontaneous symmetry breaking — hold for arbitrary $d \geq 3$.) In $d=2$, only the $\varphi \to -\varphi$ symmetry of the one-component $\lambda\varphi^4$ theory can be broken spontaneously, for ρ^2 large enough; but an $O(N)$-symmetry, with $N \geq 2$, cannot be broken (Mermin–Wagner theorem).

Proof of the Goldstone Theorem in the Euclidian region (following K. Symanzik)

We consider a relativistic QFT of some scalar field φ with a continuous internal symmetry given by a compact Lie group G with Lie algebra $\mathfrak{g}$ and corresponding conserved Noether currents j^μ_X, $X \in \mathfrak{g}$. As an example, we may think of a Lagrangian field theory of an N-component, real scalar field $\vec{\phi}$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)\cdot\left(\partial^\mu \vec{\phi}\right) + \frac{\mu^2}{2}\vec{\phi}^{\,2} - \frac{\lambda}{4!}\left(\vec{\phi}\cdot\vec{\phi}\right)^2, \qquad (3.122)$$

with $G = O(N)$ and, for $X = E_{ij} - E_{ji}$, E_{ij} = matrix unit, $i,j = 1,\dots,N$,

$$j^\mu_{ij} = \left(\partial^\mu \phi^i\right)\phi^j - \left(\partial^\mu \phi^j\right)\phi^i. \qquad (3.123)$$

The field φ transforms under a representation

ρ of G. We set $\tilde{X} := d\rho(X)$, for $X \in \mathfrak{g}$.

Let

$$Q_X := \int_{t=\text{const}} d^{d-1}x \; j^0_X(\vec{x}, t) \tag{3.124}$$

denote the <u>conserved charge</u> corresponding to the conserved current j^μ_X. Then

$$[Q_X, \varphi(x)] = (\tilde{X}\varphi)(x). \tag{3.125}$$

Let Ω_{φ_c} be a <u>vacuum</u> of the theory that breaks the symmetry G of the theory spontaneously, in the sense that

$$\langle \Omega_{\varphi_c} | \varphi(x) | \Omega_{\varphi_c} \rangle = \varphi_c \neq 0. \tag{3.126}$$

Let $H_{\varphi_c} \subset G$ denote the isotropy subgroup of φ_c, and let $\mathfrak{h}_{\varphi_c} \equiv \mathfrak{h}$ denote its Lie algebra.

Then

$$\tilde{X}\varphi_c = 0, \quad \forall X \in \mathfrak{h}. \tag{3.127}$$

For any $Y \in \mathfrak{g} \ominus \mathfrak{h}$, we then have that

$$\tilde{Y}\varphi_c \neq 0. \tag{3.128}$$

Combining (3.125) and (3.128), we find that

$$\begin{aligned}\langle \Omega_{\varphi_c} | [Q_y, \varphi(0)] | \Omega_{\varphi_c} \rangle &= \langle \Omega_{\varphi_c} | (\tilde{y}\,\varphi)(0) | \Omega_{\varphi_c} \rangle \\ &= \tilde{y}\, \langle \Omega_{\varphi_c} | \varphi(0) | \Omega_{\varphi_c} \rangle = \tilde{y}\, \varphi_c \neq 0,\end{aligned} \tag{3.129}$$

for arbitrary $y \in \mathfrak{g} \ominus \mathfrak{h}$. On the left side of (3.129), we rewrite Q_y using (3.124) and then apply formula (3.19):

$$\begin{aligned}\langle \Omega_{\varphi_c} | [Q_y, \varphi(0)] | \Omega_{\varphi_c} \rangle &= \int_{t=0} d^{d-1}x\, \langle \Omega_{\varphi_c} | [j_y^0(\vec{x},0), \varphi(0)] | \Omega_{\varphi_c} \rangle \\ &= \lim_{\varepsilon \searrow 0} \int d^{d-1}x\, \langle \Omega_{\varphi_c} | j_y^0(\vec{x},0) e^{-\varepsilon H} \varphi(0) \\ &\qquad - \varphi(0) e^{-\varepsilon H} j_y^0(\vec{x},0) | \Omega_{\varphi_c} \rangle \\ &\doteq \lim_{\varepsilon \searrow 0} \int d^{d-1}x\, \langle (j_y^0(\vec{x},\varepsilon) - j_y^0(\vec{x},-\varepsilon))\, \varphi(0) \rangle,\end{aligned} \tag{3.130}$$

where the integrand on the R.S. of (3.130) is an imaginary time (Euclidian) Green- or Schwinger function, which, in a Lagrangian scalar field theory, such as that in (3.122), can be expressed by a Wick-rotated $\left(\theta = \frac{\pi}{2}\right)$ functional integral; see (3.16), (3.17). We set

$$W^{\mu}_{y}(x) := \left\langle j^{\mu}_{y}(\vec{x}, t)\, \varphi(0) \right\rangle \qquad (3.131)$$

$$= \begin{cases} \langle \Omega_{\varphi_c} | j^{\mu}_{y}(\vec{x}, it)\, \varphi(0) | \Omega_{\varphi_c} \rangle, & t > 0 \\ \langle \Omega_{\varphi_c} | \varphi(0)\, j^{\mu}_{y}(\vec{x}, it) | \Omega_{\varphi_c} \rangle, & t < 0. \end{cases}$$

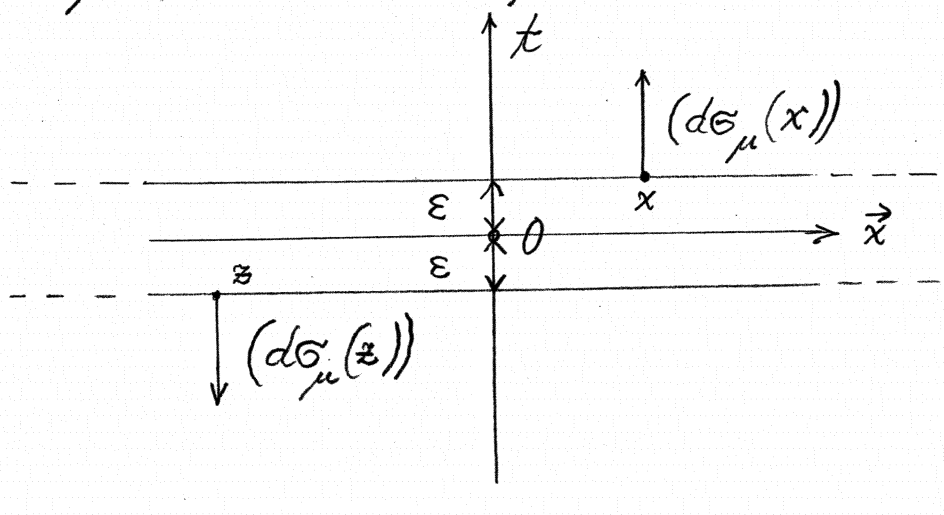
Then, combining (3.129) – (3.131), we find that

$$M_y := \tilde{y}\varphi_c = \lim_{\varepsilon \searrow 0} \int_{S_\varepsilon} d\sigma_\mu(x)\, W^{\mu}_{y}(x) \neq 0, \qquad (3.132)$$

where S_ε is the slab

$$S_\varepsilon = \left\{ x \in \mathbb{R}^d \,\middle|\, x = (\vec{x}, t),\ t = \pm \varepsilon \right\}, \qquad (3.133)$$

and $d\sigma_\mu(x)$ is the surface element on S_ε:



We now use the fact that j^{μ}_{y} is a <u>conserved current</u>, i.e.,

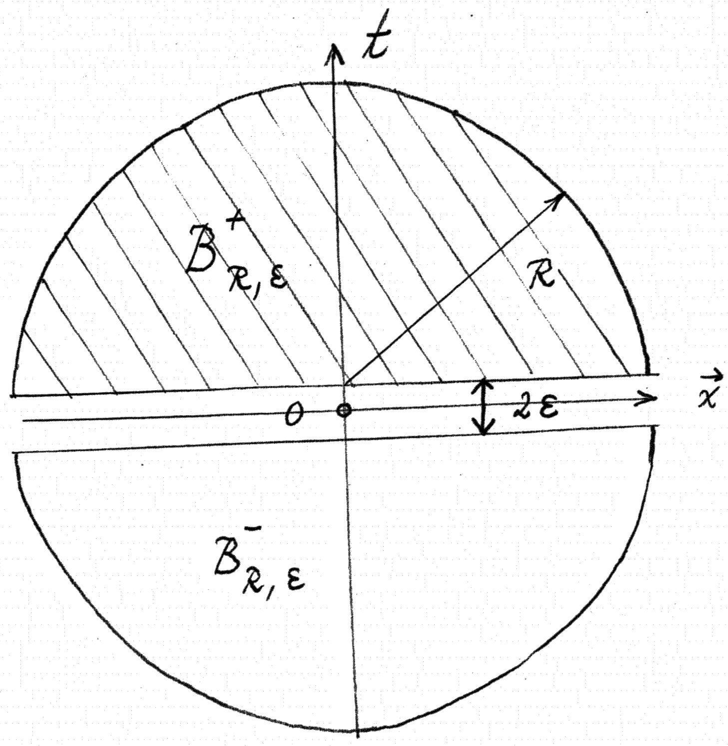
$$\partial_\mu j^{\mu}_{y}(x) = 0, \quad \forall y \in \mathfrak{g}; \qquad (3.134)$$

(Noether's theorem; see Chapter 5 of QFT I).

In the Euclidian (imaginary time, $\theta = \frac{\pi}{2}$) region, this implies that

$$\partial_\mu W^{\mu}_{y}(x) = 0, \text{ for } x \neq 0. \qquad (3.135)$$

Equation (3.135) permits us to apply <u>Gauss' theorem</u>:

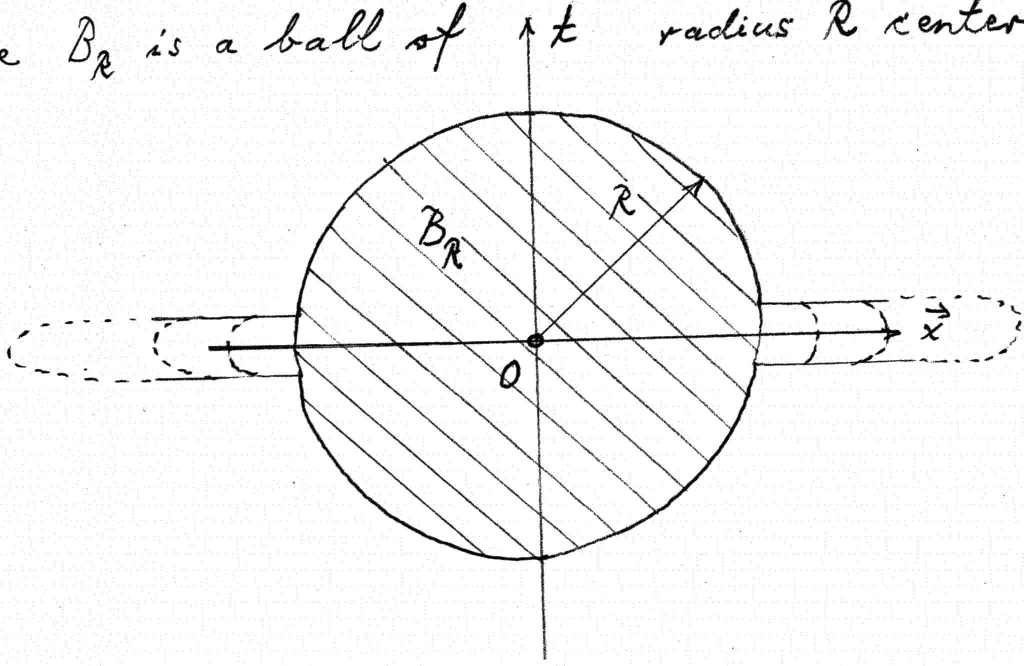


Note that, by (3.135), Gauss' theorem can be applied on $B^{+}_{R,\varepsilon} \cup B^{-}_{R,\varepsilon}$. It yields

$$0 \neq M_y = \lim_{\varepsilon \downarrow 0} \int_{S_\varepsilon} d\sigma_\mu(x)\, W_y^\mu(x) + \underbrace{\sum_{\delta=\pm} \int_{\partial B^\delta_{R,\varepsilon}} d\sigma_\mu(x)\, W_y^\mu(x)}_{=\,0,\ \text{by Gauss}}$$

$$= \int_{\partial B_R} d\sigma_\mu(x)\, W_y^\mu(x), \qquad (3.134)$$

for any $R > 0$, (using again Gauss' theorem), where B_R is a ball of radius R centered at 0

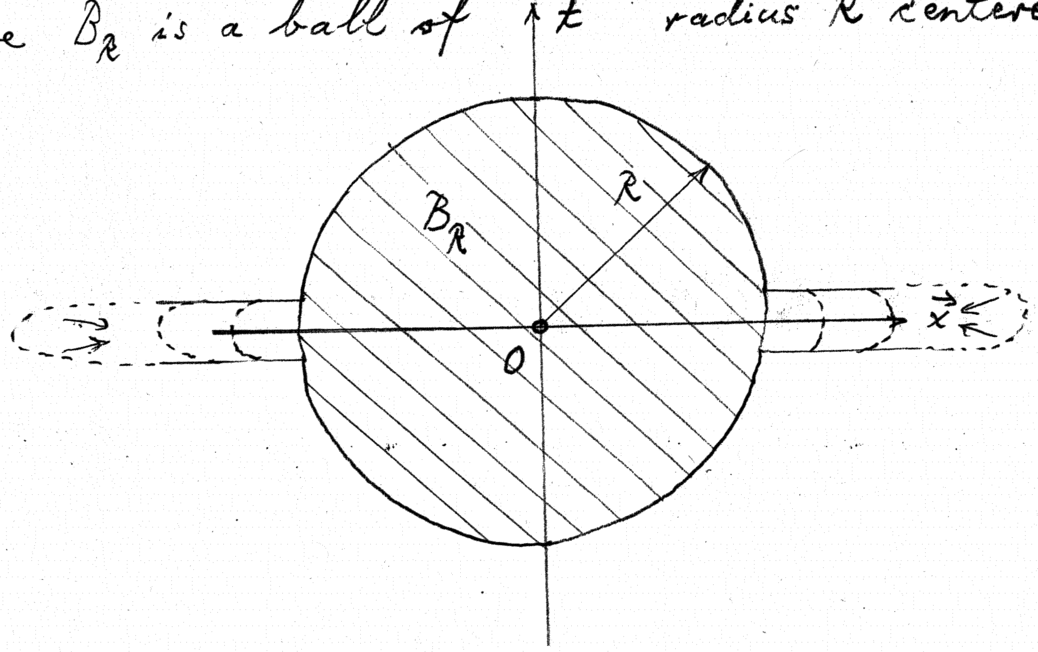


$$0 \neq M_y = \lim_{\varepsilon \searrow 0} \int_{S_\varepsilon} d\sigma_\mu(x)\, W_y^\mu(x)$$

$$+ \underbrace{\sum_{\delta=\pm} \int_{\partial B^\delta_{R,\varepsilon}} d\sigma_\mu(x)\, W_y^\mu(x)}_{=0,\ \text{by Gauss}}$$

$$= \int_{\partial B_R} d\sigma_\mu(x)\, W_y^\mu(x), \qquad (3.134)$$

for any $R > 0$, (using again Gauss' theorem), where B_R is a ball of radius R centered at 0



Because the vacuum Ω_{φ_c} is Poincaré-invariant, definition (3.131) implies that

$$R^\mu{}_\nu W_y^\nu(Rx) = W_y^\mu(x),$$

for arbitrary $R \in SO(d)$. Thus $x_\mu W_y^\mu(x)$ is

<u>invariant</u> under arbitrary rotations of Euclidian space-time, E^d. Hence

$$W^\mu_y(x) = x^\mu f(|x|), \tag{3.135}$$

(with $|x| = \sqrt{x^2}$). By (3.134), we then find that

$$W^\mu_y(x) = \text{const.}\ \frac{x^\mu}{|x|^d}, \tag{3.136}$$

which also follows from (3.135) and the fact that $\partial_\mu W^\mu_y(x) = 0$, for $x \neq 0$.

Eqs. (3.131) and (3.136) show that φ and j^0_y couple the vacuum Ω_{φ_c} to a <u>massless one-particle state</u> describing a "Goldstone boson".

In two space-time dimensions, there is no scalar relativistic field, φ, coupling the vacuum to a zero-mass one-particle state, and we therefore conclude that a continuous internal symmetry generated by charges associated with Poincaré-covariant conserved currents cannot be broken spontaneously; ("Mermin - Wagner theorem").

For more details, see S. Weinberg, "The Quantum Theory of Fields", Vol. II, pages 163 - 191.

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4. Path integrals for Fermi fields

Time-ordered vacuum expectation values of products of Fermi fields are anti-symmetric under exchanging the order of the Fermi fields; see QFT I. For this reason, they cannot be the moments of a path-space (functional) measure of the sort studied in Chapters 2 and 3.

We have already noticed in Chapter 5 of QFT I that Fermi fields must anti-commute with each other in the formal classical limit $\hbar \searrow 0$; i.e.,

$$\{\psi^{\#}(x), \psi^{\#}(y)\} = 0, \ \{\psi(x), \psi^{*}(y)\} = 0, \text{ as } \hbar \searrow 0.$$

Thus, in classical field theory, Fermi fields should be treated as Grassmann variables ("anti-commuting c-numbers"). We must therefore learn how to define path integrals over Grassmann variables. A convenient approach to solving this problem is to use coherent states. We start by defining coherent states for <u>bosons</u>.

<u>Coherent states for one quantum-mechanical "bosonic" degree of freedom.</u>

Instead of the operators Q and P (position and momentum, respectively), we make use of creation- and annihilation operators

$$a = \frac{1}{\sqrt{2}}(Q+iP)\,, \quad a^* = \frac{1}{\sqrt{2}}(Q-iP), \qquad (4.1)$$

with $[a,a]=[a^*,a^*]=0$, $[a,a^*]=1$. The groundstate of the harmonic-oscillator Hamiltonian $H = a^*a = \frac{1}{2}(P^2+Q^2-1)$ corresponding to the eigenvalue 0 is denoted by $|0\rangle$, and

$$a|0\rangle = 0 \qquad (4.2)$$

We define the <u>coherent state</u>

$$|z\rangle := \exp(z a^*)|0\rangle, \text{ so that } a|z\rangle = z|z\rangle. \qquad (4.3)$$

Then the scalar product, $\langle z_1|z_2\rangle$, is given by

$$\langle z_1|z_2\rangle = e^{\bar{z}_1 z_2}, \qquad (4.4)$$

with $\langle 0|0\rangle = 1$; see QM I. With a state $|\psi\rangle \in \mathcal{H} := L^2(Q, dQ)$ we associate the wave function

$$\psi(\bar{z}) := \langle z | \psi \rangle; \tag{4.5}$$

$\psi(\bar{z})$ is an anti-holomorphic function of z, because $\langle \cdot | \cdot \rangle$ is anti-linear in the first argument.

The scalar product of vectors $|\psi\rangle$ and $|\chi\rangle$ can be inferred from the formula

$$\int e^{-|z|^2} |z\rangle\langle z| \frac{d\bar{z} \wedge dz}{2\pi i} = \mathbb{1}. \tag{4.6}$$

This formula can be proven by expanding a general vector $|\psi\rangle \in \mathcal{H}$ in terms of powers of a^*, applied to $|0\rangle$:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n \frac{(a^*)^n}{\sqrt{n!}} |0\rangle, \tag{4.7}$$

with

$$\langle \psi | \chi \rangle = \sum_{n=0}^{\infty} \bar{\psi}_n \chi_n . \tag{4.8}$$

Details will be straightened out in the exercises.

With (4.5) and (4.6) we now find that

$$\langle \psi | \chi \rangle = \int \langle \psi | z \rangle \langle z | \chi \rangle e^{-|z|^2} \frac{d\bar{z} \wedge dz}{2\pi i}$$

$$= \int \bar{\psi}(z)\, \chi(\bar{z})\, e^{-|z|^2} \frac{d\bar{z} \wedge dz}{2\pi i}, \tag{4.9}$$

where $\bar{\psi}(z) = \overline{\psi(\bar{z})}$. The Hilbert space, $\mathcal{H}$, of a single quantum-mechanical degree of freedom can thus be viewed as the completion of the linear space of entire anti-holomorphic functions, $\psi(\bar{z})$, w.r. to the norm

$$\|\psi\| = \left(\int |\psi(\bar{z})|^2 e^{-|z|^2} \frac{d\bar{z} \wedge dz}{2\pi i} \right)^{1/2}. \quad (4.10)$$

Creation- and annihilation operators act on $|\psi\rangle$ as follows:

$$\left.\begin{aligned} (a\psi)(\bar{z}) &= \frac{\partial}{\partial \bar{z}} \psi(\bar{z}) \\ (a^*\psi)(\bar{z}) &= \bar{z}\, \psi(\bar{z}). \end{aligned}\right\} \quad (4.11)$$

The representation (4.5) – (4.11) is called the <u>Bargmann representation</u> (in honor of Valentin Bargmann, who introduced it).

The <u>proof</u> of (4.11) goes as follows.

$$\langle z|a|\psi\rangle = \langle 0|e^{a\bar{z}} a|\psi\rangle = \frac{\partial}{\partial \bar{z}} \langle 0|e^{a\bar{z}}|\psi\rangle$$

$$= \frac{\partial}{\partial \bar{z}} \psi(\bar{z}); \quad \text{and}$$

$$\langle z|a^*|\psi\rangle = \langle 0|[e^{a\bar{z}}, a^*]|\psi\rangle = \bar{z} \langle 0|e^{a\bar{z}}|\psi\rangle$$

$$= \bar{z}\, \psi(\bar{z}).$$

Let A be an arbitrary bounded operator on $\mathcal{H}$. With A we associate the symbol $\langle z|A|z'\rangle$, so that

$$(A\psi)(\bar{z}) = \langle z|A|\psi\rangle$$
$$= \int \langle z|A|z'\rangle\, \psi(\bar{z}')\, e^{-|z'|^2} \frac{d\bar{z}' \wedge dz'}{2\pi i}$$

We consider the example of an operator A given by a normal-ordered polynomial in a^* and a:

$$A \equiv A(a^*, a) = \sum_{i,j} \alpha_{ij} (a^*)^i a^j.$$

Then

$$\langle z|A|z'\rangle = e^{\bar{z}z'} A(\bar{z}, z'). \tag{4.12}$$

To the product, $A \cdot B$, of operators A and B, there corresponds the "convolution" of their symbols:

$$\langle z|A\cdot B|z'\rangle = \int \langle z|A|z''\rangle \langle z''|B|z'\rangle e^{-|z''|^2} \frac{d\bar{z}'' \wedge dz''}{2\pi i}. \tag{4.13}$$

The Bargmann representation sketched here can be used as a starting point to derive formal path-integral expressions for Green functions of anharmonic oscillators: Let $H(a^*, a)$ be the normal-ordered Hamiltonian of an anharmonic oscillator. Then we have that

$$\langle z | e^{-itH} | z' \rangle = \langle z | \left(e^{-i\varepsilon H}\right)^N | z' \rangle \,, \quad \left(\varepsilon = \frac{t}{N}\right)$$

$$= \int_{\substack{z_N = z \\ z_0 = z'}} \left(\prod_{j=1}^{N-1} \frac{d\bar{z}_j \wedge dz_j}{2\pi i} e^{-|z_j|^2} \right) \left(\prod_{i=0}^{N-1} \langle z_{i+1} | e^{-i\varepsilon H} | z_i \rangle \right)$$

$$\simeq \int \left(\prod_{j=1}^{N-1} \frac{d\bar{z}_j \wedge dz_j}{2\pi i} e^{-|z_j|^2} \right) \left(\prod_{i=0}^{N-1} \langle z_{i+1} | (\mathbb{1} - i\varepsilon H) | z_i \rangle \right)$$

$$\simeq \int \left(\prod_{j=1}^{N-1} \frac{d\bar{z}_j \wedge dz_j}{2\pi i} e^{-|z_j|^2} \right) \left(\prod_{i=0}^{N-1} e^{\bar{z}_{i+1} z_i - i\varepsilon H(\bar{z}_{i+1}, z_i)} \right)$$

$$= \int \left(\prod_{j=1}^{N-1} \frac{d\bar{z}_j \wedge dz_j}{2\pi i} \right) e^{\frac{1}{2} \sum_{i=1}^{N-1} (\bar{z}_{i+1} - \bar{z}_i) z_i + \frac{1}{2} \sum_{i=1}^{N-1} \bar{z}_i (z_{i-1} - z_i)} \times$$

$$\times e^{\frac{1}{2}(\bar{z}_1 z' + \bar{z} z_{N-1}) - i\varepsilon \sum_{i=0} H(\bar{z}_{i+1}, z_i)}$$

$$\underset{\substack{\varepsilon \searrow 0 \\ (N \to \infty)}}{\simeq} \int_{\substack{z(t) = z \\ z(0) = z'}} \mathcal{D}\bar{z} \wedge \mathcal{D}z \, e^{\frac{1}{2}(\bar{z}' z' + \bar{z} z)} \times$$

$$\times \exp \int_0^t ds \left\{ \frac{1}{2} \left(\dot{\bar{z}}(s) z(s) - \bar{z}(s) \dot{z}(s) \right) - i H(\bar{z}(s), z(s)) \right\} \tag{4.14}$$

This formula may be derived (up to minor subtleties involving normal ordering and boundary terms) from expressions (2.9), (2.10) for the path integral over phase space by replacing $q + ip$ by z and $q - ip$ by $\bar{z}$.

In order to make sense of (4.14), we really ought to complexify the time variable $t \mapsto t e^{-i\theta}$, $\theta > 0$, using that $H \geq \text{const.} > -\infty$; see (2.20), (2.21), etc.

Then

$$\langle z | e^{-ite^{-i\theta}H} | z' \rangle$$

$$= \int_{\substack{z(t)=z \\ z(0)=z'}} \mathcal{D}\bar{z} \wedge \mathcal{D}z \; e^{\frac{1}{2}\left(\bar{z}' z' + \bar{z} z\right)} \times$$

$$\times \exp \int_0^t ds \left\{ \frac{1}{2}\left(\dot{\bar{z}}(s) z(s) - \bar{z}(s)\dot{z}(s)\right) - i e^{-i\theta} H\left(\bar{z}(s), z(s)\right)\right\} \tag{4.15}$$

As an example, we calculate the two-point function of a harmonic oscillator with Hamiltonian $H = \omega_0 a^* a$. We set

$$\left.\begin{aligned} a_t &:= e^{itH} a \, e^{-itH} = e^{-i\omega_0 t} a, \\ a_t^* &:= e^{itH} a^* e^{-itH} = e^{i\omega_0 t} a^*. \end{aligned}\right\} \tag{4.16}$$

Using that $a|z\rangle = z|z\rangle$, $\langle z | a^* = \bar{z}\langle z|$, and repeating the calculations leading to (4.14), we find that

$$e^{-i(t_1 - t_2)\omega_0} \underset{t_1 > t_2}{=} \langle 0 | a_{t_1} a_{t_2}^* | 0 \rangle$$

$$= \lim_{\theta \searrow 0} Z_\theta^{-1} \int \mathcal{D}\bar{z} \wedge \mathcal{D}z \; z(t_1) \bar{z}(t_2) \exp \int_{-\infty}^{\infty} ds \left\{ \frac{1}{2}\left(\dot{\bar{z}}(s) z(s) - \bar{z}(s)\dot{z}(s)\right) - i e^{-i\theta} \omega_0 |z(s)|^2 \right\} \tag{4.17}$$

To evaluate the R.S. of (4.17) we use Fourier transformation in the time variable s and then apply formula (2.19) for complex Gaussian integrals. This yields

$$\langle 0 | a_{t_1} a^*_{t_2} | 0 \rangle \underset{t_1 > t_2}{=} \lim_{\theta \searrow 0} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{i(\omega - e^{-i\theta}\omega_0)}$$

$$= e^{-i(t_1 - t_2)\omega_0}, \tag{4.18}$$

and, for $t_1 < t_2$,

$$\langle 0 | T[a_{t_1} a^*_{t_2}] | 0 \rangle = \langle 0 | a^*_{t_2} a_{t_1} | 0 \rangle$$

$$\underset{t_1 < t_2}{=} \lim_{\theta \searrow 0} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{i(\omega - e^{-i\theta}\omega_0)} = 0, \tag{4.19}$$

as expected; (see also (2.42), (2.43)).

Formulae (4.14) – (4.19) easily generalize to systems with arbitrarily many degrees of freedom, including bosonic field theories expressed in terms of creation- and annihilation operators; (see, e.g., the book by Faddeev and Slavnov). Here are some important formulae: We consider a scalar Bose field

$$\varphi(\vec{x}, 0) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(\vec{k})}} \left[a^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} + a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \right],$$

where $a^*(\vec{k})$ and $a(\vec{k})$ are the creation- and annihilation operators familiar from QFT I, and $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, where m is the mass of the field quanta. The free-field Hamiltonian, H_0, is given by

$$H_0 = \int d^3k \; a^*(\vec{k})\, \omega(\vec{k})\, a(\vec{k}). \qquad (4.20)$$

The interaction Hamiltonian is chosen to be

$$H_I = \int d^3x \; g(\vec{x}) : U(\varphi_\kappa(\vec{x})):, \qquad (4.21)$$

where g is a space cutoff, U is a polynomial that is bounded from below $\varphi_\kappa(\vec{x}) = \int d^3y \, \kappa(\vec{x} - \vec{y})\, \varphi(\vec{y})$, where κ is a smooth approximate δ-function (ultraviolet cutoff), and $:(\cdot):$ indicates Wick ordering (all a^*'s to the left of all a's). We expand H_I in Wick-ordered products of a^*'s and a's and write

$$H_I = V(a^*, a), \quad H := H_0 + V(a^*, a). \qquad (4.22)$$

We introduce coherent states

$$|\alpha\rangle := \exp\left(\int d^3k \; \alpha(\vec{k})\, a^*(\vec{k})\right)|0\rangle, \qquad (4.23)$$

where $|0\rangle$ is the Fock vacuum, with $a(\vec{k})|0\rangle = 0$.

Then

$$\left.\begin{aligned} a(\vec{k})\,|\alpha\rangle &= \alpha(\vec{k})\,|\alpha\rangle, \\ \langle\alpha|\,a^*(\vec{k}) &= \overline{\alpha(\vec{k})}\,\langle\alpha|. \end{aligned}\right\} \tag{4.24}$$

As in (4.14), we find that

$$U(\bar{\alpha},\alpha';t',t) := \lim_{\substack{\theta\searrow 0\\ t'>t}} \langle\alpha|e^{-ie^{-i\theta}(t'-t)H}|\alpha'\rangle,$$

with

$$\langle\alpha|e^{-ie^{-i\theta}(t'-t)H}|\alpha'\rangle$$

$$= \int_{\substack{\bar{\alpha}(t',\vec{k})=\overline{\alpha(\vec{k})}\\ \alpha(t,\vec{k})=\alpha'(\vec{k})}} \mathcal{D}\bar{\alpha}\wedge\mathcal{D}\alpha\ \exp\frac{1}{2}\int d^3k\left[\bar{\alpha}(\vec{k})\,\alpha(\vec{k},t')+\bar{\alpha}(\vec{k},t)\,\alpha'(\vec{k})\right]\times$$

$$\times\ \exp\int_t^{t'} ds\left[\int d^3k\left\{\frac{1}{2}\left(\dot{\bar{\alpha}}(\vec{k},s)\,\alpha(\vec{k},s)-\bar{\alpha}(\vec{k},s)\,\dot{\alpha}(\vec{k},s)\right)\right.\right.$$

$$\left.\left.-\,ie^{-i\theta}\omega(\vec{k})\,\bar{\alpha}(\vec{k},s)\,\alpha(\vec{k},s)\right\}-ie^{-i\theta}V(\bar{\alpha}(\cdot,s),\alpha(\cdot,s))\right]$$

$$\tag{4.25}$$

Defining the scattering matrix, S, by

$$S = \lim_{\substack{t'\to\infty\\ t\to-\infty}} e^{it'H_0}\,e^{-i(t'-t)H}\,e^{-itH_0}, \tag{4.26}$$

we find the path-integral expression for S:

$$S(\bar{\alpha}, \alpha') := \langle \alpha | S | \alpha' \rangle$$

$$= \lim_{\substack{t' \to \infty \\ t \to -\infty}} \lim_{\theta \searrow 0} \int \mathcal{D}\bar{\alpha} \wedge \mathcal{D}\alpha \, \exp \frac{1}{2} \int d^3k \left[\bar{\alpha}_{t'}(\vec{k}) \alpha(\vec{k}, t') + \bar{\alpha}(\vec{k}, t) \alpha'_t(\vec{k}) \right]$$

$$\times \exp \int_t^{t'} ds \left[\int d^3k \left\{ \frac{1}{2} \left(\dot{\bar{\alpha}}(\vec{k}, s) \alpha(\vec{k}, s) - \bar{\alpha}(\vec{k}, s) \dot{\alpha}(\vec{k}, s) \right) \right. \right.$$

$$\left. \left. - i e^{-i\theta} \omega(\vec{k}) \bar{\alpha}(\vec{k}, s) \alpha(\vec{k}, s) \right\} - i e^{-i\theta} V(\bar{\alpha}(\cdot, s), \alpha(\cdot, s)) \right],$$

(4.27)

with boundary conditions

$$\left. \begin{aligned} \bar{\alpha}(\vec{k}, t') &= e^{-it'\omega(\vec{k})} \overline{\alpha(\vec{k})} =: \bar{\alpha}_{t'}(\vec{k}), \\ \text{and} \quad \alpha'(\vec{k}, t) &= e^{it\omega(\vec{k})} \alpha'(\vec{k}) =: \alpha'_t(\vec{k}) \end{aligned} \right\} \quad (4.28)$$

From expression (4.27) one can derive the LSZ reduction formulae by manipulating path integrals in the Bargmann representation. This is a somewhat non-trivial exercise, and I won't do it here.

Instead, we move on to discussing Fermi fields in a similar spirit.

Coherent states for a quantum-mechanical fermionic degree of freedom.

A single quantum-mechanical fermionic degree of freedom is the same as an $s=\frac{1}{2}$ quantum-mechanical spin. The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$\left[\frac{\sigma_x}{2}, \frac{\sigma_y}{2}\right] = i\,\frac{\sigma_z}{2}, \quad \& \text{ cycl.}$$

We introduce the usual raising and lowering operators

$$\psi^* := \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi := \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad (4.29)$$

and the "vacuum"

$$|0\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (4.30)$$

We observe that

$$\psi|0\rangle = 0, \quad \{\psi^\#, \psi^\#\} = 0, \quad \{\psi, \psi^*\} = \mathbb{1} \qquad (4.31)$$

and

$$\sigma_z = 2\psi^*\psi - \mathbb{1}. \qquad (4.32)$$

Fermionic Fock space is two-dimensional, $\cong \mathbb{C}^2$, with basis $|0\rangle, \psi^*|0\rangle$. A general vector in $\mathbb{C}^2$ has the form $|f\rangle = f_0|0\rangle + f_1\psi^*|0\rangle$. Assuming that $\langle 0|0\rangle = 1$, we find from (4.31) that

$$\langle f|g\rangle = \bar{f}_0 g_0 + \bar{f}_1 g_1. \qquad (4.33)$$

Coherent states for fermions are defined by

$$|\theta\rangle = \exp(\theta\psi^*)|0\rangle = |0\rangle + \theta\psi^*|0\rangle \qquad (4.34)$$

Given a vector $|f\rangle \in \mathbb{C}^2$, we can associate with $|f\rangle$ a "wave function", $f(\bar\theta)$, in the "fermionic Bargmann representation" by setting

$$f(\bar\theta) = \langle\theta|f\rangle = f_0 + \bar\theta f_1 . \qquad (4.35)$$

Apparently, all such wave functions are linear functions of $\bar\theta$. This enables us to interpret $\bar\theta$ as a Grassmann variable ("anti-commuting c-number"/ $\{A,B\} := AB + BA$):

$$\{\bar\theta, \bar\theta\} = 2\bar\theta^2 = 0, \quad \bar\theta^n = 0, \text{ for } n \geq 2. \qquad (4.36)$$

In the representation of $\mathbb{C}^2$ given by (4.35), we have that

$$\psi^* = \bar\theta, \quad \psi = \frac{\partial}{\partial\bar\theta}, \qquad (4.37)$$

with

$$\left\{\frac{\partial}{\partial\bar\theta}, \frac{\partial}{\partial\bar\theta}\right\} = 2\frac{\partial^2}{\partial\bar\theta^2} = 0, \qquad (4.38)$$

$\left(\frac{\partial^n}{\partial\bar\theta^n} = 0, \text{ for all } n \geq 2\right)$.

We would like to reproduce the scalar product (4.33) by "integration of $f(\theta)g(\bar\theta)$ over θ and $\bar\theta$".

For this purpose, we introduce a second Grassmann variable, θ, and require that

$$\{\theta,\theta\} = \{\theta,\bar{\theta}\}=0, \ \left\{\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right\}=\left\{\frac{\partial}{\partial\theta},\frac{\partial}{\partial\bar{\theta}}\right\}=0. \quad (4.39)$$

We define <u>Berezin integration</u> as follows: We introduce differentials $d\theta$, $d\bar{\theta}$ dual to $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\bar{\theta}}$, respectively, (i.e., $\left\langle\frac{\partial}{\partial\theta}, d\theta\right\rangle = 1$, $\left\langle\frac{\partial}{\partial\theta}, d\bar{\theta}\right\rangle = 0$), with $\{\theta, d\theta\} = \{\bar{\theta}, d\bar{\theta}\} = \{\theta, d\bar{\theta}\} = \{\bar{\theta}, d\theta\} = 0$,

and
$$\{d\theta, d\theta\} = \{d\theta, d\bar{\theta}\} = \{d\bar{\theta}, d\bar{\theta}\} = 0. \quad (4.40)$$

Then
$$\int d\theta = \int d\bar{\theta} = 0, \quad \int \theta\, d\theta = \int \bar{\theta}\, d\bar{\theta} = 1. \quad (4.41)$$

With these rules and (4.35) we find that

$$\begin{aligned}\langle f|g\rangle &:= \int e^{-\bar{\theta}\theta}\, \overline{f(\bar{\theta})}\, g(\bar{\theta})\, d\bar{\theta}\, d\theta \\ &= \int (1-\bar{\theta}\theta)(\bar{f}_0+\bar{f}_1\theta)(g_0+\bar{\theta}g_1)\, d\bar{\theta}\, d\theta \\ &= \int (\bar{f}_0 g_0 + \bar{f}_1 g_1)\, \theta\bar{\theta}\, d\bar{\theta}\, d\theta \\ &\overset{(4.41)}{=} \bar{f}_0 g_0 + \bar{f}_1 g_1, \end{aligned} \quad (4.42)$$

as desired.

An arbitrary 2×2 Matrix, A, can be expanded in a quadratic polynomial in ψ and ψ^*:

$$A = A(\psi^*,\psi) = \alpha_{00} + \alpha_{10}\psi^* + \alpha_{01}\psi + \alpha_{11}\psi^*\psi . \qquad (4.43)$$

With this polynomial we can associate the symbol

$$\langle\theta|A|\eta\rangle = e^{\bar\theta\eta} A(\bar\theta,\eta), \qquad (4.44)$$

where $\theta, \bar\theta, \eta$ and $\bar\eta$ are four anti-commuting, independent Grassmann variables. Then

$$(Af)(\bar\theta) = \int e^{-\bar\eta\eta}\langle\theta|A|\eta\rangle f(\bar\eta)\, d\bar\eta\, d\eta \qquad (4.45)$$

This formula is a consequence of

$$\int e^{-\bar\eta\eta}|\eta\rangle\langle\eta|\, d\bar\eta\, d\eta = \mathbb{1}, \qquad (4.46)$$

which follows from (4.34) and (4.40), (4.41):

$$\int e^{-\bar\eta\eta}|\eta\rangle\langle\eta|\, d\bar\eta\, d\eta$$

$$= \int (1-\bar\eta\eta)\big(|0\rangle + \eta\psi^*|0\rangle\big)\big(\langle 0| + \langle 0|\psi\bar\eta\big) d\bar\eta\, d\eta$$

$$= |0\rangle\langle 0| + \psi^*|0\rangle\langle 0|\psi = \mathbb{1}.$$

Then

$$(Af)(\bar\theta) = \langle\theta|A|f\rangle \overset{(4.46)}{=} \int e^{-\bar\eta\eta}\langle\theta|A|\eta\rangle\langle\eta|f\rangle\, d\bar\eta\, d\eta$$

$$= \int e^{-\bar\eta\eta} e^{\bar\theta\eta}\big(\alpha_{00} + \alpha_{10}\bar\theta + \alpha_{01}\eta + \alpha_{11}\bar\theta\eta\big) f(\bar\eta)\, d\bar\eta\, d\eta$$

$$= \int (1-\bar{\eta}\eta)(1+\bar{\theta}\eta)\left(\alpha_{00}+\alpha_{10}\bar{\theta}+\alpha_{01}\eta+\alpha_{11}\bar{\theta}\eta\right)\times$$
$$\times\left(f_0+f_1\bar{\eta}\right)d\bar{\eta}\,d\eta$$

$$= \alpha_{00}f_0+\alpha_{01}f_1+\bar{\theta}\left(\alpha_{00}f_1+\alpha_{10}f_0+\alpha_{11}f_1\right) \quad (4.47)$$

This must be compared with

$$\langle\theta|A|f\rangle = \langle 0|(1+\psi\bar{\theta})\left(\alpha_{00}+\alpha_{10}\psi^*+\alpha_{01}\psi+\alpha_{11}\psi^*\psi\right)\times$$
$$\times\left(f_0+f_1\psi^*\right)|0\rangle$$

$$= \alpha_{00}f_0+\bar{\theta}\alpha_{00}f_1+\alpha_{01}f_1+\bar{\theta}\alpha_{10}f_0+\bar{\theta}\alpha_{11}f_1,$$

which agrees with (4.47).

From (4.45) and (4.46) we find that

$$\langle\theta|A\cdot B|\eta\rangle = \int e^{-\bar{\xi}\xi}\langle\theta|A|\xi\rangle\langle\xi|B|\eta\rangle\,d\bar{\xi}\,d\xi$$

$$= \int e^{-\bar{\xi}\xi+\bar{\theta}\xi+\bar{\xi}\eta}A(\bar{\theta},\xi)B(\bar{\xi},\eta)\,d\bar{\xi}\,d\xi. \quad (4.48)$$

Next, let $H(\psi^*,\psi)$ be the normal-ordered Hamiltonian of one $s=\frac{1}{2}$ quantum-mechanical spin. We propose to derive a Berezin-path-integral expression for the propagator $\exp(-ite^{-i\theta}H)$, $(\hbar=1,\ t>0,\ \theta>0)$. Our derivation follows the ideas used in (4.14):

$$\langle \xi | e^{-ite^{-i\theta}H} | \eta \rangle = \langle \xi | (e^{-i\varepsilon e^{-i\theta}H})^N | \eta \rangle, \quad \varepsilon = \frac{t}{N},$$

$$\overset{(4.48)}{=} \int_{\substack{\bar{\xi}_N = \bar{\xi} \\ \xi_0 = \eta}} \prod_{j=1}^{N-1} d\bar{\xi}_j \, d\xi_j \, e^{-\sum_{i=1}^{N-1} \bar{\xi}_i \xi_i} \prod_{i=0}^{N-1} \langle \xi_{i+1} | e^{-i\varepsilon e^{i\theta} H} | \xi_i \rangle$$

$$\underset{N\to\infty}{\simeq} \int \prod_{j=1}^{N-1} d\bar{\xi}_j \, d\xi_j \, e^{-\sum_{i=1}^{N-1} \bar{\xi}_i \xi_i} \prod_{i=0}^{N-1} \langle \xi_{i+1} | \mathbb{1} - i\varepsilon e^{-i\theta} H | \xi_i \rangle$$

$$\overset{(4.44)}{=} \int \prod_{j=1}^{N-1} d\bar{\xi}_j \, d\xi_j \, e^{-\sum_{i=1}^{N-1} \bar{\xi}_i \xi_i} \prod_{i=0}^{N-1} e^{\bar{\xi}_{i+1} \xi_i} \left(1 - i\varepsilon e^{-i\theta} H(\bar{\xi}_{i+1}, \xi_i)\right)$$

$$\underset{N\to\infty}{\simeq} \int_{\substack{\bar{\xi}(t) = \bar{\xi} \\ \xi(0) = \eta}} \mathcal{D}\bar{\xi} \, \mathcal{D}\xi \, e^{\frac{1}{2}\left(\bar{\xi}\xi(t) + \overline{\xi(0)}\eta\right)} \times$$

$$\times \, e^{\int_0^t ds \left\{ \frac{1}{2}\left(\dot{\bar{\xi}}(s)\xi(s) - \bar{\xi}(s)\dot{\xi}(s)\right) - ie^{-i\theta} H(\bar{\xi}(s), \xi(s)) \right\}} \tag{4.49}$$

Apparently, the Lagrangian of a single fermionic degree of freedom with Hamiltonian $H(\psi^*, \psi)$ must be defined to be

$$L_\theta(\bar{\xi}, \xi, \dot{\bar{\xi}}, \dot{\xi}) = i\bar{\xi}\dot{\xi} - e^{-i\theta} H(\bar{\xi}, \xi), \tag{4.50}$$

where $\xi = (\xi(s))_{s\in[0,t]}$, $\bar{\xi} = (\bar{\xi}(s))_{s\in[0,t]}$ are "Grassmann-variable valued paths" on $[0,t]$; with $\{\xi^\#(s), \xi^\#(s')\} = \{\xi(s), \bar{\xi}(s')\} = 0$, $s, s' \in [0,t]$. Then

$$\langle \xi | e^{-ite^{-\theta}H} | \eta \rangle$$

$$= \int_{\substack{\bar\xi(t)=\bar\xi \\ \xi(0)=\eta}} \mathcal{D}\bar\xi\, \mathcal{D}\xi \; \exp i \int_0^t ds\, L_\theta\big(\bar\xi(s), \xi(s), \dot{\bar\xi}(s), \dot\xi(s)\big) \tag{4.51}$$

Time-ordered vacuum expectation values are introduced as in bosonic systems: Let $\hat{O}_i$, $i = 1, \ldots, n$, be operators written as polynomials in ψ^* and ψ. Let

$$\hat{O}_i(t) := e^{itH}\, \hat{O}_i\, e^{-itH}. \tag{4.52}$$

Then

$$\langle \Omega | T[\hat{O}_n(t_n) \cdots \hat{O}_1(t_1)] | \Omega \rangle$$

$$= \lim_{\theta \searrow 0} \frac{1}{Z_\theta} \int \mathcal{D}\bar\xi\, \mathcal{D}\xi \; O_n(\bar\xi(t_n), \xi(t_n)) \cdots O_1(\bar\xi(t_1), \xi(t_1)) \times$$

$$\times \exp i \int_{-\infty}^{\infty} ds\, L_\theta\big(\bar\xi(s), \xi(s), \dot{\bar\xi}(s), \dot\xi(s)\big), \tag{4.53}$$

(a limit, as a time-cutoff $T \to \infty$, is understood, and it is assumed that the groundstate, $|\Omega\rangle$, of the Hamiltonian H is <u>not</u> orthogonal to $|0\rangle$). Wick rotation to imaginary time corresponds to the choice $\theta = \frac{\pi}{2}$. Of course, formula (4.53) is an absurdity: It replaces simple calculations with 2×2 matrices (L.S.) by a rather formal functional integral (R.S.). That this absurdity has its virtues will become apparent when we consider

<u>Systems of many fermionic degrees of freedom.</u>

We consider a system of $N = 1,2,3,\cdots$ $s=\frac{1}{2}$ quantum-mechanical spins on a Hilbert space $(\mathbb{C}^2)^{\otimes N} \simeq \mathbb{C}^{2^N}$.

We introduce the Pauli matrices

$$\sigma_j^{\ell} := \mathbb{1}_2 \otimes \cdots \otimes \underset{\substack{\uparrow \\ \ell^{th}\ \text{position}}}{\sigma_j} \otimes \cdots \otimes \mathbb{1}_2\,, \quad j = x,y,z$$

We define raising and lowering operators by

$$a_\ell^* = \sigma_x^\ell + i\sigma_y^\ell\,, \quad a_\ell = \sigma_x^\ell - i\sigma_y^\ell\,.$$

We consider the <u>Klein-Jordan-Wigner transformed variables</u>:

$$\psi_\ell^* := a_\ell^* \prod_{k=1}^{\ell-1} \sigma_z^k\,, \quad \psi_\ell := \left(\prod_{k=1}^{\ell-1} \sigma_z^k\right) a_\ell \qquad (4.54)$$

Note that

$$\{\sigma_z^\ell, a_\ell\} = \{\sigma_z^\ell, a_\ell^*\} = 0;$$

but
$$[\sigma_z^\ell, a_k] = [\sigma_z^\ell, a_k^*] = 0, \quad \text{for } k \neq \ell.$$

Furthermore, $\{a_\ell^\#, a_\ell^\#\} = 0$, but $\{a_\ell, a_\ell^*\} = \mathbb{1}$, and $[a_\ell^\#, a_k^\#] = [a_\ell, a_k^*] = 0$, for $\ell \neq k$. It then follows from (4.54) that

$$\{\psi_\ell^\#, \psi_k^\#\} = 0, \ \{\psi_\ell, \psi_k^*\} = \delta_{\ell k}, \tag{4.55}$$

for all $\ell, k = 1, \dots, N$.

Arbitrary operators on $\mathcal{F}_N := \mathbb{C}^{2^N}$ can be expanded in Wick-ordered polynomials in $\psi_1^*, \dots, \psi_N^*, \psi_1, \dots, \psi_N$. Let $H(\psi^*, \psi)$ be a Hamiltonian on $\mathcal{F}_N$. We are interested in deriving a Berezin-path-integral expression for $\exp(-itH)$. For this purpose, we introduce the Grassmann algebra, G_{2n}, in $2n$ generators $\xi_1, \bar{\xi}_1, \dots, \xi_n, \bar{\xi}_n$, $n = 1, 2, 3, \dots$, with relations

$$\{\xi_i^\#, \xi_j^\#\} = \{\xi_i, \bar{\xi}_j\} = 0, \quad \text{for all } i, j. \tag{4.56}$$

Mathematically,

$$G_{2n} = \bigoplus_{\ell=0}^{2n} \Lambda^\ell (V \oplus \bar{V}), \tag{4.57}$$

with $V \simeq \bar{V} \simeq \mathbb{C}^n$.

In (4.57),

$$\Lambda^0(V \oplus \bar{V}) = \mathbb{C}, \quad \Lambda^1(V \oplus \bar{V}) = V \oplus \bar{V},$$

and $\Lambda^l(V \oplus \bar{V})$ is the space of totally anti-symmetric tensors in $(V \oplus \bar{V}) \otimes \cdots \otimes (V \oplus \bar{V})$, ($l$ factors). Let $\xi_1, \cdots, \xi_n$ be a basis of V, and $\bar{\xi}_1 \equiv \xi_{n+1}, \cdots, \bar{\xi}_n \equiv \xi_{2n}$ a basis of $\bar{V}$. A vector space basis of G_{2n} is given by the totally anti-symmetric tensors

$$\xi_{i_1} \wedge \cdots \wedge \xi_{i_k} \wedge \bar{\xi}_{j_1} \wedge \cdots \wedge \bar{\xi}_{j_l},$$

where
$$\xi_{l_1} \wedge \cdots \wedge \xi_{l_m} = \frac{1}{m!} \sum_{\pi \in S_m} \xi_{l_{\pi(1)}} \otimes \cdots \otimes \xi_{l_{\pi(m)}},$$

$l_1, \cdots, l_m \in \{1, \cdots, 2n\}$. It is obvious that

$$\dim\left(\Lambda^l(V \oplus \bar{V})\right) = \binom{2n}{l},$$

with
$$\Lambda^l(V \oplus \bar{V}) = \{0\}, \text{ for } l > 2n.$$

Hence
$$\dim G_{2n} = 2^{2n}.$$

The vector space G_{2n} becomes a (tensor) algebra over $\mathbb{C}$ with the product $\wedge$. We will omit the symbol $\wedge$, but impose the relations

$$\{\xi_i, \xi_j\} = \{\bar{\xi}_i, \bar{\xi}_j\} = \{\xi_i, \bar{\xi}_j\} = 0. \qquad (4.58)$$

We may then view G_{2n} as the algebra of polynomials in the anti-commuting variables $\xi_1, \dots, \xi_n, \bar{\xi}_1, \dots, \bar{\xi}_n$ with complex coefficients and involution

$$P(\xi_i, \bar{\xi}_j)^{+} := \bar{P}(\xi_j, \bar{\xi}_i) \tag{4.59}$$

For $n = N$, we have an isomorphism between the fermionic Fock space $\mathcal{F}_N$ and the subspace of G_{2N} spanned by polynomials in $\bar{\xi}_1, \dots, \bar{\xi}_N$:

$$\left.\begin{aligned} &\psi^*_{i_1} \cdots \psi^*_{i_k} |0\rangle \mapsto \bar{\xi}_{i_1} \cdots \bar{\xi}_{i_k}, \\ &\text{with } \psi_i = \frac{\partial}{\partial \bar{\xi}_i}, \ \psi^*_i = \bar{\xi}_i, \ i = 1, \dots, N. \end{aligned}\right\} \tag{4.60}$$

Next, we introduce <u>Berezin integration</u>:

$$\int d\xi_i = \int d\bar{\xi}_i = 0, \quad \int \xi_i \, d\xi_i = \int \bar{\xi}_i \, d\bar{\xi}_i = 1, \tag{4.61}$$

$i = 1, \dots, n$; and

$$\int \prod_{i=1}^{n} f_i(\xi_i) \prod_{j=1}^{n} g_j(\bar{\xi}_j) \, d\bar{\xi}_1 \cdots d\bar{\xi}_n \, d\xi_1 \cdots d\xi_n$$

$$:= \prod_{i=1}^{n} \int f_i(\xi_i) \, d\xi_i \prod_{j=1}^{n} \int g_j(\bar{\xi}_j) \, d\bar{\xi}_j . \tag{4.62}$$

This definition is consistent with the rules

$$\{\xi_i^{\#}, d\xi_j^{\#}\} = \{\xi_i, d\bar{\xi}_j\} = \{d\xi_i^{\#}, d\xi_j^{\#}\} = \{d\xi_i, d\bar{\xi}_j\} = 0. \tag{4.63}$$

Henceforth, we use the short-hand notation

$$\mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi := d\bar{\xi}_1 \cdots d\bar{\xi}_n \, d\xi_1 \cdots d\xi_n . \tag{4.64}$$

Let $\bar{\xi}\cdot\xi := \sum_{i=1}^{N} \bar{\xi}_i \xi_i$. Then the scalar product on $\mathcal{F}_N$ is given by

$$\langle f | g \rangle = \int e^{-\bar{\xi}\cdot\xi} \, \bar{f}(\xi) \, g(\bar{\xi}) \, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi . \tag{4.65}$$

We also impose rules on the formal vector fields $\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \bar{\xi}_i}$: $\left\{ \frac{\partial}{\partial \xi_i^{\#}}, \frac{\partial}{\partial \xi_j^{\#}} \right\} = \left\{ \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \bar{\xi}_j} \right\} = 0$, and

$$\left\langle \frac{\partial}{\partial \xi_i^{\#}}, d\xi_j^{\#} \right\rangle = \delta_{ij} , \quad \left\langle \frac{\partial}{\partial \xi_i}, d\bar{\xi}_j \right\rangle = 0, \tag{4.66}$$

for all i, j. Then

$$\int f(\xi, \bar{\xi}) \, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi = \frac{\partial}{\partial \bar{\xi}_1} \cdots \frac{\partial}{\partial \bar{\xi}_n} \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_n} f(\xi, \bar{\xi}) \tag{4.67}$$

In order to carry out explicit calculations with Berezin integrals, we need some formulae for <u>Gaussian Berezin integrals</u> and for <u>changes of variables</u>.

(1) Let A be an $n \times n$ complex matrix, and let $\eta_1, \dots, \eta_n, \bar{\eta}_1, \dots, \bar{\eta}_n$ be Grassmann variables generating $\mathcal{G}_{2n}$. Then

$$\int e^{-\bar{\xi}^T A \xi + i\bar{\eta}^T \xi + i\bar{\xi}^T \eta} \, \partial\bar{\xi} \wedge \partial\xi$$

$$= \det A \cdot \exp[-\bar{\eta}^T A^{-1} \eta]. \qquad (4.68)$$

Proof. The dependence of the L.S. of (4.68) on

$$\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \quad \text{and} \quad \bar{\eta}^T = (\bar{\eta}_1, \cdots, \bar{\eta}_n)$$

follows by quadratic completion in the exponent of the integrand on the L.S. of (4.68), provided we can show that

$$\int f(\xi+\eta, \bar{\xi}+\bar{\eta}) \, \partial\bar{\xi} \wedge \partial\xi = \int f(\xi, \bar{\xi}) \, \partial\bar{\xi} \wedge \partial\xi. \quad (4.69)$$

This identity follows from (4.61) by expanding f in its Taylor series,

$$f(\xi+\eta, \bar{\xi}+\bar{\eta}) = f^{(0)} + \sum_i \{ f^{(1)}_i (\xi_i + \eta_i) + f^{(1)}_{\bar{i}} (\bar{\xi}_{\bar{i}} + \bar{\eta}_{\bar{i}}) \}$$

$$+ \cdots + f^{(2n)}_{1,\cdots,n,\bar{1},\cdots,\bar{n}} \prod_{i=1}^{n} (\xi_i + \eta_i) \prod_{\bar{j}=1}^{n} (\bar{\xi}_{\bar{j}} + \bar{\eta}_{\bar{j}}).$$

Then the L.S. and the R.S. of (4.69) are both equal to $f^{(2n)}_{1,\cdots,n,\bar{1},\cdots,\bar{n}}$.

Hence it suffices to show that

$$\int e^{-\bar{\xi}^T A \xi} \, \partial\bar{\xi} \wedge \partial\xi = \det A.$$

To show this, we expand $\exp[-\bar{\xi}^T A \xi]$ in its Taylor series. Clearly, by (4.61),

$$\int \frac{(-1)^m}{m!} \bar{\xi}_{i_1} A^{i_1 j_1} \xi_{j_1} \cdots \bar{\xi}_{i_m} A^{i_m j_m} \xi_{j_m} \, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi = 0,$$

for all $m \neq n$, and

$$\int \frac{(-1)^n}{n!} \bar{\xi}_{i_1} A^{i_1 j_1} \xi_{j_1} \cdots \bar{\xi}_{i_n} A^{i_n j_n} \xi_{j_n} \, d\bar{\xi}_1 \cdots d\bar{\xi}_n \, d\xi_1 \cdots d\xi_n$$

$$= \int (-1)^n \bar{\xi}_1 A^{1 j_1} \xi_{j_1} \cdots \bar{\xi}_n A^{n j_n} \xi_{j_n} \, d\bar{\xi}_1 \cdots d\bar{\xi}_n \, d\xi_1 \cdots d\xi_n$$

$$= \sum_{\pi \in S_n} \operatorname{sig} \pi \, A^{1\pi(1)} \cdots A^{n\pi(n)} \underbrace{\int (-1)^n \bar{\xi}_1 \xi_1 \cdots \bar{\xi}_n \xi_n \, d\bar{\xi}_1 \cdots d\bar{\xi}_n \times d\xi_1 \cdots d\xi_n}_{=1}$$

$$= \det A .$$

(We have used the summation convention.)

Formula (4.68) is the Berezin analogue of formula (2.19) for Gaussian integrals on $\mathbb{C}^n$. There is also an analogue to formula (2.16) for Gaussian integrals on $\mathbb{R}^n$: Let n be even, and let A be an anti-symmetric $n \times n$ matrix. Then

$$\int e^{-\xi^T A \xi} \, d\xi_1 \cdots d\xi_n = 2^{\frac{n}{2}} \sqrt{\det A}, \qquad (4.70)$$

or, more generally,

$$\int e^{-\xi^T A \xi + i\eta^T \xi} d\xi_1 \cdots d\xi_n = e^{-\frac{1}{4}\eta^T A^{-1}\eta} 2^{\frac{n}{2}} \sqrt{\det A} \quad (4.71)$$

The proof will be found in the exercises; (see also the book by Pierre Ramond).

(2) Let J be a regular $n \times n$ matrix. Then

$$\int f(J\xi)\, d\xi_1 \cdots d\xi_n = \det J \int f(\xi)\, d\xi_1 \cdots d\xi_n \quad (4.72)$$

Proof. Follows from (4.67); (see exercises).

We are now prepared to apply the general formalism developed above to fermionic systems with ∞ many degrees of freedom. We propose to generalize formulae (4.49) – (4.51). Given a Dirac field $\psi(x)$, with $\bar{\psi}(x)$ the conjugate field, we introduce a formal, ∞ - dimensional Grassmann algebra

$$\bigwedge_{x \in \mathbb{R}^d} G_{2 \cdot 2^{[\frac{d}{2}]}}(x),$$

with generators $\Psi(x)$, $\bar{\Psi}(x)$, $x \in \mathbb{R}^d$, satisfying

$$\{\Psi(x), \Psi(y)\} = \{\bar{\Psi}(x), \bar{\Psi}(y)\} = \{\Psi(x), \bar{\Psi}(y)\} = 0,$$

for arbitrary x, y in $\mathbb{R}^d$. The Lagrangian density is given by

$$\mathcal{L}_\theta(\bar{\Psi}(x), \Psi(x)) := i\bar{\Psi}(x)\dot{\Psi}(x) - e^{-i\theta}\mathcal{H}(\bar{\Psi}(x), \Psi(x)) \quad (4.73)$$

where $\mathcal{H}(\bar{\Psi}(x), \Psi(x))$ is ~~ob~~tained from the Hamiltonian density $\mathcal{H}(\bar{\psi}(\vec{x},0), \psi(\vec{x},0))$ by replacing $\bar{\psi}(\vec{x},0)$ by $\bar{\Psi}(x)$ and $\psi(\vec{x},0)$ by $\Psi(x)$. Let $\hat{O}_i$, $i = 1, \dots, n$, be polynomials in $\bar{\psi}(\vec{x},0), \psi(\vec{x},0)$, and $\hat{O}_i(t) = e^{itH}\hat{O}_i e^{-itH}$, where

$$H = \int d^{d-1}x \; \mathcal{H}(\bar{\psi}(\vec{x},0), \psi(\vec{x},0))$$

is the Hamiltonian of the theory. Let $|\Omega\rangle$ be the groundstate of H, and assume that $\langle 0|\Omega\rangle \neq 0$, where $|0\rangle$ is the Fock vacuum. Then, by analogy with (4.49) – (4.53), we find that

$$\langle\Omega| T[\hat{O}_n(t_n) \cdots \hat{O}_1(t_1)] |\Omega\rangle$$

$$= \lim_{\theta \searrow 0} \frac{1}{Z_\theta} \int \mathcal{D}\bar{\Psi} \wedge \mathcal{D}\Psi \; O_n(\bar{\Psi}(\cdot, t_n), \Psi(\cdot, t_n)) \cdots$$

$$\times\, O_1(\bar{\Psi}(\cdot, t_1), \Psi(\cdot, t_1)) \exp i \int_{\mathbb{R}^d} d^d x \, \mathcal{L}_\theta(\bar{\Psi}(x), \Psi(x)) \quad (4.74)$$

Let us consider the example of a free, massive Dirac field. As discussed in QFT I (see also Exercises), the Dirac spinor field $\psi(\vec{x},t)$ and its adjoint $\psi^*(\vec{x},t)$ satisfy the following equal-time canonical anti-commutation relations:

$$\left.\begin{aligned} \{\psi^{\#}_{\alpha}(\vec{x},t), \psi^{\#}_{\beta}(\vec{y},t)\} &= 0, \\ \{\psi_{\alpha}(\vec{x},t), \psi^{*}_{\beta}(\vec{y},t)\} &= \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y}), \end{aligned}\right\} \quad (4.75)$$

$(\hbar=1)$. These anti-commutation relations generalize the relations (4.55) to a system with ∞ many fermionic degrees of freedom.

The Lagrangian of a Dirac field has the form

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}(\partial_{\mu}-iA_{\mu})\psi - m\bar{\psi}\psi - V(\bar{\psi},\psi,\varphi), \quad (4.76)$$

where $\bar{\psi} := \psi^{*}\gamma^{0}$ is the conjugate spinor, A_{μ} is a gauge field, e.g., the electromagnetic vector potential of QED, and $V(\bar{\psi},\psi,\varphi)$ describes the coupling of the Dirac field to a scalar field φ, (e.g., a Higgs field). In this chapter, we interpret A_μ and φ as (classical) external fields, and we may think of V as describing some Yukawa coupling:

$$V(\bar{\psi}, \psi, \varphi) = g\,\bar{\psi}\psi\,\varphi,$$

where g is a coupling constant.

The Lagrangian (4.76) is invariant under <u>local gauge transformations</u>:

$$\left.\begin{aligned} \psi(x) &\mapsto \psi^{(\alpha)}(x) := e^{i\alpha(x)}\psi(x), \\ \bar{\psi}(x) &\mapsto \bar{\psi}^{(\alpha)}(x) := \bar{\psi}(x)\,e^{-i\alpha(x)}, \\ A_\mu(x) &\longmapsto A_\mu(x) + \partial_\mu \alpha(x). \end{aligned}\right\} \qquad (4.77)$$

The Lagrange functional, $L(\bar{\psi}, \psi; A_\mu, \varphi)$, is defined to be

$$L(\bar{\psi}, \psi; A_\mu, \varphi) = \int d^3x\; \mathcal{L}\big(\bar{\psi}(\vec{x},t), \psi(\vec{x},t); A_\mu(\vec{x},t), \varphi(\vec{x},t)\big)$$

In the temporal gauge, $A_0 = 0$, L has the form

$$L = \int d^3x\; \psi^*(\vec{x},t)\, i\frac{\partial}{\partial t}\psi(\vec{x},t) - H\big(\psi^*(\cdot,t), \psi(\cdot,t)\big),$$

where H is the <u>Hamiltonian</u> of the theory.

The action functional of the theory is given by

$$S = \int d^4x \; \mathcal{L}(\bar{\psi}(x), \psi(x); A_\mu(x), \varphi(x))$$

$$= \int dt \; L(\bar{\psi}(\cdot,t), \psi(\cdot,t); A_\mu(\cdot,t), \varphi(\cdot,t))$$

In order to come up with a path integral representation of the time-ordered Green functions of Dirac- and conjugate Dirac spinor fields $(\psi, \bar{\psi})$, we propose to replace $\psi(x)$ by $\Psi(x)$ and $\bar{\psi}(x)$ by $\bar{\Psi}(x)$, where $\Psi(x)$ and $\bar{\Psi}(x)$ are the generators of an ∞ dimensional Grassmann algebra; see (4.73). Then we have that

$$\langle 0 | T[\psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_1}(y_1) \cdots \psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_n}(y_n) \times$$

$$\exp i \int d^4x \; \mathcal{L}_I(\bar{\psi}(x), \psi(x); A_\mu(x), \varphi(x))] | 0 \rangle$$

$$= Z(A_\mu = 0; \varphi = 0)^{-1} \int \mathcal{D}\bar{\Psi} \wedge \mathcal{D}\Psi \; e^{i\int d^4x \, \mathcal{L}(\bar{\Psi}, \Psi; A_\mu, \varphi)} \times$$

$$\times \Psi_{\alpha_1}(x_1)\bar{\Psi}_{\beta_1}(y_1) \cdots \Psi_{\alpha_n}(x_n)\bar{\Psi}_{\beta_n}(y_n), \quad (4.78)$$

where

$$\mathcal{L}_I(\bar{\psi}, \psi; A_\mu, \varphi) = \mathcal{L}(\bar{\psi}, \psi; A_\mu, \varphi) - \mathcal{L}(\bar{\psi}, \psi; A_\mu = 0, \varphi = 0),$$

and

$$Z(A_\mu, \varphi) := \int \mathcal{D}\bar{\Psi} \wedge \mathcal{D}\Psi \; e^{i\int d^4x \, \mathcal{L}(\bar{\Psi}, \Psi; A_\mu, \varphi)} ; \quad (4.79)$$

see (4.74).

In order to define (4.78) and (4.79) more precisely, we must perform a Wick rotation, $t \mapsto t e^{-i\theta}$, $\theta > 0$, and introduce a space cutoff ($A_\mu = 0$, $\varphi = 0$, outside a compact region Ω of space-time) and an ultraviolet regularization (e.g. Pauli-Villars). One must then remove the UV regularization, let $\Omega \nearrow M^4$ and pass to the limit $\theta \searrow 0$.

Wick's theorem (see QFT I) follows from the observation that the Grassmann integral on the R.S. of (4.78) is Gaussian. If $A_\mu = 0$, $\varphi = 0$ then (4.78) yields:

$$\langle 0 | T[\psi_{\alpha_1}(x_1) \cdots \bar{\psi}_{\beta_n}(y_n)] | 0 \rangle$$

$$= \sum_{\pi \in S_n} \operatorname{sig} \pi \prod_{j=1}^{n} S_{\alpha_j \beta_{\pi(j)}}(x_j - y_{\pi(j)}), \quad (4.80)$$

where

$$S_{\alpha\beta}(x) = -i \lim_{\varepsilon \searrow 0} \int \frac{(\not{p} - m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} \, e^{ip\cdot x} \frac{d^4p}{(2\pi)^4}, \quad (4.81)$$

as follows from formula (4.68) in the limit where $N \to \infty$.

We define an "effective action" by setting

$$S_{eff}(A_\mu, \varphi) := -i \ln Z(A_\mu, \varphi). \tag{4.82}$$

Recalling the gauge invariance of the Lagrange density,

$$\mathcal{L}(\bar{\psi}^{(\alpha)}, \psi^{(\alpha)}; A_\mu + \partial_\mu \alpha, \varphi) = \mathcal{L}(\bar{\psi}, \psi; A_\mu, \varphi), \tag{4.83}$$

with $\psi^{(\alpha)}$ and $\bar{\psi}^{(\alpha)}$ as in (4.77), and noticing that

$$\mathcal{D}\bar{\psi}^{(\alpha)} \wedge \mathcal{D}\psi^{(\alpha)} = \mathcal{D}\bar{\psi} \wedge \mathcal{D}\psi, \tag{4.84}$$

see (4.72) (with $N \to \infty$), we find that

$$S_{eff}(A_\mu + \partial_\mu \alpha, \varphi) = S_{eff}(A_\mu, \varphi), \tag{4.85}$$

i.e., the effective action is gauge-invariant.

5. Functional-integral quantization of (non-abelian) gauge theories

The prototype of a gauge theory is QED, as discussed in QFT I. It describes the interactions between charged leptons, e, μ and τ, and photons, γ. The field describing e.g. electrons and positrons is a massive Dirac field ψ, as considered in the last chapter. The photons are described by the quantized electromagnetic vector potential, A_μ. The Lagrangian of QED is given by

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu \nabla_\mu \psi - m\bar{\psi}\psi, \quad (5.1)$$

where $\bar{\psi} = \psi^* \gamma^0$ is the conjugate spinor,

$$\nabla_\mu = \partial_\mu - iA_\mu \quad (5.2)$$

is the covariant derivative, and

$$F_{\mu\nu} = i[\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (5.3)$$

is the electromagnetic field tensor.

$\mathcal{L}$ is invariant under gauge transformations,

$$M^4 \ni x \longmapsto e^{i\alpha(x)} \in U(1).$$

Under such gauge transformations, A_μ and ψ transform as follows:

$$\left.\begin{aligned} \psi(x) &\mapsto \psi^{(\alpha)}(x) := e^{i\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\mapsto \bar{\psi}^{(\alpha)}(x) := \bar{\psi}(x)e^{-i\alpha(x)} \\ A_\mu(x) &\mapsto A_\mu^{(\alpha)}(x) := A_\mu(x) + \partial_\mu\alpha(x) \end{aligned}\right\} \quad (5.4)$$

In this course, we study non-abelian generalizations of QED, in particular the theory of strong interactions: QCD. In order to prepare the grounds for our analysis, we recall the definition of Lie groups, Lie algebras, principal bundles and associated vector bundles.

5.1. Semi-simple (compact) Lie groups and their Lie algebras

A Lie group, G, is a continuous group that is a smooth manifold. Standard examples are

$G = U(1) \simeq S^1$, $G = SU(2) \simeq S^3$, $G = SO(3) \simeq S^3/\mathbb{Z}_2$.

The group operations,

$$g : G \to G, \quad g' \mapsto g \cdot g', \quad g' \in G$$

and

$$g \to g^{-1},$$

$g \in G$ are assumed to be differentiable maps. Left translations, $L_g : g' \mapsto gg'$, and right translations, $R_g : g' \mapsto g'g$, $g \in G$, are surjective and injective. They act transitively on G. Any $g \in G$ can be mapped to the identity element, e, by a unique left- and right translation. Let $T_g G$ denote the tangent space to G at the point $g \in G$. By the differentials of the left- or right translations, $L_{g^{-1}}$, $R_{g^{-1}}$, $T_g G$ is mapped to $T_e G$. A basis of vectors in $T_g G$ is then mapped to a basis in $T_e G$, and, thanks to the group multiplication law, there is no non-trivial holonomy in this parallel transport. One thus obtains a global moving frame from a basis in $T_g G$, (for any $g \in G$). It follows, that Lie groups are what one calls parallelizable manifolds; in particular they are orientable. Furthermore, by left- or right translations, Lie groups admit (bases of) global, nowhere vanishing vector fields, i.e., the tangent bundle, TG, of G is trivial. Global vector fields constructed by left- or right translations from a vector $X \in T_g G$ are left-(translation) invariant, right-(translation) invariant, resp., by construction. They span a vector space of dimension $\dim (T_g G) = \dim G$.

Given two arbitrary vector fields, X and Y, on a a manifold M, one can form their Lie bracket, $[X, Y]$, which is again a vector field on M. Thus, vector fields on a smooth manifold form an ∞ dimensional Lie algebra. If M is a Lie group, G, then the left- (or right-) invariant vector fields form a finite- (dim G-) dimensional Lie-subalgebra, denoted $\mathfrak{g}$. By construction, $\mathfrak{g} \simeq T_e G \simeq T_g G$, as a linear space, for any $g \in G$. The Lie bracket on TG equips $\mathfrak{g}$, and hence $T_e G$, with a bracket, $[\cdot,\cdot]$. This Lie bracket is <u>bilinear</u>, <u>anti-symmetric</u> and satisfies the <u>Jacobi identity</u>. Equipped with $[\cdot,\cdot]$, $\mathfrak{g}$ is called the <u>Lie algebra</u> of the Lie group G.

Given an abstract, finite-dimensional, real Lie algebra, $\mathfrak{g}$, there is a unique <u>connected</u>, <u>simply connected</u> Lie group, $\tilde{G}$, with the property that $T_e \tilde{G} \simeq \mathfrak{g}$. ($M$ is connected $\Leftrightarrow$ $\pi_0(M) = \{id.\}$; M is simply connected $\Leftrightarrow$ $\pi_1(M) = \{id.\}$.) If G is any other <u>connected</u> Lie group with Lie algebra $\mathfrak{g}$ then there is a surjective group homomorphism $\varphi : \tilde{G} \to G$ with the property that

$\ker\varphi \subseteq Z(\tilde{G})$, where $Z(\tilde{G})$ is the center of $\tilde{G}$; $\tilde{G}$ is called the universal covering group correspon- ding to $\mathfrak{g}$.

A Lie group G is called compact iff it is a compact manifold.

The exponential map

A one parameter subgroup $\{g(t) \in G \mid t \in \mathbb{R}\}$ is a subgroup of G $(\pi_0(G) = \{id.\})$ with the property that

$$g(t) \cdot g(s) = g(t+s), \quad t, s \in \mathbb{R}, \qquad (5.5)$$
$$(g(0) = e).$$

A one-parameter subgroup $\{g(t) \mid t \in \mathbb{R}\}$ is uniquely determined by its derivative in t at $t=0$:

$$X := \left.\frac{dg(t)}{dt}\right|_{t=0} \in \mathfrak{g} \qquad (5.6)$$

We define the exponential map $\exp: \mathfrak{g} \to G$ by setting

$$\exp(tX) := g(t), \qquad (5.7)$$

with X as in (5.6).

Note that when G is a matrix group "exp" is the usual exponential power series of a matrix. We also remark that G can be equipped with a Riemannian metric with the property that the one-parameter subgroups are geodesics. In this metric, "exp" is the usual exponential map of Riemannian geometry.

For a finite-dimensional Lie group G, there is an open neighborhood, N, of $e \in G$ such that every $g \in N$ can be uniquely written as

$$g = \exp X \,, \quad X \in \mathfrak{g} \,; \tag{5.8}$$

i.e., different one-parameter subgroups of G do not intersect each other in N, except at $e \in N$.

For compact Lie groups, "exp" is surjective.

If $\tilde{G}$ is compact the representations, ρ, of $\tilde{G}$ are in one-one correspondence with representations, $d\rho$, of $\mathfrak{g}$; ρ and $d\rho$ act on the same vector space.

There is a particular representation, Ad, the so-called adjoint representation of G, on its Lie algebra $\mathfrak{g}$ defined as follows:

$$Ad(g)\, X := g X g^{-1} \,, \quad g \in G, \; X \in \mathfrak{g}, \tag{5.9}$$

with

$$gXg^{-1} = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\left[g \exp(\varepsilon X) g^{-1} - e\right] \quad (5.10)$$

Ad corresponds to a representation, ad, of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad}(X)Y := \frac{d}{d\varepsilon} \mathrm{Ad}(\exp(\varepsilon X))Y\Big|_{\varepsilon=0}$$
$$= [X, Y]. \quad (5.11)$$

The Lie algebra $\mathfrak{g}$ is equipped with an invariant, symmetric bilinear form, the so-called Killing form, $(\cdot\,,\cdot)$, which is defined by

$$(X,Y) := \mathrm{Tr}\left(\mathrm{ad}(X)\cdot \mathrm{ad}(Y)\right) \equiv \mathrm{Tr}_{\mathrm{ad}}(X\cdot Y) \quad (5.12)$$

The invariance of the Killing form follows from the cyclicity of the trace:

$$(\mathrm{ad}(X)Y, Z) + (Y, \mathrm{ad}(X)Z)$$
$$= \mathrm{Tr}_{\mathrm{ad}}([X,Y]Z) + \mathrm{Tr}_{\mathrm{ad}}(Y[X,Z])$$
$$= \mathrm{Tr}_{\mathrm{ad}}([XY - YX]Z + Y[XZ - ZX]) = 0$$

Hence

$$(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = (X,Y), \; g \in G. \quad (5.13)$$

The center, $Z(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$ is the set $\{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$. An abelian Lie algebra $\mathfrak{g}$ is one satisfying $Z(\mathfrak{g}) = \mathfrak{g}$. An ideal $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

A simple Lie algebra $\mathfrak{g}$ is one that is not abelian and does not contain any proper ideals. A Lie algebra $\mathfrak{g}$ is called semi-simple iff it is a direct sum of pairwise orthogonal (w.r. to the Killing form) simple ideals:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}] = \mathfrak{g}_i \qquad (5.14)$$

The finite-dimensional, real simple Lie algebras have been classified by E. Cartan: There are four infinite series: $A_r \ (r \geq 1)$, B_r, C_r, D_r,

where
$$A_r \simeq su(r+1), \quad B_r \simeq so(2r+1), \qquad (5.15)$$
$$C_r \simeq sp(r), \quad D_r \simeq so(2r).$$

These are the so-called classical Lie algebras.

We have that $A_1 \simeq B_1 \simeq C_1 \simeq D_1$,

$$B_2 \simeq C_2, \quad D_2 \simeq A_1 \oplus A_1, \quad D_3 \simeq A_3$$

Furthermore, there are five exceptional Lie algebras,

E_6, E_7, E_8, G_2 and F_4. (5.16)

The subscripts denote the rank of the Lie algebra, i.e., the dimension of a Cartan subalgebra (= Lie algebra of maximal torus in the corresponding Lie group).

For semi-simple Lie algebras, the Killing form $(\cdot,\cdot)$ is non-degenerate, i.e., if $(X,Y)=0, \forall Y$, then $X=0$. If G is compact $(\cdot,\cdot)$ is negative-definite; (hence $-(\cdot,\cdot)$ determines a Riemannian metric on G). We can choose a basis, $(T^A)_{A=1\dots\dim G}$, of $\mathcal{G}$ with the property that

$$(T^A, T^B) = \mathrm{Tr}_{ad}(T^A T^B) = -2\delta^{AB}. \tag{5.17}$$

The "structure constants", f^{AB}_{C}, of $\mathcal{G}$ are defined by

$$[T^A, T^B] = \sum_C f^{AB}_{C} T^C \tag{5.18}$$

By (5.17)

$$f^{AB}_{C} = -\frac{1}{2}\mathrm{Tr}_{ad}\left([T^A, T^B]\, T^C\right)$$

It follows that f^{AB}_{C} is totally anti-symmetric in A, B, C. The Jacobi identity for the Lie bracket,

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0,$$

implies that

$$f^{AB}_{\ \ D} f^{DC}_{\ \ E} + f^{CA}_{\ \ D} f^{DB}_{\ \ E} + f^{BC}_{\ \ D} f^{DA}_{\ \ E} = 0. \qquad (5.19)$$

A representation, $d\rho$, of $\mathfrak{g}$ on a Hilbert space V is called <u>unitary</u> iff $d\rho(X)^* = -d\rho(X)$, $\forall X \in \mathfrak{g}$. For example, ad is unitary on $\mathfrak{g}_{\mathbb{C}}$ (with scalar product given by $-(\cdot,\cdot)$).

A compact Lie group, G, is equipped with a unique left- and right translation-invariant probability measure: the <u>Haar measure</u>, dg; (dg is the Riemannian volume form w.r. to the Riemannian metric mentioned above). We also recall the <u>Peter–Weyl theorem</u>, ...

For further facts, see e.g. the book by Fuchs and Schweigert.

5.2. <u>Principal and associated vector bundles.</u>

A <u>fibre bundle</u>, E, is a topological space with the following properties:

Locally, E is homeomorphic to a Cartesian product

$$E \simeq F \times B, \tag{5.20}$$

where F is the <u>fibre</u> and B the <u>base space</u>.

If B is topologically non-trivial (5.20) does, in general, <u>not</u> hold globally.

A formal definition of a fibre bundle is as follows:

E is a fibre bundle over the base space B, with fibre F iff E is a Hausdorff space with a continuous projection $\pi : E \to B$ such that, for every $x \in B$, $\exists$ open neighborhood $U \ni x$ and a homeomorphism $\Phi : F \times U \xrightarrow{\simeq} \pi^{-1}(U) \subseteq E$ with the property that

$$\pi(\Phi(f,y)) = y, \tag{5.21}$$

for all $f \in F$ and all $y \in U$.

(U, Φ) provides a local coordinate chart for E.

We say that E is trivial iff $E \simeq F \times B$, globally.

<u>Transition functions</u>

On $U_1 \cap U_2 \subset B$, we define the <u>transition function</u>

$$\Phi_{12} := \Phi_2^{-1} \circ \Phi_1 : F \times (U_1 \cap U_2) \to F \times (U_1 \cap U_2).$$

By (5.21),

$$\Phi_{12}(f, x) = (\varphi_{12,x}(f), x), \quad x \in U_1 \cap U_2, \qquad (5.22)$$

where $\varphi_{12,x} : F \xrightarrow{\simeq} F$ is continuous in x.

E is uniquely determined by F, $\{U_i\}_{i=1,2,\dots}$ with $\bigcup_i U_i = B$ and all the transition functions $\{\Phi_{ij} \mid U_i \cap U_j \neq \emptyset\}$. The transition functions, Φ_{ij}, have the properties: Φ_{ii} = identity on $F \times U_i$, $\forall i$, and, on $U_i \cap U_j \cap U_k$, $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$.

A <u>section</u>, s, of E is a map $s : B \to E$, with the property that $\pi(s(x)) = x$, $x \in B$. If s is only defined locally it is called a <u>local section</u>. We define

$$\Gamma(E) = \text{space of (local) sections of } E.$$

<u>Special examples of fibre bundles.</u>

I. <u>Vector bundles.</u> For vector bundles, $F \simeq V$, where V is a (finite-dim.) vector space.

One then requires that the transition functions are given by linear maps

$$\Phi_{ij}(v, x) = (M_{ij,x}\, v, x) \tag{5.23}$$

where $M_{ij,x}$ is a linear isomorphism : $V \to V$, for all $x \in U_i \cap U_j$; i.e., $M_{ij,x} \in GL_n(K)$, if V is an n-dim. vector space over the field $K = \mathbb{R}, \mathbb{C}$.

Vector bundles always admit global sections. We say that s does not vanish anywhere if $s(x) \neq 0$, $\forall x \in B$.

Examples.

(i) $B = M$ (a smooth manifold)

$E = TM$, the tangent bundle of M

$\Gamma(E) = C^\infty$ vector fields on M.

(ii) The Möbius strip

. . . . . .

Note that $\Gamma(E)$ is always a projective module for the algebra $C(B)$, of continuous (smooth) functions on B.

II. Principal bundles

A principal bundle, P, with structure group G (usually a Lie group) over a smooth manifold M is a fibre bundle with fibre G and a continuous projection $\pi : P \to M$, as above. There is a smooth right-action, R, of G on P. If s is a local section of P over some open subset $U \subset M$ then

$$(g, s(x)) \longmapsto R_{s(x)}(g)s(x) = s(x)g^{-1}, \quad g \in G, x \in M,$$
$$G \times U \longrightarrow \pi^{-1}(U)$$

is a local diffeomorphism.

Transition functions of a principal bundle are G-valued functions, $g_{ij}(\cdot)$, with

$$g_{ij} : (g, x) \to (g\, g_{ij}^{-1}(x), x), \qquad (5.24)$$

for $x \in U_i \cap U_j$, $g \in G$; then

$$s_j(x)\, g_{ij}^{-1}(x) = s_i(x).$$

Example. $P = S^3 \simeq SU(2)$, $G = U(1)$, $B = G/H = S^2$. This is the Hopf fibration (Dirac monopole on S^2).

Let P be a principal bundle with structure group G, and let TP be its tangent bundle. Every point $p \in P$ lies on an orbit surface of the right action, R, of G on P:

$$R_p : g \longmapsto p \cdot g^{-1} \qquad (5.25)$$
$$G \to P.$$

R_p imbeds a copy of G into P, with $R_p(e)p = p$, for every p in P. The differential (tangent map), $d_g R_p$, maps $T_e G \simeq \mathfrak{g}$ to a subspace of $T_p P$:

$$d_g R_p : \mathfrak{g} \ni X \longmapsto d_g R_p (X) \in T_p P \qquad (5.26)$$
$$\mathfrak{g} \to T_p P$$

The image of $\mathfrak{g}$ under $d_g R_p$, $Im(d_g R_p)$, is called <u>vertical tangent space</u> in p. A vector field, Z, on P is called <u>fundamental</u> iff $Z(p) \in Im(d_g R_p)$, for all $p \in P$. Defining

$$[d_g R_p(X), d_g R_p(Y)] := d_g R_p([X,Y]),$$

we see that $Im(d_g R_p)$ is a Lie algebra homomorphic to $\mathfrak{g}$. The Lie bracket of two fundamental vector fields is again a fundamental vector field.

A <u>connection</u> on P is a decomposition of TP into vertical and "<u>horizontal</u>" subbundles,

$$T_pP = Im(dR_p) \dot{+} H_p, \quad p \in P,$$

with $H_{pg^{-1}} = d_pR_p(g^{-1})H_p$. The space $H = \bigcup_{p \in P} H_p$ is the horizontal subbundle of TP. A connection is uniquely determined by a 1-form, $\mathcal{A}$, on P with values in $\mathfrak{g}$ by

$$H_p = \{ Y \in T_pP \mid \langle \mathcal{A}(p), Y \rangle = 0 \},$$

and for any $Z = Z(X) \in Im(dR_p)$ corresponding to an element $X \in \mathfrak{g}$, $\langle \mathcal{A}(p), Z(X) \rangle = X$.

Let (U, Φ) be a local coordinate chart of P, i.e.,

$$\Phi : G \times U \longrightarrow \pi^{-1}(U) \subset P,$$

with $\pi(\Phi(g,x)) = x$. Then the pull back, $\Phi^*(A)$, of $\mathcal{A}$ is given by

$$\Phi^*(\mathcal{A})(g,x) = g\,dg^{-1} + A(x), \quad x \in U, \qquad (5.27)$$

where $A(x) = \sum_\mu A_\mu(x)\,dx^\mu$ is a 1-form on U with values in $\mathfrak{g}$.

A principal bundle, P, with structure group G and base manifold M is trivial, i.e.,

$$P \simeq G \times M$$

iff P has a globally defined section.

III. Associated vector bundles.

Let P be a principal bundle with structure group G, and let V be a vector space carrying a representation, ρ, of G. We consider the trivial vector bundle $V \times P$ with fibre V and base space P. It carries a G-action defined by

$$g(v,p) := (\rho(g)v, pg^{-1})$$

This allows us to pass to the quotient

$$V \times_G P,$$

where

$$V \times_G P = \left\{ [(v,p)] \,\middle|\, (v_1,p_1) \sim (v_2,p_2) \text{ iff } (v_1,p_1) = (\rho(g)v_2, p_2 g^{-1}),\ g \in G \right\} \tag{5.28}$$

It turns out that $V \times_G P$ is a vector bundle with fibre V and base space P/G. It is called the vector bundle associated to P and the representation ρ of G. If $\{g_{ij}(\cdot)\}$ are the transition functions of P then $\{\rho(g_{ij}(x)) \mid x \in U_i \cap U_j\}$ are the transition functions of $V \times_G P$. Locally, $V \times_G P$ is homeomorphic to $V \times U$, U an open subset of the base space, M, of P.

Sections of $V \times_G P$ can be identified with G-equivariant functions, $f : P \to V$, with

$$f(p \cdot g^{-1}) = \rho(g) f(p). \tag{5.29}$$

Thus

$$\Gamma(V \times_G P) = \mathrm{Map}_G(P \to V). \tag{5.30}$$

Let $\Lambda^{\cdot}(M)$ be the Grassmann algebra of differential forms on M, and let

$$\Lambda^{\cdot}(M, \Gamma(V \times_G P)) := \Lambda^{\cdot}(M) \otimes_{C^\infty(M)} \Gamma(V \times_G P) \quad (5.31)$$

denote the $C^\infty(M)$-module of $\Gamma(V \times_G P)$-valued differential forms;

$$\Lambda^{\cdot}(M, \Gamma(V \times_G P)) = \bigoplus_{p=0}^{n} \Lambda^p(M, \Gamma(V \times_G P)),$$

$n = \dim M$, $\Lambda^0(M, \Gamma(V \times_G P)) = \Gamma(V \times_G P)$.

The Grassmann algebra $\Lambda^{\cdot}(M)$ acts on $\Lambda^{\cdot}(M, \Gamma(V \times_G P))$ by

$$(\alpha, s) \longmapsto \alpha \wedge s, \quad (5.31)$$

where $\alpha \in \Lambda^{\cdot}(M)$, $s \in \Gamma(V \times_G P)$, and $\wedge$ is exterior multiplication.

<u>Definition.</u> A <u>covariant derivative</u>, or <u>connection</u>, ∇, on $V \times_G P$ is a linear map

$$\nabla : \Lambda^{\cdot}(M, \Gamma(V \times_G P)) \longrightarrow \Lambda^{\cdot}(M, \Gamma(V \times_G P))$$

with the properties

(i) $\nabla : \Lambda^p(M, \Gamma(V \times_G P)) \longrightarrow \Lambda^{p+1}(M, \Gamma(V \times_G P))$,

for all $0 \leq p \leq n$, $(\Lambda^m(M, \Gamma(V \times_G P)) = 0$, for

$m > n = \dim M\big)$,

(ii) $$\nabla(\alpha \wedge s) = d\alpha \wedge s + (-1)^{\deg\alpha}\alpha \wedge \nabla s; \qquad (5.32)$$

for all forms α of fixed degree, $\deg \alpha$, and all $s \in \Lambda^{\cdot}(M, \Gamma(V \times_G P))$; ($d$ denotes the exterior derivative).

(iii) Let $h: \pi^{-1}(U) \longrightarrow V \times U$,

$U \subset M$ be a local trivialization of $V \times_G P$. Then there is a 1-Form, A, defined on U with values in $\mathfrak{g}$ such that, for an arbitrary $s \in \Gamma(V \times_G P)$ with $\operatorname{supp} s \subset U$,

$$h(\nabla s) = dh(s) + d\rho(A)\,h(s), \qquad (5.32)$$

where $d\rho(A)$ is the representative of A in the representation $d\rho$ of $\mathfrak{g}$.

Let U_i and U_j be two coordinate patches in M, with $U_i \cap U_j \neq \emptyset$, and let $g_{ij}(x)$, $x \in U_i \cap U_j$, be the transition function of P. Let $y \in U_i \cap U_j$; then

$$h_i(\nabla s(y)) = \rho(g_{ij}(y))\, h_j(\nabla s(y))$$

$$= \rho(g_{ij}(y))\left\{dh_j(s(y)) + d\rho(A^{(j)}(y))\, h_j(s(y))\right\}$$

$$\stackrel{!}{=} dh_i(s(y)) + d\rho(A^{(i)}(y))\, h_i(s(y))$$

$$= d(\rho(g_{ij}(y)) h_j(s(y))) + d\rho(A^{(i)}(y)) \times$$
$$\times \rho(g_{ij}(y))\, h_j(s(y)).$$

Comparing the second and the last line, we find that

$$A^{(j)}(y) = \rho(g_{ij}^{-1}(y))\left\{d\rho(g_{ij}(y)) + A^{(i)}(y)\rho(g_{ij}(y))\right\} \tag{5.33}$$

The forms A are inherited from a connection on P.

The <u>curvature tensor</u>, $F(\nabla)$, of a connection ∇ on $V \times_G P$ is defined by

$$F(\nabla) = \nabla^2 \tag{5.34}$$

on $\Lambda^{\cdot}(M, \Gamma(V \times_G P))$. We show that $F(\nabla)$ is a <u>tensor</u>, i.e., a $C^\infty(M)$ - <u>linear</u> map from $\Lambda^{\cdot}(M, \Gamma(V \times_G P))$ into itself:

$$F(\nabla)(\alpha \wedge s) = \nabla(\nabla(\alpha \wedge s))$$
$$= \nabla(d\alpha \wedge s + (-1)^{\deg \alpha} \alpha \wedge \nabla s)$$
$$= d^2\alpha \wedge s + (-1)^{\deg\alpha + 1} d\alpha \wedge \nabla s$$

$$+ (-1)^{\deg\alpha}\, d\alpha \wedge \nabla s + \alpha \wedge \nabla^2 s$$

$$= \alpha \wedge F(\nabla) s \, ; \qquad (5.35)$$

i.e., $F(\nabla)$ is even a $\Lambda^{\cdot}(M)$- linear map of $\Lambda^{\cdot}(M, \Gamma(V \times_G P))$ into itself.

Let $\{s_1, \cdots, s_m\}$, $m = \dim V$, be a local basis of sections of $V \times_G P$. For every $i = 1, \cdots, m$, ∇s_i is then an element of $\Lambda^1(M) \otimes_{C^\infty(M)} \Gamma(V \times_G P)$. It can be expanded in the basis $\{s_1, \cdots, s_m\}$:

$$\nabla s_i = \sum_{j=1}^{m} A^j_{\rho i} \otimes s_j \, , \qquad (5.36)$$

where $A^j_{\rho i} \in \Lambda^1(M)$, for all i, j; with $A_\rho \equiv d\rho(A)$.

The curvature tensor $F(\nabla) = \nabla^2$ is a $C^\infty(M)$- linear map from $\Gamma(V \times_G P)$ to $\Lambda^2(M) \otimes_{C^\infty(M)} \Gamma(V \times_G P)$.

Thus

$$F(\nabla) s_i = \sum_{j=1}^{m} F^j_i \otimes s_j \, , \quad F^j_i \in \Lambda^2(M). \quad (5.37)$$

Combining (5.36) with (5.37) and the definition $F(\nabla) = \nabla^2$, we obtain that

$$\sum_j F^j_i \otimes s_j = F(\nabla) s_i = \nabla(\nabla s_i)$$
$$= \sum_j \nabla\left(A^j_{\rho i} \otimes s_j\right)$$
$$= \sum_j \left(dA^j_{\rho i} \otimes s_j - A^j_{\rho i} \otimes \nabla s_j\right)$$
$$= \sum_j \left(dA^j_{\rho i} - \sum_k A^k_{\rho i} \wedge A^j_{\rho k}\right) \otimes s_j ,$$

and hence

$$F^j_i = dA^j_{\rho i} + \sum_k A^j_{\rho k} \wedge A^k_{\rho i} \qquad (5.38)$$

Taking the exterior derivative of (5.38), we find that

$$dF^j_i = d^2 A^j_{\rho i} + dA^j_{\rho k} \wedge A^k_{\rho i} - A^j_{\rho k} \wedge dA^k_{\rho i}$$
$$= F^j_k \wedge A^k_{\rho i} - A^j_{\rho k} \wedge A^k_{\rho l} \wedge A^l_{\rho i}$$
$$- A^j_{\rho k} \wedge F^k_i + A^j_{\rho k} \wedge A^k_{\rho l} \wedge A^l_{\rho i}$$
$$= F^j_k \wedge A^k_{\rho i} - A^j_{\rho k} F^k_i , \qquad (5.39)$$

where we have used the summation convention. This is the second <u>Bianchi identity</u>.

<u>Physics.</u>

We will now study rather general non-abelian gauge theories, with gauge group, G, some compact Lie group, on a space-time manifold M. Typically, $M = M^4$. Geometrically, we will consider a principal bundle, P, with fibre G and base space M, along with associated vector bundles $V \times_G P$.

<u>Gauge fields</u> are connections on P. Given a local trivialization of P, a connection corresponds to a 1-form, A, on an open subset of M with values in the Lie algebra, $\mathfrak{g}$, of G. The affine space of all connections on the principal bundle P is denoted by $\mathcal{A}_P$. This space carries an action of the infinite-dimensional group, $\mathcal{G}$, of gauge transformations given by

$$\mathcal{G} = \{\text{smooth maps } g : x \in M \mapsto g(x) \in G\}.$$

Under a gauge transformation $g \in \mathcal{G}$, a gauge field A transforms as follows:

$$A \rightarrow {}^g A = g A g^{-1} + g\, d g^{-1}, \tag{5.40}$$

see (5.33). Thus, if A_1 and A_2 are two gauge fields then

$$ {}^gA_1 - {}^gA_2 = g\,(A_1 - A_2)\,g^{-1}, $$

i.e., the difference, $A_1 - A_2$, transforms homogeneously under gauge transformations.

<u>Matter fields</u> are sections of a vector bundle $V \times_G P$ associated to P and a representation, ρ, of G on V. Gauge transformations of a matter field $\psi \in \Gamma(V \times_G P)$ are given by

$$ \psi(x) \mapsto {}^g\psi(x) := \rho(g(x))\,\psi(x), \quad g(\cdot) \in \mathcal{G}. $$

5.3. Non-abelian gauge (Yang-Mills-) theories

Let G be a compact Lie group, which we choose as the gauge group of a gauge field theory on space-time M^4. A connection on a principal G-bundle, $P \simeq G \times M^4$, over M^4 is determined by a non-abelian vector potential or gauge field, $A_\mu(x)$, with values in the Lie algebra, $\mathfrak{g}$, of G; i.e., by a $\mathfrak{g}$-valued 1-form

$$A(x) = \sum_{\mu=0}^{3} A_\mu(x) dx^\mu \in \Lambda^1(M^4) \otimes \mathfrak{g}.$$

A gauge field $A(x)$ determines a notion of parallel transport: Let

$$\gamma = \left\{ x(t) \in M^4 \,\middle|\, 0 \leq t \leq 1, x(0) = x, x(1) = y \right\}$$

be a curve connecting the point x to the point y. With γ we associate the parallel transporter (holonomy operator)

$$U(\gamma) := P\left(\exp \int_\gamma A\right) \in G, \qquad (5.41)$$

where P indicates "<u>path ordering</u>": $U(\gamma)$ is given by

$$U(\gamma) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \; A(x(t_1)) \cdot \dot{x}(t_1) \cdots A(x(t_n)) \dot{x}(t_n);$$

("Dyson series"). It defines a map from the fibre over x to the fibre over y by multiplying a group element in the fibre over x from the right by $U(\gamma)$ and interpreting the result as a group element in the fibre over y. If $\{g(x) \in G \mid x \in M^4\}$ is a <u>gauge transformation</u> acting on the fibres G of P by right-multiplication (see (5.25)) then we must apparently require that $U(\gamma)$ transforms under $g(\cdot)$ by

$$U(\gamma) \longmapsto {}^{g}U(\gamma) = g(x)\, U(\gamma)\, g(y)^{-1}, \qquad (5.42)$$

for any path γ connecting x to y. Eq. (5.42) is equivalent to

$$A_\mu(x) \longmapsto {}^{g}A_\mu(x) = g(x)\, A_\mu(x)\, g(x)^{-1} + g(x)\, \partial_\mu g(x)^{-1}, \qquad (5.43)$$

see (5.40)!

<u>Curvature</u>, or <u>field strength</u>, manifests itself in the circumstance that $U(\gamma)$ depends non-trivially on γ, <u>not</u> just on its endpoints: If γ_1 and γ_2 are two different paths from x to y and γ_2^- denotes the path from y to x obtained by reversing the orientation of γ_2 then

$$U(\gamma_1) \neq U(\gamma_2)$$

implies that the curvature, F, of A does not vanish identically on any surface Σ bounded by the loop $\gamma_1 \circ \gamma_2^-$. However, if $F \equiv 0$ then $U(\gamma)$ only depends on the endpoints of γ.

Let $\sigma^{\mu\nu}$ be a surface element in the (μ, ν)-plane, and let $\gamma_{\mu\nu} = \partial\sigma^{\mu\nu}$ be the loop starting and ending at a point $x \in \partial\sigma^{\mu\nu}$. Then

$$U(\gamma_{\mu\nu}) \simeq \mathbb{1} + F_{\mu\nu}(x)\,\sigma^{\mu\nu} + O(|\sigma^{\mu\nu}|^2) \tag{5.44}$$

with

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - [A_\mu(x), A_\nu(x)].$$

Under a gauge transformation $g(\cdot)$, $U(\gamma_{\mu\nu})$ transforms as $U(\gamma_{\mu\nu}) \mapsto {}^{g}U(\gamma_{\mu\nu}) = g(x)U(\gamma_{\mu\nu})g(x)^{-1}$; thus

$$F_{\mu\nu}(x) \longmapsto {}^{g}F_{\mu\nu}(x) = g(x)F_{\mu\nu}(x)\,g(x)^{-1}. \tag{5.45}$$

All these claims can be verified by simple, explicit calculations, which are presumably well known to the reader. Physicists call the holonomy operators $U(\gamma)$ <u>Wilson lines</u>, and if γ is a loop and ρ is a representation of G they call

$$W_\rho(\gamma) = tr\big(\rho(U(\gamma))\big)$$

a <u>Wilson loop</u>.

Let $\psi(x)$ be a matter field with values in a representation space, V, of G corresponding to the representation ρ; (i.e., ψ is a section of a vector bundle associated to P and ρ).

Under a gauge transformation $g(\cdot) \in \mathcal{G}$ (= group of all gauge transformations – see (5.40)), $\psi(x)$ transforms as

$$\psi(x) \longmapsto {}^{g}\psi(x) = \rho(g(x))\,\psi(x), \qquad (5.46)$$

and

$$\psi^*(x) \longmapsto {}^{g}\psi^*(x) = \psi^*(x)\,\rho(g(x))^*$$

Covariant differentiation of ψ is defined by

$$(\nabla_\mu \psi)(x) = (\partial_\mu - d\rho(A_\mu(x)))\,\psi(x) \qquad (5.47)$$

If A_μ transforms under gauge transformations as specified in (5.43) then it follows that

$$({}^{g}\nabla_\mu\, {}^{g}\psi)(x) = {}^{g}(\nabla_\mu \psi)(x), \qquad (5.48)$$

with

$${}^{g}\nabla_\mu = g(x)\,\nabla_\mu\, g(x)^{-1},$$

as an easy calculation shows; (use that

$$0 = \partial_\mu e = \partial_\mu (g(x)\,g(x)^{-1}) = (\partial_\mu g)(x)\,g(x)^{-1} + g(x)\,\partial_\mu g(x)^{-1}.)$$

It is easy to see that the field strength,

$F_{\mu\nu}$, satisfies

$$d\rho(F_{\mu\nu}) = [\nabla_\mu, \nabla_\nu]. \qquad (5.49)$$

Example. We let ψ be a Dirac spinor field with values in a representation space V of G carrying a unitary representation ρ of G, (i.e. $\rho(g)^* = \rho(g)^{-1} = \rho(g^{-1})$). Let $\bar\psi(x) = \overset{*}{\psi}(x)\gamma_0$. Since $\rho(\cdot)$ commutes with the Dirac matrices, we find that the Lagrangian density

$$\mathcal{L} = i\bar\psi(x)\gamma^\mu \nabla_\mu \psi(x) - m\bar\psi(x)\cdot\psi(x) \qquad (5.50)$$

is gauge-invariant.

We propose to look for a Lagrangian density for the gauge field $A(x)$. In analogy with electrodynamics, and using (5.45), we choose

$$\mathcal{L}_{YM} = -\frac{1}{8g^2} \operatorname{Tr}_{ad}\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right), \qquad (5.51)$$

where g is the gauge coupling constant; (g is analogous to the elementary electric charge e).

By (5.45), $\mathcal{L}_{YM}$ is gauge-invariant.

If
$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n ,$$
where the $\mathfrak{g}_i$'s are Lie algebras of Lie groups G_i, we decompose $A(x)$ as $A(x) = A^{(1)}(x) \oplus \cdots \oplus A^{(n)}(x)$, where $A^{(i)}$ is $\mathfrak{g}_i$-valued. Then a more general Lagrangian density is given by

$$\mathcal{L}_{YM} = -\sum_{j=1}^{n} \frac{1}{8 g_j^2} \, Tr_{ad} \left(F^{(j)}_{\mu\nu}(x) \, F^{(j)\mu\nu}(x) \right) \qquad (5.52)$$

where the gauge coupling constants g_j can all be different from each other. This generalization is important in connection with the <u>Standard Model</u>.

A typical Lagrangian density of a gauge theory with a Dirac field ψ and a scalar field ϕ has the form

$$\mathcal{L} = -\frac{1}{8g^2} Tr_{ad}\left(F_{\mu\nu} F^{\mu\nu}\right) + i\bar{\psi}\gamma^\mu \nabla_\mu \psi - m\bar{\psi}\psi$$

$$+ (\nabla_\mu \phi)^* \cdot (\nabla^\mu \phi) - U(\phi^*, \phi, \bar{\psi}, \psi),$$

where U is a gauge-invariant local function of $\phi^*, \phi, \bar{\psi}$ and ψ; e.g.,

$$U = \frac{\lambda}{4!}\left(|\phi|^2 - \mu^2\right)^2 + \text{Yukawa couplings.}$$

Let us recall the Bianchi identies satisfied by $F_{\mu\nu}$, which are non-abelian generalizations of the homogeneous Maxwell equations:

$$\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0, \qquad (5.53)$$

where ∇ here denotes covariant differentiation in the adjoint representation, i.e.,

$$\nabla_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - [A_\lambda, F_{\mu\nu}] \qquad (5.54)$$

Eq. (5.53) follows from the Jacobi identity for ∇ and the eq. $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$.

To do explicit calculations, e.g., in perturbation theory, it is sometimes useful to rescale the gauge field: $A_\mu = g\tilde{A}_\mu$, and to expand $\tilde{A}_\mu$ in a basis $\{T^A\}$ of $\mathfrak{g}$:

$$\tilde{A}_\mu = \tilde{A}_{B\mu}\, T^B, \quad \tilde{F}_{\mu\nu} = \tilde{F}_{B\mu\nu}\, T^B,$$

where

$$\tilde{F}_{B\mu\nu} = \partial_\mu \tilde{A}_{B\nu} - \partial_\nu \tilde{A}_{B\mu} - g f_B^{\ CD}\, \tilde{A}_{C\mu} \tilde{A}_{D\nu} \qquad (5.55)$$

Equations of motion (analogous to the inhomogeneous Maxwell equations and the covariant Dirac equation) can be derived from the Lagrangian densities introduced above and will be studied, in more detail, in the exercises.

5.4. Hamiltonian quantization of pure Yang-Mills theory

The quantization of gauge theories is notoriously tricky. The simplest example for this is the quantum theory of the free electromagnetic field. The Lagrangian of pure QED is given by

$$\mathcal{L} = \frac{1}{4}\left(F_{\mu\nu} F^{\mu\nu}\right), \tag{5.56}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We split A_μ into transversal and longitudinal components:

$$A_\mu = A_\mu^T + A_\mu^L = \left(P_\mu^{T\nu} + P_\mu^{L\nu}\right) A_\nu,$$

where $P_\mu^{T\nu} = \delta_\mu^\nu - k_\mu k^\nu / k^2$, $P_\mu^{L\nu} = k_\mu k^\nu / k^2$, in momentum space. The action corresponding to (5.56) is

$$S = \frac{1}{2}\int d^4x \left[\sum_{\mu,\nu} \left(\partial_\mu A_\nu^T\right)^2\right]$$

It is independent of A^L. The propagator for A_μ^T in momentum space is then given by

$$D_{\mu\nu}^T(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{-i}{k^2 + i0}.$$

This can be seen, e.g., by evaluating a Gaussian path integral corresponding to S. However, since

S is independent of A^L, the propagator, $D^L_{\mu\nu}(k)$, of A^L is ill-defined, i.e., diverges. This pathology is caused by the gauge invariance of $\mathcal{L}$ and S: When one integrates over all configurations of A then, by gauge invariance, one sums over every "physical history" infinitely many times.

In canonical quantization, the problem shows up in that $\mathcal{L}$ and S do not contain terms depending on $\dot{A}_0$, and hence there is no momentum field canonically conjugate to A_0.

The solution to these problems is to fix some gauge, e.g. $A^L \equiv 0$ (Lorentz gauge), or $A_0 \equiv 0$ and $\vec{\nabla}\cdot\vec{A} \equiv 0$ (Coulomb gauge).

In order to arrive at a <u>formally</u> well defined Hamiltonian path integral quantization of a pure non-abelian Yang-Mills theory, one will also try to choose a gauge condition that is satisfied by one gauge field on every orbit of the action of the group of gauge transformations, $\mathcal{G}$, on the space, $\mathcal{A}$, of all gauge fields. The goal of fixing a gauge condition is to arrive at a Lagrangian for the remaining degrees of freedom that enables us to pass to the Hamiltonian formalism with a phase space that is equipped with a non-degenerate symplectic 2-form. If we succeed in this we have two options: We may either quantize canonically, à la Heisenberg-Dirac, or quantize the theory by Hamiltonian path integrals.

The Arnowitt-Fickler-, or axial gauge is such a gauge condition. It is given by the equation

$$A_{B3}(x) \equiv 0, \quad \forall B. \tag{5.57}$$

It is an elementary exercise to verify that, given an arbitrary gauge field, $A(x)$, we can find a gauge transformation, $g(x)$, such that

$$^{g}A_3(x) = 0 ;$$

(use Wilson line operators for paths γ starting, e.g. in the 3-plane $x^3 = 0$ and parallel to the 3-axis — ↗ exercises).

We rewrite the Lagrangian (5.51) of pure Yang-Mills theory in first-order form, using (5.44) and (5.57):

$$\mathcal{L} = \frac{1}{2g^2}\left(F_{B\mu\nu}\right)^2 - \frac{1}{4g^2}\Big\{F_B^{ij}\left(\partial_i A_{Bj} - \partial_j A_{Bi} + f_B^{CD} A_{Ci} A_{Dj}\right)$$

$$+ F_B^{0i}\left(\partial_0 A_{Bi} - \partial_i A_{B0} + f_B^{CD} A_{C0} A_{Di}\right)$$

$$+ F_B^{03}\left(-\partial_3 A_{B0}\right) + F_B^{i3}\left(-\partial_3 A_{Bi}\right)\Big\} \qquad (5.58)$$

where $i, j = 1, 2$. Obviously, manifest Lorentz- and rotation invariance are lost.

The action S is given by
$$S = \int d^4x \, \mathcal{L},$$
and we may freely integrate by parts when evaluating S. We then observe that no time-derivatives of A_{B0}, F_B^{ij} and F_B^{03} ever appear. Thus, their Euler-Lagrange equations do not involve time-derivatives of these fields; they are constraints fixing A_{B0}, F_B^{ij} and F_B^{03} on the initial-value surface in terms of A_{Bi} and F_B^{0i}. We can then eliminate A_{B0}, F_B^{ij} and F_B^{03} by solving their equations of motion and plugging the result back into (5.58). This yields an action, $\tilde{S}$, only depending on A_{Bi}, $\dot{A}_{Bi}$ and F_B^{0i}, with A_{Bi} interpreted as configuration space coordinates (q) and F_B^{0i} as their conjugate momenta (p). $\tilde{S}$ is of the form

$$\tilde{S} = \int dt \int d^3x \left\{ F_B^{0i} \dot{A}_{Bi} - \mathcal{H}\left(F_B^{0i}, A_{Bi}\right)\right\}.$$

$$\left(\sim \int dt \left\{ p\dot{q} - H(p,q)\right\}\right).$$

Adding a source term, e.g.,

$$\langle J^i, A_i \rangle = \sum_B \int d^4x\, J_B^i(x) A_{Bi}(x),$$

we can use $\tilde{S}$ as our starting point for a path-integral quantization on phase space:

$$e^{iW(J)} = \mathcal{N} \int \prod_B \prod_{i=1}^{2} \mathcal{D}A_{Bi}\, \mathcal{D}F_B^{0i}\, e^{i(\tilde{S} + \langle J^i, A_i \rangle)}, \quad (5.59)$$

where $\mathcal{N}$ is a normalization factor; see (2.9), (2.10), (3.10), (3.11).

However, the constrained variables, A_{B0}, F_B^{ij} and F_B^{03} enter (5.58) only <u>linearly</u> and <u>quadratically</u>. Moreover, the Lagrangian $\mathcal{L}$ in (5.58) is obtained from the original Lagrangian

$$\mathcal{L}_{YM} = \frac{1}{2g^2}\left(F_{B\mu\nu}\right)^2 - \frac{1}{4g^2} F_B^{\mu\nu}\left(\partial_\mu A_{B\nu} - \partial_\mu A_{B\nu} + f_B^{CD} A_{C\mu} A_{D\nu}\right) \quad (5.60)$$

simply by imposing the gauge condition (5.57), i.e. $A_{B3}(x) \equiv 0$, $\forall B$. Hence we can rewrite (5.59) as

$$e^{iW(J)} = \mathcal{N} \int \prod_{B} \Big(\prod_{\mu=0}^{2} \mathcal{D}A_{B\mu} \prod_{\mu,\nu} \mathcal{D}F_{B\mu\nu} \Big) e^{i(S + \langle J^i; A_i \rangle)}$$

$$= \mathcal{N} \int \mathcal{D}A \, \mathcal{D}F \prod_{B,x} \delta\big(A_{B3}(x)\big) e^{i(S_{YM} + \langle J^i; A_i \rangle)}, \tag{5.61}$$

where $S_{YM} = \int d^4x \, \mathcal{L}$.

Unfortunately, the Arnowitt-Fickler gauge (5.57) and the path integral (5.61) are inconvenient to do practical calculations. We therefore would like to explore other gauge choices and convince ourselves that the gauge-invariant information encoded into $W(J)$ does not depend on the choice of the gauge condition.

5.4. The Faddeev-Popov construction

Generalizing the Lagrangian path-integral quantization of Chapter 3 formally to gauge fields, we are led to consider the path integrals

$$\frac{1}{Z}\int e^{iS(A)} \prod_j O_j(A)\, \mathcal{D}A, \tag{5.62}$$

with

$$S(A) = -\frac{1}{8g^2}\int d^4x\, Tr_{ad}\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right),$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - [A_\mu(x), A_\nu(x)],$$

$$\text{"}\, Z = \int e^{iS(A)}\, \mathcal{D}A \,\text{"},$$

where $O_1(A), \dots, O_n(A)$ are gauge-invariant functionals of the gauge field A_μ, e.g.,

$$O_i(A) = Tr_{ad}\left(F_{\mu\nu}(x_i) F^{\mu\nu}(x^i)\right)$$

(or $O_i(A) =$ Wilson loop $W_{\rho_i}(\gamma_i), \dots$). The path integral (5.62) is a formal expression for the time-ordered Green functions of the "fields" $O_i(A)$.

The formal path integral (5.61) is ill-defined, because gauge-equivalent field configurations,

$$A_\mu \text{ and } {}^gA_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1},$$

all contribute with the same weight to (5.61). Since the Lie algebra of the group of gauge transformations is non-compact, this tends to cause divergences. The problem of divergences can be solved, formally, by fixing a gauge; see, e.g., (5.57). Then the path integral is rewritten as an integral over a surface of A's in the space of all gauge-field configurations that obey a gauge-fixing condition times an integral over the group of gauge transformations. This last integral turns out to yield a factor that can be cancelled by a normalization factor.

We now introduce some notions and concepts to render these remarks more precise.

Orbit space and gauge fixing

Let $\mathcal{A} := \{A_\mu(\cdot)\}$ be the space of all possible gauge field configurations. This is an affine space modelled on the vector space of 1-forms with values in the Lie algebra, $\mathfrak{g}$, of the gauge group G: If A_1 and A_2 are two gauge fields in $\mathcal{A}$ then $A_1 - A_2 \in \Lambda^1(M, \mathfrak{g})$, and

$$^g A_1(x) - {}^g A_2(x) = g(x)\big(A_1(x) - A_2(x)\big) g^{-1}(x),$$

where $g(\cdot) \in \mathcal{G}$ is a gauge transformation. Here

$$\mathcal{G} = \{ g(\cdot) \mid g(x) \in G,\ \forall x \in M^4,\ (\partial_\mu g)(x) \text{ continuous} \}$$

is the group of all gauge transformations. It acts as a transformation group on $\mathcal{A}$ by

$A \mapsto {}^g A = g A g^{-1} + g\, d g^{-1}$, for $A \in \mathcal{A}$. The orbit, $[A]$, of A under $\mathcal{G}$ is defined by

$$[A] = \{ A' \mid A' = {}^g A, \text{ for some } g(\cdot) \in \mathcal{G} \} \qquad (5.63)$$

The orbit space, $\mathcal{B}$, is the space of all orbits $[A]$, for $A \in \mathcal{A}$; i.e.,

$$"\mathcal{B} = \mathcal{A}/\mathcal{G}".$$

If G is non-abelian $\mathcal{G}$ does not act freely on $\mathcal{A}$, and $\mathcal{B}$ turns out to be a singular space that does not admit an integration measure (volume form).

The action S_{YM}, see (5.60), (5.62), and the fields $O_i(A)$ are constant along $\mathcal{G}$-orbits, i.e.,

$$S_{YM}(A) = S_{YM}([A]), \quad O_i(A) = O_i([A]).$$

Thus, it would be nice if (5.62) could be reinterpreted as an integral over the orbit space $\mathcal{B}$. We have learned in Chapter 2, (2.50) – (2.61), that, starting with path integrals over <u>phase space</u>, with the action written in the first-order formalism (see (5.60)), "$\mathcal{D}A$" must be interpreted as the formal Riemannian volume form corresponding to the Riemannian metric used in the definition of the "kinetic energy term", (i.e., the term quadratic in $F_{\mu\nu}$ in (5.60)). Formally,

$$\mathcal{D}A = \prod_{\mu, x} dA_\mu(x). \tag{5.64}$$

Unfortunately, $\mathcal{D}A$ does not determine a well-defined integration measure on the orbit space $\mathcal{B}$, and we must do something more clever to define (5.62). However, formally, in a vicinity of the origin $A \equiv 0$ in $\mathcal{A}$, things turn out to be quite simple.

The Lie algebra, $\mathcal{X} := Lie(\mathcal{G})$, of $\mathcal{G}$ is the space of $\mathfrak{g}$-valued continuous functions on M^4. Ideally, a gauge-fixing condition, F, would be an $\mathcal{X}$-valued function on $\mathcal{A}$ with the property that the surface given by the equation

$$F(A) = C \in \mathcal{X} \tag{5.65}$$

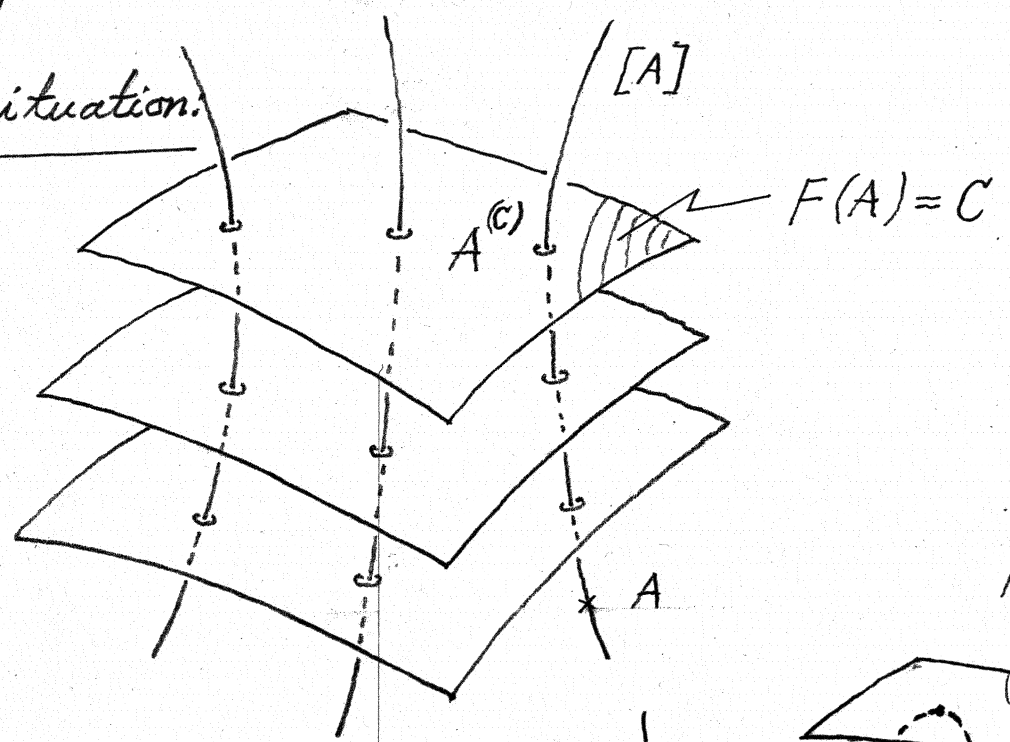
cuts every orbit $[A] \in \mathcal{B}$ in <u>precisely one</u> gauge field $A^{(C)} \in [A]$, (for an arbitrary $C \in \mathcal{X}$). Unfortunately, for non-abelian gauge fields, it is difficult to find gauge-fixing conditions, F, with this property that are convenient for concrete calculations. This is the problem of <u>Gribov copies</u>: Usually, eq. (5.65) has an odd number > 1 of solutions for a given orbit $[A] \in \mathcal{B}$. However, if we restrict our attention to gauge fields, A, with small curvature or field strength, eq. (5.65) tends to have a unique solution (in a suitable range of C's), and this is all that counts in <u>perturbation theory</u>.

A necessary condition for a gauge-fixing condition F on $\mathcal{A}$ to be admissible in a perturbative analysis is the condition of <u>transversality</u>

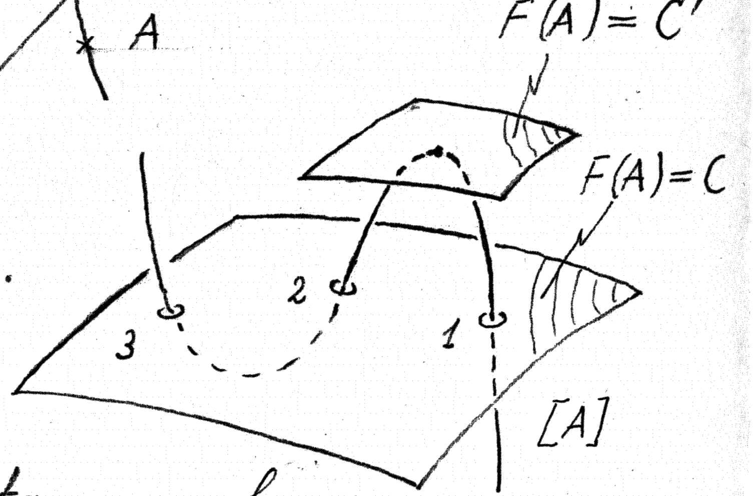
$$\left.\frac{\delta F({}^{g}A^{(C)})}{\delta g}\right|_{g=1} \quad \underline{\text{non-singular}}, \qquad (5.66)$$

for $C \in \mathcal{X}$, $[A] \in \mathcal{B}$. We must clarify what (5.66) means!

<u>Ideal situation:</u>



<u>Gribov copies:</u>



If an orbit $[A]$ intersects a surface $F(A) = C$ in several Gribov copies $A_1^{(C)}, \ldots, A_{2n+1}^{(C)}$ then $[A]$ intersects a neighboring surface, $F(A) = C'$, <u>non-transversally</u>; (see picture).

<u>Meaning of (5.66)</u>: For $g(\cdot) \in \mathcal{G}$ in a small neighborhood of the identity $e \in \mathcal{G}$ we have that

$$g(x) = e^{\varepsilon X(x)}, \quad X(\cdot) \in \mathcal{X} = Lie(\mathcal{G}), \qquad (5.67)$$

$|\varepsilon|$ small. Then we find that

$${}^{g}A_\mu = g A_\mu g^{-1} + g \partial_\mu g^{-1}$$

$$= A_\mu + \varepsilon [X, A_\mu] - \varepsilon \partial_\mu X + O(\varepsilon^2)$$

$$= A_\mu - \varepsilon \nabla_\mu X + O(\varepsilon^2), \qquad (5.68)$$

where $\nabla_\mu = \partial_\mu + ad(A_\mu)$ is the covariant derivative on a vector bundle associated with the adjoint representation of G on $\mathfrak{g}$; (here $\mathcal{X}$). Then

$$M(A^{(C)})X := \frac{\delta F(^g A^{(C)})}{\delta g}\Big|_{g=1}$$

$$\equiv \frac{d}{d\varepsilon} F(A^{(C)} - \varepsilon \nabla X)\Big|_{\varepsilon=0}$$

$$= -\frac{\delta F}{\delta A_\mu}(A^{(C)}) \cdot \nabla_\mu X \qquad (5.68)$$

Obviously, the operator $M(A)$ defined in (5.68) is a <u>linear operator</u> from $\mathcal{X}$ to $\mathcal{X}$. It is called the <u>gauge variation of the gauge condition</u> at $A \in \mathcal{A}$. The meaning of the transversality condition (5.66) is that $M(A^{(C)})$ should be <u>invertible</u> at $A^{(C)}$, for $C \in \text{Image}\, F \subseteq \mathcal{X}$.

Usually, gauge conditions will be <u>local</u>, i.e.,

$F(A)(x)$ only depends on $A(x)$ and finitely many derivatives of A in x.

<u>Example.</u> <u>The Coulomb gauge</u> (condition)

$$\sum_{j=1}^{3} \partial_j A_{Bj}(x) \equiv 0 \qquad (5.69)$$

In this example, we find that

$$M(A) = \sum_{i=1}^{3} \partial_i \cdot \nabla_i \qquad (5.70)$$

Gribov has shown that if G is non-abelian $M(A)$, as defined in (5.70), has 0-eigenvalues, i.e. $M(A)$ is not invertible, for certain "large" A's, and there are orbits $[A]$ cutting the surface determined by (5.69) several times ("Gribov copies"). However, for "small" A's, $M(A)$ is invertible: Let $\mathcal{M}_{CD}(x,y)$ denote the integral operator corresponding to $M(A)^{-1}$. Then, by (5.70),

$$\left(\Delta_x \mathcal{M}_{BC} + f_{BD}^{\;\;E} A_{Ej}(x) \cdot \partial_j \mathcal{M}_{DC}\right)(x,y) = \delta_{BC}\, \delta(x-y),$$

so that $\mathcal{M}_{BC}(x,y)$ solves the integral equation

$$\mathcal{M}_{BC}(x,y) = \frac{\delta_{BC}}{4\pi|\vec{x}-\vec{y}|} - \int_{z^0=x^0} f_{BD}{}^{E}\, \frac{A_{Ej}(z)}{4\pi|\vec{x}-\vec{z}|}\, \partial_j \mathcal{M}_{DC}(z,y)\, d^3z. \tag{5.71}$$

For small A's (perturbation theory!), eq. (5.71) has a unique solution.

Henceforth, we will assume that the gauge variations, $M(A)$, of our gauge conditions, $F(A)=C$, are <u>invertible</u>. F is then called a <u>regular gauge condition</u>. (An example is the Arnowitt-Fickler gauge.) We proceed to discussing the

<u>Faddeev-Popov construction.</u>

We choose a real-valued function f on the image, $Rg(F) \subseteq \mathcal{X}$, of F, where $F: \mathcal{A} \to \mathcal{X}$ is a regular gauge condition. The constant $\mathcal{F}$ is defined by

$$\mathcal{F} := \int_{Rg(F)} f(C)\, \mathcal{D}C. \tag{5.72}$$

For an orbit $[A] \in \mathcal{B}$, $A^{(0)}$ is the gauge field in $[A]$ that solves the equation $F(A)=0$; (we assume here w.l.o.g. that $0 \in Rg(F)$). Since F is regula any point $A \in [A]$ can be parametrized <u>either</u> by $C \in \mathcal{X}$, with $F(A) = C$, <u>or</u> by the gauge transformation $g(\cdot) \in \mathcal{G}$ with the property that $A = {}^{g}A^{(0)}$. For a given orbit $[A]$, the Jacobi matrix for the change of variables from $g(\cdot)$ to C in the point $A = A^{(C)} = {}^{g}A^{(0)}$ is given by the operator $M(A) \equiv DF(A)$, with

$$M(A)X = \frac{\delta F({}^{h}A)}{\delta h}\bigg|_{h=1} = \frac{\delta F({}^{g'}A^{(0)})}{\delta g'}, \qquad (5.73)$$

for $h(x) = \exp \varepsilon X(x)$, $g' = h \cdot g$; see (5.68). Thus

$$\mathcal{F} = \int f(C)\,\mathcal{D}C = \int f\left(F({}^{g'}A^{(0)})\right) \det\left(DF({}^{g'}A^{(0)})\right) \mathcal{D}g'$$

$$= \int f\left(F({}^{h}A)\right) \det\left(DF({}^{h}A)\right) \mathcal{D}h, \qquad (5.74)$$

for any $A \in [A]$, where $\mathcal{D}g'$, $\mathcal{D}h$ are the formal Haar measures on $\mathcal{G}$. In the last equation, we have used the right-invariance of the Haar measure: If φ is a function on $\mathcal{A}$ then

$$\int \varphi({}^{g'}A^{(0)})\, \mathcal{D}g' = \int \varphi({}^{hg}A^{(0)})\, \mathcal{D}(hg)$$

$$= \int \varphi({}^{h}A)\, \mathcal{D}(hg)$$

$$= \int \varphi({}^{h}A)\, \mathcal{D}h \qquad (5.75)$$

We now insert the identity (5.74) into the formal path integral (5.62):

$$\mathcal{F}^{-1} \int f(C) \mathcal{D}C \int e^{iS(A)} \prod_j O_j(A)\, \mathcal{D}A$$

$$= \mathcal{F}^{-1} \int \left[\int f(F({}^{h}A))\, \det(DF({}^{h}A)) \mathcal{D}h \right] e^{iS(A)} \times$$

$$\times \prod_j O_j(A)\, \mathcal{D}A$$

$$\overset{\text{Fubini}}{=} \mathcal{F}^{-1} \int \left[\int f(F({}^{h}A))\, \det(DF({}^{h}A))\, e^{iS(A)} \times \right.$$

$$\left. \times \prod_j O_j(A)\, \mathcal{D}A \right] \mathcal{D}h$$

$$= \mathcal{F}^{-1} \int \left[f(F({}^{h}A))\, \det(DF({}^{h}A))\, e^{iS({}^{h}A)} \times \right.$$

$$\left. \times \prod O_j({}^{h}A)\, \mathcal{D}A \right] \mathcal{D}h$$

↑ gauge invariance of S and of $O_1, \dots, O_n$

$$= \mathcal{F}^{-1} \left(\int \mathcal{D}h \right) \int f(F(A))\, \det(DF(A))\, e^{iS(A)} \prod_j O_j(A)\, \mathcal{D}A$$

↑ $\mathcal{D}A = \mathcal{D}\,{}^{h}A$

$$= \mathcal{F}^{-1} \mathrm{vol}(\mathcal{G}) \int e^{iS(A)} f(F(A)) \det \underbrace{(DF(A))}_{\equiv M(A)} \prod_j O_j(A) \mathcal{D}A \tag{5.76}$$

Faddeev and Popov propose to interpret (5.62) as

$$" \frac{1}{Z} \int e^{iS(A)} \prod_j O_j(A) \mathcal{D}A "$$

$$:= \frac{1}{\tilde{Z}} \int e^{iS(A)} f(F(A)) \det M(A) \prod_j O_j(A) \mathcal{D}A, \tag{5.77}$$

with $\tilde{Z} := \int e^{iS(A)} f(F(A)) \det M(A) \mathcal{D}A.$ (5.78)

If we set $f(C) = \delta(C - C_0)$ then the integrations in (5.77), (5.78) extend over the gauge-fixing surface given by the equation $F(A) = C_0$. Another popular choice for f is a Gaussian.

It is, perhaps, useful to consider a finite-dimensional analogue of the Faddeev-Popov procedure. Let $(y, x) \in \mathbb{R}^{M+N}$, and let S be a function <u>only depending on</u> x. (We can think of x as being points $[A]$ in orbit space and of y as gauge transformations. Moreover, S plays the rôle of the Yang-Mills action.) A gauge condition is imposed by equations $F^a(y,x) = C_0^a$, $a = 1, \cdots, M$; or

$$F(y,x) = C_0, \tag{5.79}$$

with $F = (F_1, \cdots, F_M)$. If the matrix

$$DF(y,x) = \left(\frac{\partial F^a(y,x)}{\partial y^b}\right)$$

is invertible in an open neighborhood, U, of a point (y_0, x_0) with $F(y_0, x_0) = C_0$, then, by the implicit function theorem, there is an R^M-valued function $\varphi(x) = (\varphi_1(x), \cdots, \varphi_M(x))$ such that

$$F(y,x) = C_0 \iff y = \varphi(x), \quad x \in \overline{U},$$

where $\overline{U}$ is an open neighborhood of x_0; i.e.

$$F(\varphi(x), x) = C_0, \quad x \in \overline{U}.$$

Next,

$$F^a(y,x) = F^a(y,x) - F^a(\varphi(x),x) + C_0^a$$

$$= C_0^a + \sum_{b=1}^{M} \frac{\partial F^a}{\partial y^b}(\varphi(x), x)\cdot\left(y^b - \varphi^b(x)\right) + h.o. \tag{5.80}$$

(Taylor expansion around $(\varphi(x), x)$!)

We set

$$M(y,x) = DF(y,x) = \left(\frac{\partial F^a}{\partial y^b}(y,x)\right),$$

which is invertible on U. Then, by (5.80),

$$y^b - \varphi^b(x) = M^{-1}(\varphi(x),x)^b{}_a\left(F^a(y,x) - C_0^a\right) + h.o.$$

Let us choose $f(C) = \delta^{(M)}(C - C_0)$. Then

$$\int_{\overline{U}} e^{iS(x)}\, d^N x$$

$$= \int_{\mathbb{R}^M \times \overline{U}} e^{iS(x)}\, \delta^{(M)}\left(y - \varphi(x)\right) d^M y\, d^N x$$

$$= \int_{\mathbb{R}^M \times \overline{U}} e^{iS(x)}\, \delta^{(M)}\left(M^{-1}(\varphi(x),x)\left(F(y,x) - C_0\right)\right) d^M y\, d^N x$$

$$= \int_{\mathbb{R}^M \times \overline{U}} e^{iS(x)} \det\left(M(\varphi(x),x)\right) \delta\left(F(y,x) - C_0\right) d^M y\, d^N x$$

$$= \int_{\mathbb{R}^M \times \overline{U}} e^{iS(x)} \det\left(M(y,x)\right) \delta\left(F(y,x) - C_0\right) d^M y\, d^N x$$

To carry out the integral over all of R^{M+N} we make use of a partition of unity. This reproduces a finite-dimensional version of (5.78), with $f(C) = \delta(C - C_0)$. (For general f, use that $f(F(A)) = \int f(C)\,\delta(F(A) - C)\,\mathcal{D}C$.)

Examples.

(1) Arnowitt Fickler gauge: $A_3(x) \equiv 0$. In this gauge $M(A) = -\partial_3$ is independent of A, i.e. $\det M(A) = \text{const}$. Choosing $f(C) = \delta(C)$, we see that (5.77), (5.78) reproduce (5.61), after the Gaussian integration over $F_{\mu\nu}$ has been carried out in (5.61)!

(2) Lorentz gauge: We choose

$$F_B(A)(x) = \partial^\mu A_{B\mu}(x), \tag{5.81}$$

and

$$f(C) = \exp\left(\frac{i}{4\alpha}\int \mathrm{Tr}_{ad}\left(C(x)^2\right) d^4x\right)$$

$$= \exp\left(-\frac{i}{2\alpha}\sum_B \int \left(C_B(x)\right)^2 d^4x\right). \tag{5.82}$$

The gauge variation of the gauge condition is

$$(M(A)\, X)(x) = \partial^\mu \nabla_\mu X \qquad (5.83)$$

Thus, the Faddeev-Popov path integral is given by

$$\langle O_1 \cdots O_n \rangle = \frac{1}{\tilde{Z}} \int e^{iS(A) - \frac{i}{2\alpha} \sum_B \int (\partial^\mu A_{B\mu})^2(x)\, d^4x} \times$$

$$\det(\partial^\mu \nabla_\mu) \prod_j O_j(A)\, \mathcal{D}A, \qquad (5.84)$$

with ∇ the covariant derivative in the adjoint representation. For an <u>abelian</u> gauge theory, such as QED, the adjoint representation is trivial, and $\nabla_\mu = \partial_\mu$. Hence $\det(\partial^\mu \nabla_\mu) = \det(\Box)$ is a (divergent) constant <u>independent</u> of A, which disappears in normalized expectation values. However, in <u>non-abelian</u> gauge theories, $\det(\partial^\mu \nabla_\mu)$ depends on A in a non-trivial manner.

<u>Faddeev-Popov ghosts</u>

It is convenient to rewrite the so-called <u>Faddeev-Popov determinant</u>, $\det(M(A))$, as a <u>Berezin integral</u>

We introduce Grassmann fields ξ and $\bar{\xi}$ with values in the Lie algebra $\mathfrak{g}$ of the gauge group G. (We sometimes also use the notation $\bar{\xi} =: b$, $\xi := c$.) Let $\int (\cdot)\, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi$ denote the usual Berezin integral introduced in Chapter 4. Then, by (4.68),

$$\det M(A) = \int e^{-\langle \bar{\xi}, M(A)\xi \rangle} \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi \qquad (5.85)$$

Changing variables, $\bar{\xi} \mapsto i\bar{\xi}$, $\xi \mapsto \xi$, we find:

$$\langle O_1 \cdots O_n \rangle = \frac{1}{Z_{gf}} \int e^{i\int \mathcal{L}_{gf}(A; \bar{\xi}, \xi) d^4x} \prod_j O_j(A) \times$$
$$\times\, \mathcal{D}A\, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi, \qquad (5.86)$$

where

$$\mathcal{L}_{gf} = -\frac{1}{8} Tr_{ad}(F_{\mu\nu} F^{\mu\nu}) - i \ln f(F(A)) + \frac{1}{2} Tr_{ad}(\bar{\xi} M(A) \xi), \qquad (5.87)$$

and

$$Z_{gf} = \int e^{i\int \mathcal{L}_{gf}(A; \bar{\xi}, \xi) d^4x} \mathcal{D}A\, \mathcal{D}\bar{\xi} \wedge \mathcal{D}\xi .$$

In the Lorentz gauge, we find that

$$\mathcal{L}_{gf}(A;\bar{\xi},\xi) = -\frac{1}{8}\,Tr_{ad}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{4\alpha}\,Tr_{ad}\left(\partial^\mu A_\mu\right)^2$$

$$-\frac{1}{2}\,Tr_{ad}\left(\bar{\xi}\,\partial^\mu \nabla_\mu \xi\right)$$

$$= \frac{1}{4}F_{B\mu\nu}F^{B\mu\nu} + \frac{1}{2\alpha}\left(\partial^\mu A_{B\mu}\right)^2$$

$$+\ \bar{\xi}_B\,\partial^\mu\left(\nabla_\mu \xi\right)_B . \tag{5.88}$$

Because the Faddeev – Popov ghosts, $\bar{\xi}$ and ξ, are Fermi fields with spin 0, they violate the usual connection between spin and statistics. Since they are not physical fields, that isn't a problem!
It is not a particularly good idea to attempt to quantize $\mathcal{L}_{gf}$ canonically, using the operator formalism, (although that has been tried, too; ↗ Kugo–Ojima, Scharf et al.). We will always work with path integrals.

For the purposes of perturbation theory, we split $\mathcal{L}_{gf}$ into a quadratic term, $\mathcal{L}^{(2)}_{gf}$, and a higher-order term:

In the Lorentz gauge,

$$\mathcal{L}^{(2)}_{gf} = +\frac{1}{2}\partial_\mu A_{B\nu}\,\partial^\mu A^\nu_B - \frac{1}{2}\partial_\mu A_{B\nu}\,\partial^\nu A^\mu_B$$

$$+\frac{1}{2\alpha}\partial_\mu A^\mu_B\,\partial_\nu A^\nu_B + \bar{\xi}_B\,\Box\,\xi_B . \qquad (5.89)$$

After an integration by parts, the quadratic term in the action is given by

$$S^{(2)}_{gf} = \int \mathcal{L}^{(2)}_{gf}\,d^4x$$

$$= -\frac{1}{2}\int A_{B\mu}\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu(1-\alpha^{-1})\right)A_{B\nu}\,d^4x$$

$$+\int \bar{\xi}_B\,\Box\,\xi_B\,d^4x \qquad (5.90)$$

Thus, the <u>propagators</u> of the theory are given by

$$\langle A_{B\mu}(x)\,A_{C\nu}(y)\rangle_0 = \delta_{BC}\,i\,D_{\mu\nu}(x-y),$$

where $D_{\mu\nu}$ solves the equation

$$\left[\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu(1-\alpha^{-1})\right]D_{\nu\lambda}(x) = \delta^\mu_\lambda\,\delta(x) \qquad (5.91)$$

After Fourier transformation, and taking into account the Wick rotation, $t \mapsto e^{-i\theta}t$, $\theta \searrow 0$, we obtain

$$D_{\mu\nu}(x) = -\lim_{\varepsilon\searrow 0}\int \frac{\eta_{\mu\nu} - (1-\alpha)\,p_\mu p_\nu/p^2}{p^2+i\varepsilon}\,e^{ip\cdot x}\,\frac{d^4p}{(2\pi)^4} \qquad (5.92)$$

The ghost propagator is given by

$$\left\langle \bar{\xi}_B(x)\, \xi_C(y) \right\rangle_0 = i\,\delta_{BC}\, D(x-y), \tag{5.93}$$

where

$$D(x) = -\lim_{\varepsilon \searrow 0} \int \frac{e^{ip\cdot x}}{p^2 + i\varepsilon} \frac{d^4 p}{(2\pi)^4}. \tag{5.94}$$

Expressions for the vertices are obtained from the terms in $\mathcal{L}_{gf}$ of order 3 an 4 in the fields:

$$\begin{aligned} \mathcal{L}^I_{gf} &:= \mathcal{L}_{gf} - \mathcal{L}^{(2)}_{gf} \\ &= g f^{BCD} \partial_\mu A_{B\nu} A^\mu_C A^\nu_D - \frac{g^2}{4} f^{BCD} f^{BEF} A_{C\mu} \times \\ &\quad A_{D\nu} A^\mu_E A^\nu_F + g\, \bar{\xi}_B f^{BCD} \partial^\mu \left(A_{C\mu} \xi_D \right) \end{aligned} \tag{5.95}$$

It is easy to generalize these considerations to the full Lagrangian of QCD:

5.5 Lagrangian of SU(N) gauge theory

5.5.1 Notations, definitions

Lie algebra of SU(N), representations R = fund, adj

$$[t^a_R, t^b_R] = i f^{abc} t^c_R \,, \qquad a,b,c = 1,2,\ldots, N^2-1 \qquad (5.5.1)$$

fundamental representation

$$(t^a_{fund})_{ij} = (T^a)_{ij} \,, \qquad a = 1,2,\ldots N^2-1,\; i,j = 1,2,\ldots N \qquad (5.5.2)$$

adjoint representation

$$(t^a_{ad})_{bc} = (F^a)_{bc} \qquad a,b,c = 1,2,\ldots, N^2-1 \qquad (5.5.3)$$

SU(2): $N=2$

$$\left.\begin{array}{ll} (T^a)_{ij} = \frac{1}{2}\sigma^a_{ij}\,, & \sigma^a: \text{ Pauli-matrices} \\ (F^a)_{bc} = i\,\epsilon^{bac}\,, & \epsilon^{abc} = \text{Levi-Civita tensor} \end{array}\right\} \qquad (5.5.4)$$

SU(3): $N=3$

$$\left.\begin{array}{ll} (T^a)_{ij} = \frac{1}{2}\lambda^a_{ij}\,, & \lambda^a: \text{ Gell-Mann matrices} \\ (F^a)_{bc} = i f^{bac}\,, & f^{abc}: \text{ structure constant tensor of SU(3)} \end{array}\right\} \qquad (5.5.5)$$

fermion fields are in fundamental representations
gauge field are in adjoint representations, they are elements of the Lie-algebra

Abstract form of the transformation of matter fermions

$$\psi(x) \to {}^g\psi(x) = \rho(g(x))\,\psi(x) \qquad \text{(see 5.46)}$$

The unitary transformation in the fundamental representation

$$U_{ij}(x) = \left(e^{ig_s \alpha^a(x) T^a}\right)_{ij}, \quad (U^+U)_{ij} = \delta_{ij} \qquad (5.5.6)$$

$$\left.\begin{aligned} \psi_i(x) &\to {}^g\psi_i(x) = U_{ij}(x)\,\psi_j(x) \\ \bar\psi_i(x) &\to {}^g\bar\psi_j(x) = \bar\psi_j(x)\,U^+_{ji} \end{aligned}\right\} \qquad (5.5.7)$$

The gauge-field is element of the Lie-algebra

$$(A_\mu(x))_{ij} = (t^c)_{ij}\,A^c_\mu(x) \qquad (5.5.8)$$

Transformation of the gauge field

$$A_\mu(x)_{ij} \to {}^gA_\mu(x)_{ij} = \left(U A_\mu U^+\right)_{ij} + \frac{i}{g_s}\left((\partial^\mu U(x))\,U^+\right)_{ij} \qquad (5.5.9)$$

g_s: coupling constant (strong interaction SU(3))

$\alpha^a(x)$: parameter of the gauge transformation

The covariant derivative acting on vectors in representation R

$$\begin{aligned} (\nabla_\mu)_{ij} &= \delta_{ij}\partial_\mu + ig_s A_\mu(x)_{ij} = \partial_\mu + igA_\mu \\ &= \delta_{ij}\partial_\mu + ig_s (t^a)_{ij} A^a_\mu(x) \end{aligned} \qquad (5.5.10)$$

5.5.2 The Lagrangian

$$\mathcal{L}_{QCD} = \bar{\Psi}_f i \not{D} \psi_f - m_f \bar{\Psi}_f \psi_f - \frac{1}{2} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$
$$- \frac{1}{\alpha} \mathrm{Tr}\left(\partial_\mu A\, \partial^\mu A\right)$$
$$+ \partial^\mu \bar{\xi}^a \left(\delta^{ab} \partial_\mu - g_s f^{abc} A^b_\mu \right) \xi^c \qquad (5.5.11)$$

$$F_{\mu\nu} = T^a F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^b_\mu - g_s f^{abc} A^b_\mu A^c_\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_s \left[A_\mu, A_\nu \right]$$

where $f = u, d, s, c, b, t$, $N = 3$.

It is a good exercise to write out explicitely all suppressed indices, for example

$$\bar{\Psi}_u \psi_u = \bar{\Psi}_u(x)^{\alpha_i i} \psi^{\beta}_{u,\beta, i}(x)$$
$$\not{D} = \delta_{ij} \partial_\mu + i g_s \left(A_\mu(x)\right)_{ij}$$

The naive classical gauge invariant Lagrangian is modified by adding a gauge fixing term and a scalar fermionic field in the adjoint representation which is called the Fadeev - Popov ghost.

$$\mathcal{L}_{QCD} = \underbrace{\mathcal{L}^{(0)}_{QCD}}_{\substack{\text{bilinear} \\ \text{in the fields} \\ \text{independant from} \\ g_s}} + \underbrace{\mathcal{L}^{(I)}_{QCD}}_{\substack{\text{trilinear or quatrolinear} \\ \text{in the field} \\ \text{proportional to } g_s}} \qquad (5.5.12)$$

5.5.3 Standard weak coupling perturbation theory

Amplitudes are constructed by Feynman-rules derived by using Gell-Mann-Low formula of weak coupling perturbation theory and by the Wick's-theorem.

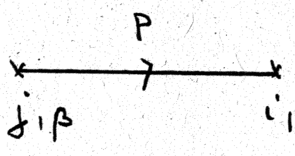
Propagators, Vertices, Feynman-rules

We can read out the propagators from the free Lagrangian obtained by $g_s = 0$

$$\mathcal{L}^{free}_{QCD} = \bar{\psi}_f(x)\, i\gamma^\mu \partial_\mu \psi(x)_f - m_f\, \bar{\psi}_f(x)\psi_f(x)$$
$$+ \frac{1}{2} A^a_\mu(x)\left(\eta^{\mu\nu}\Box - \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu\right) A^a_\nu(x)$$
$$- \bar{\xi}^a(x)\,\Box\,\xi^a(x) \qquad (5.5.13)$$

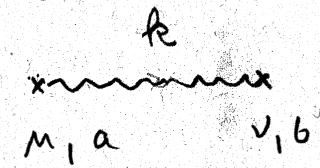
$$\left.\begin{aligned} S_F(x-y)_{\alpha\beta,ij} &= i\langle 0|T\,\psi_{\alpha,i}(x)\,\bar{\psi}_{\beta,j}(y)|0\rangle \\ D(x-y)^{ab}_{\mu\nu} &= i\langle 0|T\,A^a_\mu(x)\,A^b_\mu(y)|0\rangle \\ D_{gh}(x-y)^{ab} &= i\langle 0|T\,\xi^a(x)\,\bar{\xi}^b(y)|0\rangle \end{aligned}\right\} \quad (5.5.14)$$

Diagrammatically in Feynman-rules

(a) 

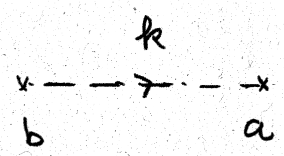
$$= S_{F\,ij,\alpha\beta}(p) = \delta_{ij}\left(\frac{i}{\not p - m + i\epsilon}\right)_{\alpha\beta}$$

$$= \delta_{ij}\,\frac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$$

(b) 

$$= D^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2 + i\epsilon}\left(-\eta_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)k_\mu k_\nu\right)$$

(5.5.16)

(c) 

$$= D^{ab}_{gh}(k) = \frac{i\delta^{ab}}{k^2 + i\epsilon}$$

Vertices

Interaction vertices of Feynman rules can be read from the interaction Lagrangian

$$\mathcal{L}_{int,\,QCD} = i g_s \bar\psi_f(x)\,\gamma_\mu A^\mu(x)\,\psi_f(x)$$

$$+ g_s \frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu,b} A^{\nu,c} f^{abc}$$

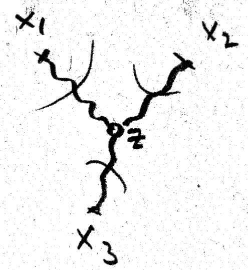
$$- g_s^2 \frac{1}{4} f^{ebc} f^{eb'c'} A^\mu_b A^\nu_c A_{\mu,b'} A_{\nu,c'}$$

$$+ g_s f^{abc} (\partial^\mu \bar\xi^a) f^{abc} \xi^b A^c_\mu \qquad (5.5.17)$$

Recall a convenient form of the Gell-Mann-Low formula (only scalar fields)

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)|0\rangle = Z[0]\,\exp\left[\frac{1}{2i}\int d^dx\,d^dy\,D(x,y)\cdot\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}\right]\cdot\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\cdot\exp\left[i\int dz\,\mathcal{L}^{int}(z)\right]$$

Leading order



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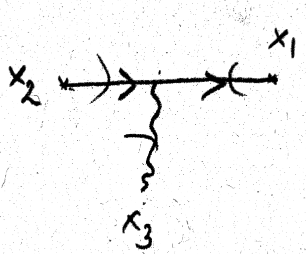
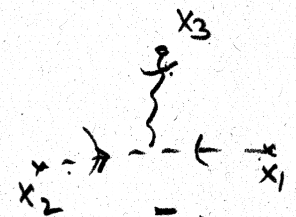
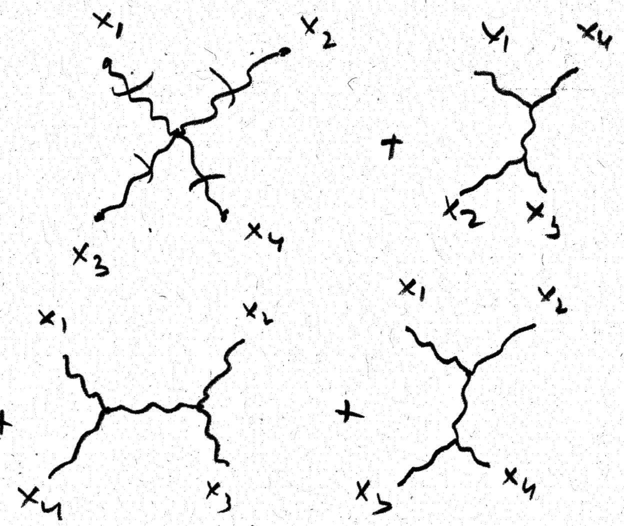


$\psi\,\bar{\psi}\,A$



$\xi\,\bar{\xi}\,A$



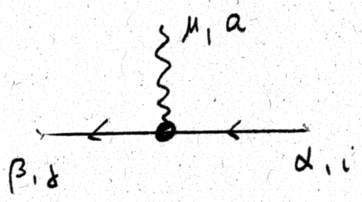
AAAA

We „ampute" the propagators and work in Fourier space

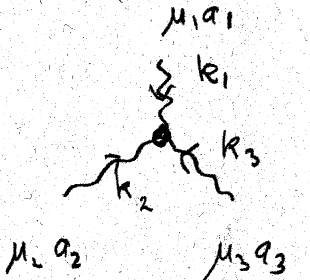
$$\psi(k) = \int d^dx\, e^{ikx}\,\psi(x)$$

and calculate the local factors.

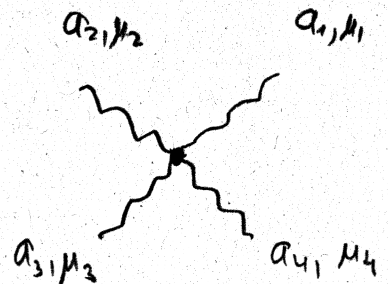
Diagrammatic representation of vertices for QCD



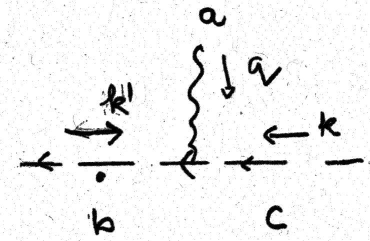
$$-ig_s\,(T^a)_{ji}\,(\gamma_\mu)_{\beta\alpha}$$



$$-g\,f^{a_1a_2a_3}\left[g^{\mu_1\mu_2}(k_1-k_2)^{\mu_3} + g^{\mu_2\mu_3}(k_2-k_3)^{\mu_1} + g^{\mu_3\mu_1}(k_3-k_2)^{\mu_2}\right]$$



$$-ig^2[\,f^{ea_1a_2}f^{ea_3a_4}(g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + f^{ea_1a_3}f^{ea_4a_2}(g^{\mu_1\mu_4}g^{\mu_2\mu_3}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}) + f^{ea_1a_4}f^{ea_2a_3}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_3}g^{\mu_2\mu_4})$$



$$g\,f^{abc}\,k'^{\mu}$$

$S_{fi} = \delta_{fi} + i\,T_{fi}$ is the definition of T-matrix

All T_{fi} matrix elements are described by Feynman diagrams

S-matrix is unitary

$\delta_{fi} = \sum_n S_{fn} S^+_{ni}$ $\qquad$ $\mathrm{Im}\,T_{fi} = \frac{1}{2}\sum_n T_{fn} T^+_{ni}$

Construct $T_{fi}^{(k)}$, $T_{fi} = g_s^{n_0} \sum_{k=0}^{\infty} g_s^{2k} T_{fi}^{(k)}$

At a given order, write down all Feynman-diagrams which give $g_s^{n_0+2k}$ coupling constant factor

Feynman-rules in momentum space

i) Identify all distinguisable scattering diagrams without vacuum bubles

ii) Associate with every line the Fourier-integrated momentum propagator

$$\int \frac{d^d k_j}{(2\pi)^d} \left\{ S_F^{\alpha\beta, ij}(k_j), \; D_{ab}^{\mu\nu}(k_j), \; D_{gh}^{ab}(k_j) \right\}$$

for quarks — for gluon — for FP-ghost

iii) Associate with each vertex the corresponding vertex function with the momentum conservation δ-function $\delta^{(d)}\left(\sum_{i \in \text{vertex}} k_i\right)(2\pi)^d$

iv) Multiply by appropriate symmetry factors

v) $-$ signs for closed fermion loops

vi) appropriate ~~external~~ polarization vectors for external particles

k, μ, s $\quad e^{\mu}(k,s)$, $\quad p, \alpha, s \quad u_\alpha(p,s)$

$p, \alpha \quad \bar{u}_\alpha(p,s)$,

8. Renormalization of gauge theories

8.1 Introductory remarks

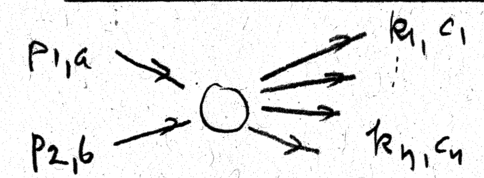
Renormalizable field theories have the distinct feature that the ultraviolet divergencies appearing in loop integrals can all be absorbed in field, coupling constant renormalization factors. This remains valid in all orders of perturbation theory such that we obtain a Lorentz-invariant, in gauge theories also gauge invariant unitary S-matrix.

To prove this for gauge theories is rather technical and requires the introduction of concepts like Wick rotations, Euclidean field theory, regularization (we discuss here only dimensional regularization), renormalization prescription, renormalization schemes, superficial degree of divergences, counter terms, BPHZ method and renormalization group. In QFT I many of these concepts have already been introduced in the case of scalar $\lambda\varphi^4$ field theory and QED. I recall the main concepts with using simple examples

The new material, not discussed in QFT I is the renormalization of gauge theories, the renormalization group, the evaluation of one-loop counter terms and the discussion of asymptotic freedom.

8.2 Review some concepts and results from QFT I

8.2.1 Scattering cross-section



$$\langle k_1, \ldots, k_n, out \mid p_1 p_2 \, in \rangle$$

$$= i (2\pi)^4 \delta (p_1 + p_2 - \Sigma k_i) \, T(p_1, p_2, \{k\}) \qquad (8.1)$$

$$T_{fi} = (2\pi)^4 \delta(P_f - P_i) \, T_{fi} \qquad (8.2)$$

$$\sigma (a+b \to c_1 + \ldots c_n)_{2\to n} = \frac{1}{\frac{1}{4}\sqrt{(p_1 p_2)^2 - M_1^2 M_2^2}} \int d\phi_n (\{k_i\}) \overline{|T(p_1, p_2, \{k\})|^2} \qquad (8.3)$$

$$d\phi_n (p_1 + p_2 \to k_1 + \ldots + k_n) = \prod_{i=1}^{n} \frac{d^{d-1} k_i}{(2\pi)^{d-1} 2k_i^0} (2\pi)^4 \delta (p_1 + p_2 - \sum_{i=1}^{n} k_i) \qquad (8.4)$$

We have uniquely defined algorithm, well-defined answer with tree amplitudes (leading order process

8.2.2 Tree and loop diagrams

In an arbitrary diagram, the number of integrals left after we used all the delta functions gives the number of loops

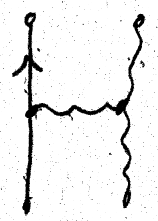
Let us assume that we have I lines, V vertices, then the number of remaining momentum integrals are

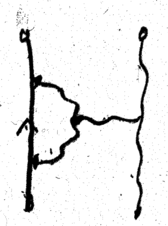
$$L = I - V + 1 \qquad (8.5)$$

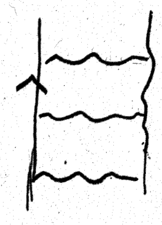
(topological result, called Euler theorem)

$L=0$ tree diagrams, $L = 1, 2, \ldots$ one, two, n-loop diagrams

Tree - graph , one - loop graph , 2-loop graph

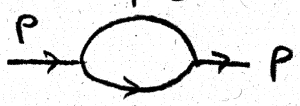


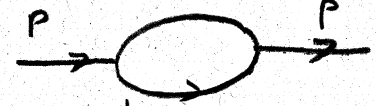




Tree-graph	one-loop graph	2-loop graph
$I = 5$ $E = 4$ $n = 5$	$I = 4$ $E = 4$, $n = 8$	$n = 11$
$V = 6$	$V = 8$	$V = 10$
$L = n - V + 1 = 0$	$L = 8 - 8 + 1 = 1$	$L = 2$

We can parametrize the flow of momenta in the graph, therefore we can determine the loop momenta directly without going through the evaluation of the δ- functions. We identify the independent loops and choose momentum flow according to momentum conservation.





$l \to l + \frac{1}{2}p$

The evaluation of Green - functions in momentum space has been reduced to calculating multidimensional integrals of products of Feynman propagators and vertex functions.

The task is not straightforward since in some cases the integrals are divergent.

8.2.3 Regularization of divergent loop integrals

Simplest example in dimensional regularization

$$I_2(p^2, m^2, d) = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - m^2 + i\epsilon][(\ell-p)^2 - m^2 + i\epsilon]} \quad (8.6)$$

d: dimension of space-time

Feynman-parametrization, threshold singularities

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \quad (8.7a)$$

$$\prod_{i=1}^N A_i^{-n_i} = \left(\prod_{i=1}^N \frac{1}{\Gamma(n_i)}\right) \Gamma\left(\sum_{i=1}^N n_i\right) \int_0^1 dx_1 \dots dx_N \frac{x_1^{n_1-1} \dots x_N^{n_N-1} \, \delta\left(1 - \sum_{i=1}^N x_i\right)}{\left[\sum x_i A_i\right]^{\sum_{i=1}^N n_i}} \quad (8.7)$$

$$I_2(p^2, m, d) = \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - 2\ell p x + x p^2 - m^2 + i\epsilon]^2} \quad (8.8)$$

Linear terms are eliminated by changing variables

$$\ell = \ell' + px$$

$$I_2(p^2, m, d) = \int_0^1 dx \int \frac{d^d \ell'}{(2\pi)^d} \frac{1}{[\ell_0'^2 - \vec{\ell}'^2 + x(1-x)p^2 - m^2 + i\epsilon]^2} \quad (8.9)$$

Denominator has poles in the ℓ_0 plane

$$\ell_0 = \pm \sqrt{\vec{\ell}^2 + M^2(x) - i\epsilon} \quad (8.10)$$

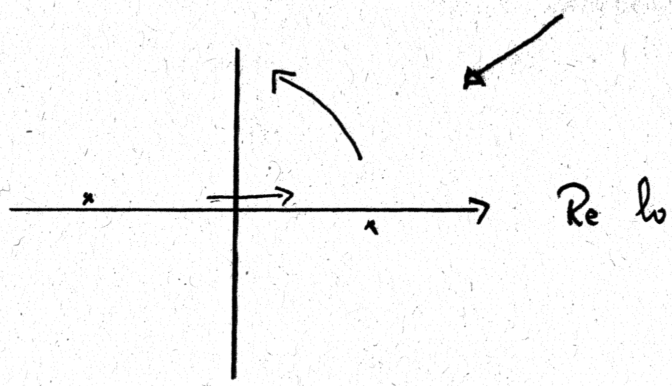
$$M^2(x) = -p^2 x(1-x) + m^2 = Q^2 x(1-x) + m^2 > 0 \quad (8.11)$$

Threshold singularities are regulated by the $+i\epsilon$ prescription of the Feynman - propagator

From (8.10) it follows that

$$l_0 = \pm(\lambda - i\epsilon) \tag{8.12}$$

The position of the poles in the complex plane are indicated in the figure below



The integration contour can be rotated to the imaginary axis (Wick-rotation).
Simultaneously we rotate the energy component of the external momenta also to imaginary value

$$l_0 = i\, l_{0E} \qquad p_0 = i p_{0E} \quad , \quad l_E^2 < 0 \quad , \quad p_E^2 < 0 \tag{8.13}$$

In $d < 4$ dimension the integral is convergent

$$I_2(p^2, m^2, d) = i \int_0^1 dx \int_{-\infty}^{\infty} d\,l_E \frac{1}{(2\pi)^d} \frac{1}{[\underbrace{-l_E^2 - M_E^2(x)}_{\text{negative definite}} + i\epsilon]^2} \tag{8.14}$$

$\epsilon \to 0$ limit is allowed

We can write

$$\int d^d \ell_E = \int_0^\infty |\vec{\ell}_E|^{d-1} \, d\ell_E \, d\Omega_d \qquad (8.12)$$

Since the integrand only depends on $|\vec{\ell}_E|^2$ we can carry out the angular integral

$$\Omega_d = \int d\Omega_d \qquad (8.12)$$

which gives the surface of a sphere in d-dimension.
We can evaluate Ω_d using a trick as follows

$$\mathcal{I}_1 = \int d^d k_E \, e^{-k_E^2} = \left(\int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^d = \pi^{d/2} \qquad (8.13)$$

We can evaluate the same integral also in polar coordinates

$$\mathcal{I}_2 = \int d^d k_E \, e^{-k_E^2} = \Omega_d \int |\vec{k}_E|^{d-1} d|k|_E \, e^{-k_E^2} = \frac{1}{2} \Omega_d \, \Gamma(d/2) \qquad (8.14)$$

wher $\Gamma(x)$ is the standard $\Gamma(x)$ function $\Gamma(x+1) = x!$

Since $\mathcal{I}_1 = \mathcal{I}_2$ we obtain

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \qquad (8.15)$$

Our derivation is valid for any positive integer value of d and the result (8.15) allows for analytic continuation to any value of d in the complex plane.

Evaluating the leading singularity

$$I_2(-p_E^2, m^2, d) = i(2\pi)^{-d}\Omega_d \int_0^\infty dz\, z^{d-1} \int_0^1 dx\, [z^2 + x(1-x)p_E^2 + m^2]^{-2} \quad (8.16)$$

$$i(2\pi)^{-d}\Omega_d \int_a^\infty dz\, z^{d-5} \sim -i \frac{1}{d-4} \frac{\Omega_d}{(2\pi)^d} A^{-2\epsilon} \sim \text{const}\left(\frac{1}{2\epsilon_{UV}} - \ln a\right)$$

(8.16)

Clearly the integral becomes divergent if

$$d = 4 - 2\epsilon_{UV} \quad (8.17)$$

goes to 4 as $\epsilon_{UV} \to 0$ (assuming $\epsilon_{UV} > 0$).

We can easily perform the integral up to $O(\epsilon)$ with using the result

$$J_2 \equiv i \int d^d \ell_E (-\ell_E^2 - M^2)^{-2} = i \pi^{d/2} \Gamma(2 - d/2)(M^2)^{d/2-2} \quad (8.18)$$

when

$$I_2(-p_E^2, m^2, d) = i(4\pi)^{-d/2} \Gamma(2-d/2) \int_0^1 dx\, [m^2 + x(1-x)p_E^2]^{d/2+2}$$

$$= i(4\pi)^{-d/2} \left\{ \frac{1}{\epsilon_{UV}} - \gamma_{Euler} + \ln m^2 - \left(1 + \frac{4m^2}{p_E^2}\right)^{1/2} \cdot \ln \frac{\left(1 + \frac{4m^2}{p_E^2}\right)^{1/2} + 1}{\left(1 + \frac{4m^2}{p_E^2}\right)^{1/2} - 1} \right\} + O(\epsilon) \quad (8.19)$$

Due to the logarithms, I_2 has an imaginary part at $-p_E^2 \geq 4m^2$

Comment 1) : Wick rotation can be carried out for any arbitrary scalar loop diagram with L loop integral, I propagator and E external leg

After Wick rotation (allowed by the sign of $i\epsilon$) the denominator becomes positive definite for Euclidean values of loop-momentum p_E^2.

Comment 2) : Arbitrary one-loop scaler integral with k propagator factors

$$I_k(p_i p_j, m, d) = (2\pi)^{-d}\,\Gamma(k) \,*$$

$$* \int_0^1 \prod_{i=1}^{k} dx_i \cdot \delta(1-\Sigma x_i)\; \mathcal{J}_k^{(d)}\left(M^2(p_i p_j, \{x_i\}, m^2)\right) \qquad (8.20)$$

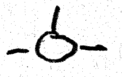
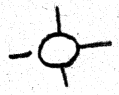
$$\mathcal{J}_k^{(d)}(M^2) \equiv i \int d^d \ell_E \,(-\ell_E^2 - M^2)^{-k}$$

$$= i\,\Omega_d \int_0^\infty dr\, r^{d-1}\,(-r^2 - M^2)^{-k} \qquad (8.21)$$

$d/2 \geq k$ $\quad \mathcal{J}_k^{(d)}(M^2)$ ultraviolet divergent

$$\mathcal{J}_k^{(d)}(M^2) = i\,(-1)^k \frac{\pi^{d/2}}{(k-d/2)} \frac{\Gamma(k-\frac{1}{2}d+1)}{\Gamma(k)} (M^2)^{d/2-k} \qquad (8.22)$$

This is the dimensionally continued form of the general one-loop integral. It is analytic in d and has simple poles at $d = 2k, 2k+2, \ldots$

$K=2$:  , $k=3$:  , $k=4$: 

Comment 3 : Renormalization and short distance physics

If the physics at extreme short distances do not influence observations significantly, then we can cut out the contributions at large momenta without modifying the results at large distances. But the loop integrals provide

$$\int^{\Lambda} \frac{dk}{k} \sim \ln \Lambda$$

Λ-dependence. The mass, coupling constant and field normalization factors in the Lagrangian of the theory are not directly measurable.

If the coupling, mass and normalization parameters depend on Λ such that they cancel the Λ-dependence generated by the ultraviolet cut-off of the loop integrals, we may obtain a well defined finite theory! Indeed this is possible for renormalizable theories.

It is hard to prove that in all orders of perturbation theory we can have a regularization of the UV singularities such that the above described cancellation occurs without violating the symmetries of the theory and the unitarity of the S-matrix.

8.3 Ultraviolet power counting

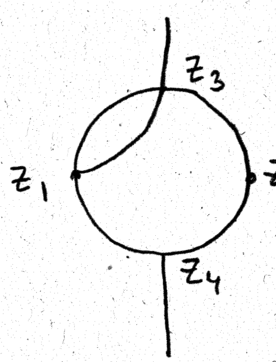
The euclidean integrals are well defined provided the integrals are convergent. At a given order of the coupling constant the contribution of a Feynman diagram is a well defined unique expression if the Feynman rules are fully fixed. In higher orders the corresponding function is represented as a (multi) loop integral. The physical dimension of the loop integral can be easily established. In general d-dimensional space time a one particle irreducible diagram has the superficial degree of divergence defined as

$$\omega(\Gamma) = d \cdot L - 2I^B - I^F + \sum_{i=1}^{M} \delta_i V_i \quad , \qquad I = I^B + I^F \qquad (8.23)$$

where L is the number of the loops, $I^{F(B)}$ is the number of internal fermion (boson) lines. We assume that we have only spin ϕ and spin 1 (gauge) bosons. V_i is the number of the vertices of type i and the type i vertices have δ_i loop-momentum factors.

First, we have the Euler-relation

$$L = I - V + 1 \qquad \text{where } \sum_{i=1}^{M} V_i = V \qquad (8.24)$$

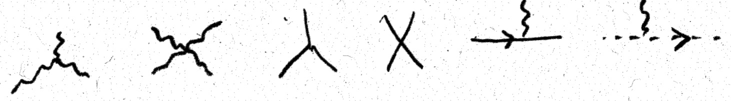


$E = 2, \quad V = 4 \quad I = 6$

example$_1$

$L = I - V + 1 = 3$

$$V_i = \{ V_{3g}, V_{4g}, V_{3\varphi}, V_{4\varphi}, V_{fg}, V_{gh} \} \qquad (8.25)$$



$$\omega(\Gamma) = d \cdot L - 2 I_G - I_f + V_{g,3} \quad \text{in QCD} \qquad (8.26)$$

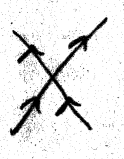
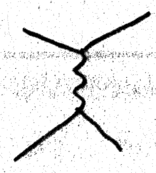
$\omega(\Gamma)$ is not the same as the dimension of the graph if we have dimension-full coupling

$\langle 0 | T \psi_1 \psi_2 \bar{\psi}_3 \psi_4 | 0 \rangle$, $\mathcal{L}_{4\psi} = \int d^4x \, (\bar{\psi}\psi)^2 G_F$

$\dim(G_F) = -2$

$$\omega(\Gamma) = -12 + 4 + 4 \cdot 3/2 = -2$$

example

 $G_F \sim \frac{g^2}{8 M_W^2}$ 

The fields have dimensions $\frac{d-2}{2}$ or $\frac{d-1}{2}$ (boson, fermion), there are 4 Fourier transformation wich has $-4 \cdot 4 = -16$ dimension, there is a momentum conserving δ-function which is dropped giving $+4$ increase.

We have $I+E$ Fourier integrals and $V+E$ δ-functions since at the vertices momentum is conserved. One δ-function gives the overall momentum conservation. Therefore the number of remaining four dimensional Fourier integrals is

$$L = (I+E) - (V+E) + 1 = I - V + 1 \qquad (8.24)$$

Furthermore

$$I^B = \sum_{i=1}^{V} \tfrac{1}{2} n_i^B V_i - \tfrac{1}{2} E^B \ , \quad I^F = \sum_{i=1}^{V} \tfrac{1}{2} n_i^F V_i - \tfrac{1}{2} E^F \qquad (8.25)$$

$$\omega(\Gamma) = d + \sum_{i=1}^{V} \Delta_i V_i - \frac{d-2}{2} E^B - \frac{d-1}{2} E^F \qquad (8.26)$$

where $I = I^B + I^F$, B denotes bosons, F denotes fermions n_i^B denotes the bosonic, n_i^F the fermionic legs of vertex type i

Δ_i denotes the physical dimension of the vertex

$$\Delta_i = \frac{d-2}{2} n_i^B + \frac{d-1}{2} n_F + \delta_i - d \qquad (8.27)$$

The integrals $\int_\mu^\infty dk\, k^{\omega(\Gamma)-1}$ are ultraviolet convergent if

$$\omega(\Gamma) < 0 \qquad (8.28)$$

$\omega(\Gamma)$ at fixed value of E^B, E^F increase with the number of those V_i vertices for which

$$\Delta_i > 0$$

Those theories for which all vertices have

$$\Delta_i \leq 0 \qquad (8.29)$$

are called renormalizable. Those theories which do not fulfil this condition are called unrenormalizable.

Theories for which all vertices have negativ Δ_i called super renormalizable.

If a theory is renormalizable

$$\omega(\Gamma) \leq d - \frac{d-2}{2} E_B - \frac{d-1}{2} E_F \qquad (8.30)$$

The number of the Green-functions which have positive overall divergence is finite since this upper limit is independent from the number of vertices.

Let us consider four dimensional renormalizable field theories. $\omega(\Gamma) \leq 4 - \frac{3}{2} E_F - \frac{1}{2} E_B$, and we get as allowed couplings:

$$\phi^3,\ \phi^4,\ \bar{\psi}\Gamma\psi\phi,\ F^a_{\mu\nu}F^{a\,\mu\nu},\ A\phi\partial_\mu\phi, \tag{8.31}$$

Other couplings are either non-gauge invariant or have $\Delta_i > 0$. $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i g\, c^{abc} A^b_\mu A^c_\nu$.

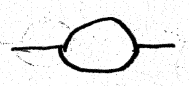
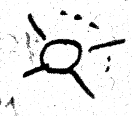
Comments

1. ϕ^3 in $d=4$ dimension is superrenormalizable

$$\Delta_{\phi_3} = \frac{2}{2}\cdot 3 - 4 = -1 = -4 + 3 = -1$$

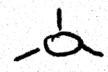
$$\omega(\Gamma)^{d=4}_{\phi_3} = 4 - V - E^B$$

Increasing V or increasing E^b $\omega(\Gamma)$ decreases.
Finite number of 1PI divergent diagrams:

 divergent, all others are finite (, )

2. ϕ^3 in $d=6$ dimension is renormalizable

$$\omega(\Gamma)^{d=6}_{\phi_3} = 6 - 2E^B$$

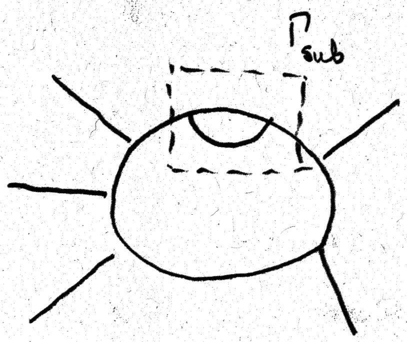
Only 2-leg and 3-leg diagrams have non-negative overall degree of divergences. , 

ϕ^3 in $D=6$ dimension is asymptotically free, therefore it is an interesting model to understand some properties of renormalizable asymptotically free theories.

iii) ϕ^4-theory in $D=4$ is renormalizable but not asymptotically free (see later).

In $d=5,6$ dimensions only ϕ^3 is renormalizable...

The overall degree of divergence gives the convergence criterium for the l-loop integrals. In the case of multi-loop diagrams, a diagram with $\omega(\Gamma) < 0$ can have sub-diagrams with $\omega(\Gamma_{sub}) \geq 0$



$$\omega(\Gamma_5) < 0 \,, \qquad \omega(\Gamma_{sub} \cup \Gamma_2) > 0$$

The renormalization theorems provide us with a simple algorithm to subtract and absorb all the divergent terms for any diagrams and subdiagrams.

The algorithm is most conveniently formulated using <u>counter terms</u>.

We shall explain the concept of counter terms in details in case of ϕ^3 theory in $d=6$ dimension.

8.4 The algorithm of renormalized perturbative calculations

Renormalization is carried out using the counter term algorithm in four steps.

i) One has to choose a regularization scheme (those which preserve the symmetries of the theory are preferred since it limits the number of the counter terms).

ii) Construct the counter terms allowed by the symmetries of the theory and the regularization scheme. If the regularization schem violates some of the symmetries then non-symmetric counter terms have to be included as well.

iii) The divergent part of the counter terms are uniquely fixed by requiring that ~~the~~ diagrams having counter term vertices cancel all divergences. The finite part of the counter term are fixed by selecting appropriate renormalization prescriptions

iv) Relate the renormalized parameters to physical observables and determine their values from the most precise observables. We can test the theory by extracting the renormalizable coupling constant and mass parameters from different measured quantities. The extracted values must agree within the experimental and theoretical error bars.

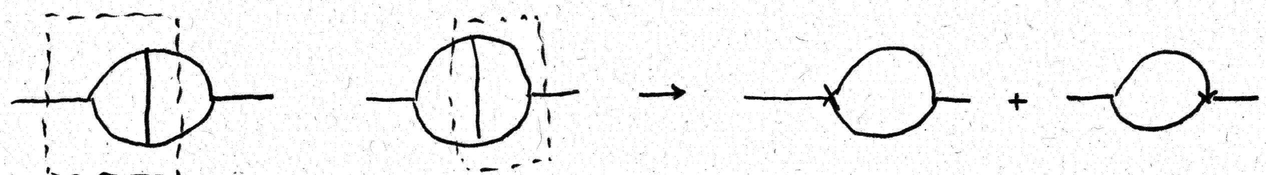
This algorith gives a well defined asymptotica series for S-matrix elements in powers of the renormalized coupling constants.

Comment

Enough to consider one-particle irreducible diagrams (1PI). The renormalization procedure is iterative: after fixing parameters of the counter terms up to (n-1) loop order we can calculate n-loop diagrams, with counter terms for all divergent subdiagrams. The result will be divergent, but the form of the divergence will have to form of counter terms of order n. We can fix the parameters of the order n counter terms from such n-loop results. Clearly, this iterative procedure leads to finite perturbation theory.

The counter term algorithm is a consequence of the renormalization theorem of Hepp, Zimmermann and Bogoliubov and Parasiuk. The have shown that subtracting from the loop integrand local terms to get finite integrals, the integral of the local subtraction terms can be reinterpreted as counter term contributions.

When also overlapping subgraphs are present



we subtract non-overlapping divergent subgraphs. This property allows us to interpret the subtraction as counter term contribution. The general proof that this is always the case is rather complex. The proof based on the BPHZ subtraction can be made regularization scheme independent.

8.5 Constructing counter terms for ϕ^3-theory in d=6 dimension

8.5.1 The form and parametrization of counter terms

$$\mathcal{L}_{bare}(\varphi, g\mu^{\frac{6-d}{2}}, m) = \mathcal{L}_{ren}(\varphi_R, g_R\mu^{\frac{6-d}{2}}, m_R) + \mathcal{L}_{CT} \qquad (8.32)$$

$$\varphi = Z_\phi^{1/2}\varphi_R, \quad m = Z_m^{1/2} m_R, \quad g = Z_g\, g_R \qquad (8.33)$$

$$\mathcal{L}_{CT} = (Z_\phi - 1)\frac{1}{2}\partial_\mu\varphi_R\partial^\mu\varphi_R - (Z_\varphi Z_m - 1)\frac{1}{2}m_R^2\varphi_R^2 + \frac{1}{6}g_R\mu^{\frac{6-d}{2}}(Z_g Z_\phi^{3/2} - 1)\phi_R^3 \qquad (8.34)$$

We simply rewrote the bare Lagrangian in terms of rescaled field, mass and coupling constant parameter. We separated terms such that $\mathcal{L}_{bare}$ has the same functional form in terms of unrenormalized quantities as $\mathcal{L}_{ren}$ in terms of renormalized quantities.

$$\mathcal{L}_{Ren}(\varphi_R, g_R\mu^{\frac{6-d}{2}}, m_R) = \frac{1}{2}\partial_\mu\varphi_R\partial^\mu\varphi_R - \frac{1}{2}m_R^2\varphi_R^2 - \frac{g_R\mu^{\frac{6-d}{2}}}{6}\varphi_R^3 \qquad (8.35)$$

We shall choose dimensional regularization. Therefor the Lagrangian and the corresponding Feynman diagrams are written in general d-dimension. The coupling constant g inserted in the form $g\mu^{\frac{6-d}{2}}$ remains dimensionless in d=6 dimension. Here μ is an arbitrary mass parameter which absorbs the physical dimension of the coupling constant.

The counter term Lagrangian (8.34) add new vertices to the set of Feynman rules we have derived with the „bare" Lagrangian.

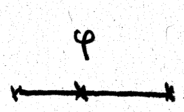
The peculiarity of these vertices that their „coupling constants are given as doppel series: Laurent series in the regularization parameter ϵ and power series in the renormalized coupling

$$(Z_\xi - 1) = \sum_{n=0}^{\infty} z_{\xi,n}(g_R)\frac{1}{\epsilon^n}, \qquad z_{\xi,n} = \sum_{l=0} \bar{g}_R^{2(n+l)} z_{\xi,n}^{(l)} \tag{8.37}$$

$$\xi = \varphi, m, g$$

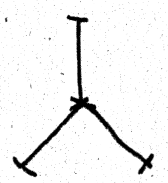
where $\bar{g}_R^2 = \dfrac{g^2}{(4\pi)^3}$. (8.38)

In next-to-leading order (at one-loop order) the new vertices are



$$-ip^2(Z_\varphi - 1) = -ip^2\left[\bar{g}_R^2\left(a_{\varphi,1}^{(1)}\frac{1}{\epsilon} + c_{\varphi,1}^{(1)}\right) + O(\bar{g}_R^4)\right]$$

$$+im_R^2(Z_\varphi Z_m - 1) = im_R^2\left[\bar{g}_R^2\left(\bar{a}_{m,1}^{(1)}\frac{1}{\epsilon} + \bar{c}_{m,1}^{(1)}\right) + O(\bar{g}_R^4)\right]$$



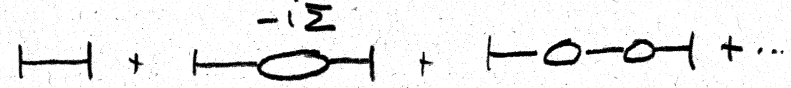
$$-ig_R(Z_g Z_\varphi^{3/2} - 1) = -ig_R\left[\bar{g}_R^2\left(\bar{a}_{g,1}^{(1)}\frac{1}{\epsilon} + \bar{c}_{g,1}^{(1)}\right) + O(\bar{g}_R^4)\right] \tag{8.39}$$

Note that to order $\bar{g}_R^4$ we can write

$$(Z_\varphi Z_m - 1) = (Z_\varphi - 1) + (Z_m - 1)$$
$$(Z_\varphi^{3/2} Z_g - 1) = \frac{3}{2}(Z_\varphi - 1) + (Z_g - 1) \tag{8.40}$$

8.5.2 Calculating $Z_\phi^{(1)}, Z_m^{(1)}$ in φ^3 theory in d = 6 dimension

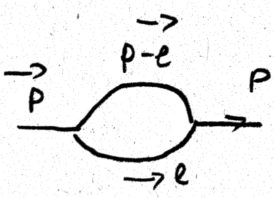
We recall the relation between the self-energy and the propagator

$\Delta(p^2) =$ 

Solving this geometric series for $-i\Sigma$ we obtain

$$i\,\Delta^{-1}(p^2) = p^2 - m_R^2 - \Sigma(p^2) \tag{8.41}$$

According the Feynman-rules



(a)

$$-i\,\Sigma(p^2) = \frac{1}{2}(-i)^2(i)^2 g_R^2\, \mu^{\frac{6-d}{2}} \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 - m_R^2 + i\epsilon][(l-p)^2 - m_R^2 + i\epsilon]} \tag{8.42}$$

We have evaluated this integral already using Feynman parametrization, making Wick rotation and using polar coordinates in the d-dimensional Euclidean-space. See (8.7), (8.14), (8.16).

Introducing

$$\epsilon = 3 - d/2 \tag{8.43}$$

we obtain

$$\Sigma^{(a)}(p^2) = \frac{1}{2}\,\frac{g_R^2}{(4\pi)^3}\,(4\pi\mu^2)^\epsilon\,\frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)}\;*$$

$$*\int_0^1 dx\,[m_R^2 - x(1-x)p^2 - i\epsilon]^{1-\epsilon} \tag{8.44}$$

In equation (8.44) the integral over x for small $\epsilon > 0$ is convergent. We expand the integrand and the prefactor to $\mathcal{O}(\epsilon)$ and we obtain

$$\Sigma^{(a)}(p^2) = -\frac{1}{2}\bar{g}_R^2 \left\{ \left[\frac{1}{\epsilon} + \ln\frac{4\pi\mu^2}{m_R^2} - \gamma_E + 1\right]\left(m_R^2 - \frac{1}{6}p^2\right) - m_R^2 \int_0^1 dx\, f(x)\ln f(x) \right\} \qquad (8.45)$$

Here the factor $1/6$ in front of p^2 was obtained as a result of the integral $\int_0^1 x(1-x)\,dx = 1/6$, furthermore we introduced the short hand notation

$$f(x) = 1 - x(1-x)\frac{p^2}{m_R^2} \qquad (8.46)$$

Let us calculate now the contribution of the counter term vertices

$$-i\Sigma^{(b)}(p^2) = \qquad (8.47)$$

$$\Sigma^{(b)}(p^2) = \left\{p^2(Z_\phi - 1) - m_R^2\left[(Z_\varphi - 1) + (Z_m - 1)\right]\right\} \qquad (8.48a)$$

$$= +\bar{g}_R^2\left\{p^2\left(a_{\varphi,1}/\epsilon + c_{\varphi,1}\right) - m_R^2\left(\bar{a}_{m,1}/\epsilon + \bar{c}_{m,1}\right)\right\} \qquad (8.48b)$$

The p^2/ϵ-term in (8.45) is cancelled if we have

$$a_{\varphi,1} = -\frac{1}{12}\frac{1}{\epsilon} \qquad \bar{a}_{m,1} = -\frac{1}{2}\epsilon\,, \quad a_{m,1} = \bar{a}_{m,1} - a_{\varphi,1} = -\frac{5}{12}\frac{1}{\epsilon} \qquad (8.49)$$

The sum of the two contribution is finite

$$\Sigma(p^2) = \Sigma_1^{(a)}(p^2) + \Sigma_1^{(b)}(p^2) =$$

$$-\frac{1}{2} g_R^2 \Big\{ \frac{1}{6} C_{\varphi,1}\, p^2 - m_R^2 \left(\frac{5}{6} C_{m,1} + \frac{1}{6} C_{\varphi,1}\right)$$
$$+ \left[m_R^2 - \frac{1}{6} p^2 \right] \left(1 - \gamma_E + \ln \frac{4\pi\mu^2}{m_R^2} \right)$$
$$- m_R^2 \int f(x) \ln f(x)\, dx$$
$$+ O(\epsilon), + O(g^4) \Big\} \qquad (8.50)$$

$$\delta m^2 = m_R^2 (Z_\phi Z_m - 1), \quad m = Z_m^{1/2} m_R, \quad \phi = Z_\phi^{1/2} \phi_R \ . \qquad (8.33)$$

The answer still ambigous: we need to fix the value of the constants $C_{\varphi,1}$ and $C_{m,1}$.

The procedure which fixes these constant is called renormalization prescription.

In the framework of dimensional regularization we can prescribe to all orders that all constant terms in all renormalization factors are vanishing

MS-scheme: $\qquad Z_{\xi,0}(g_R) = 0 \qquad (8.51)$

(see eq. (8.37)). This prescription is called the minimal subtraction scheme. By adding the counter term choosing $C_{\varphi,1}=0$ and $C_{m,1}=0$, in equation (8.50) the $\Sigma^{(b)}$ contribution subtracted only the $1/\epsilon$ pole terms of $\Sigma^{(a)}$, The origin of the term $1-\gamma_E + \ln 4\pi$ is the same trivial prefactor appearing in all one-loop contributions with the choice

$$\left.\begin{aligned} c_{\varphi,1} &= -\gamma_E + \ln 4\pi \\ c_{m,1} &= -\gamma_E + \ln 4\pi \end{aligned}\right\} \quad \underline{\overline{MS}\text{-scheme}} \qquad (8.52)$$

we eliminate this term from the finite expression. We can include this factor into the redefinition of ϵ

$$\frac{1}{\epsilon} \longrightarrow \frac{(4\pi)^{\epsilon}\,\Gamma(1+\epsilon)}{\epsilon} = \frac{1}{\bar{\epsilon}} \qquad (8.53)$$

If we expand instead of ϵ in $\bar{\epsilon}$, and we only subtract the pole terms in $\bar{\epsilon}$ we get the ($\overline{MS}$) subtraction-scheme.

A more "physical" subtraction scheme is <u>the "momentum" subtraction scheme</u>

The full propagator has the form

$$\Delta(p^2) = \frac{i}{p^2 - m_R^2 - \Sigma(p^2, m_R^2, c_\varphi, c_m)}$$

We require that at $p^2 = -M^2$

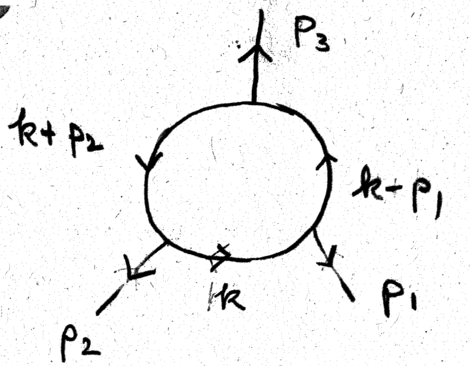
$$\left.\begin{aligned} \Sigma(p^2, m_R^2, c_\varphi, c_m)\Big|_{p^2=-M^2} &= -M^2 - m_R^2 \\ \frac{\partial \Sigma}{\partial p^2}\Big|_{p^2=-M^2} &= 1 \end{aligned}\right\} \qquad (8.54)$$

On-shell renormalization scheme is obtained within the momentum subtraction by requiring $M^2 = -m_R^2$

The two conditions of equations (8.54) give linear equations which fix the values of $c_{\phi,1}$ and $c_{m,1}$.

8.5.3 Calculating $Z_g^{(1)}$ in $\varphi^3_{d=6}$ theory

In $\phi^3_{d=6}$ theory the remaining ultraviolet diagram is the 1PI 3-point one-loop diagram



$\Gamma_3^{(a)}(p_1, p_2, p_3) =$

Very similar calculation to $-i\Sigma^{(a)}(p^2)$ except that we have two Feynman parameters (see equation (8.71))

$p_1 + p_2 + p_3 = 0$

$$\Gamma_3^{(a)}(p_1, p_2, p_3) = -g_R\mu^{\epsilon}\left\{ \bar{g}_R^2 (4\pi\mu^2)^{\epsilon}\, \Gamma(\epsilon) \int_0^1\!\!\int_0^1 \frac{dx\,dy\, \Theta(1-x-y)}{[M^2(\{p_i^2\}, x, y, m_R^2) - i\epsilon]^{\epsilon}} \right\} \quad (8.55)$$

where

$$\left.\begin{aligned} M^2(\{p_i^2\}, x, y, m_R^2) &= m_R^2 + 2xy\, p_1 p_2 - x(1-x)p_1^2 - y(1-y)p_2^2 \\ 2p_1p_2 &= p_3^2 - p_1^2 - p_2^2 \end{aligned}\right\} \quad 8.56$$

Again, the integrand is finite and we can carry out the integrations $dx\,dy$ first by expanding the integrand in ϵ to $\mathcal{O}(\epsilon)$.

$$\Gamma_3^{(a)} = (-g_R\mu^\epsilon)\left[\tfrac{1}{2}\bar{g}_R^2\right] * \left\{\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m_R^2} - \right.$$
$$\left. - 2\int_0^1 dx \int_0^1 dy\, \theta(1-x-y)\, \ln\left(\frac{M^2 - i\epsilon}{m_R^2}\right) + O(\epsilon)\right\} \qquad (8.57)$$

$$\Gamma_3^{(6)} = \Gamma_{3,CT} = g_R\mu^\epsilon\left[(Z_\varphi)^{3/2} Z_g - 1\right]$$
$$= g_R\mu^\epsilon\left[\tfrac{3}{2}(Z_\varphi - 1) + Z_g - 1\right] \qquad (8.58)$$

using our results on $(Z_\phi - 1) = \bar{g}_R^2\left[-\frac{\lambda}{12}\frac{1}{\epsilon} - \frac{1}{12}C_{\varphi,1}\right]$

we obtain for $(Z_g - 1)$

$$Z_g^{(1)} - 1 = \bar{g}_R^2\left[-\frac{3}{8}\left(\frac{1}{\epsilon} + C_{g,1}\right)\right] \qquad (8.57)$$

Again in the momentum subtraction scheme we need to impose a renormalization condition to fix $C_{g,1}$. Customarily

$$\Gamma_3 = \left\{\Gamma_3^{(a)} + \Gamma_3^{(6)}\right\}_{p_i^2 = -M_i^2} = -g_R \qquad (8.58)$$

where the subtraction is at $p_i p_j = -\frac{1}{2}M^2(3\delta_{ij} - 1)$

This condition fixes the value of $C_{g,1}$.

Note that Γ_3 depends on μ in all schemes. In the momentum subtraction scheme it is customary to make the choice $\mu^2 = -M^2$.
Physical quantities are given in terms of renormalized Green function, which depend μ (M) and g_R. The experimentally fitted value of g_R depends on the chosen value of μ or M.

8.6 The renormalization group

8.6.1 Renormalized and unrenormalized Green-functions

$$G_R(x_1,..,x_n, g_R, m_R, \mu, \epsilon) = \langle 0|T\phi_R(x_1)\,\phi_R(x_2)..\,\phi_R(x_n)|0\rangle \quad (8.59)$$

$$G_{bare}(x_1,..x_n, g, m, \epsilon) = \langle 0|T\phi(x_1)..\quad\phi(x_n)|0\rangle$$

Since the Lagrangian did not change $\mathcal{L}_{bare} = \mathcal{L}_{ren} + \mathcal{L}_{CT}$ if we evaluate G in all orders in perturbation theory

$$Z_\phi^{n/2}(g_R, m_R, \mu, \epsilon)\, G_R(x_1,..x_n, g_R, m_R, \mu, \epsilon) \quad (8.60)$$

$$= G_{bare}(x_1,..x_n, g, m, \epsilon)$$

We can make a formal proof by using the generator functional of Green functions

---

Reminder:

We are interested in connected Green-functions and one-particle irreducible Green-functions

$$Z[J] = \frac{\int \mathcal{D}\varphi\, e^{i\langle \mathcal{L}\rangle + i\langle J\varphi\rangle}}{\int \mathcal{D}\varphi\, e^{i\langle \mathcal{L}\rangle}} = e^{iW[J]} \quad (8.61)$$

$Z[J]$ = generating functional of Green-functions

$W[J]$ = generating functional of connected Green-functions

$$Z[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} dx_i\, J(x_i)\right) G(x_1,..,x_n) \quad (8.62)$$

$$W[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod dx_i\, J(x_i)\, G_c(x_1,..,x_n) \quad (8.63)$$

$$\Gamma[\phi_0] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod dx_i\, \phi_0(x_i)\, \Gamma_n(x_1 \dots x_n) = W[J] - \langle J\varphi\rangle \qquad (8.64)$$

$\Gamma[\phi_0] =:$ generating functional of 1PI Green-functions

end of reminder

$$Z_R[J] = \frac{\int \mathcal{D}\varphi_R\, e^{i\langle \mathcal{L}(\phi_R, g_R, m_R, c_i, \epsilon)\rangle + i\langle J\varphi_R\rangle}}{\int \mathcal{D}\varphi_R\, e^{i\langle \mathcal{L}(\varphi_R, m_R, c_i, \epsilon)\rangle}} =$$

$$= \frac{\int \mathcal{D}\varphi\, e^{i\langle \mathcal{L}(\phi, m, g, \epsilon)\rangle + i\langle J Z_\phi^{-1/2}, \varphi\rangle}}{\int \mathcal{D}\varphi\, e^{i\langle \mathcal{L}(\varphi, m, g, \epsilon)\rangle}}$$

$$= Z[Z_\phi^{-1/2} J] = e^{i W_R[J]} = e^{i W_R[Z_\phi^{-1/2} J]} \qquad (8.65)$$

The equation (8.60) follows by taking variations. Of course this was expected since $\mathcal{L}$ did not change

Clearly (8.60) remains true also for connected Green functions

Now we can easily prove that

$$Z_\phi^{-n/2}\, \Gamma_R(x_1 \dots x_n, g_R, m_R, \mu, \epsilon) = \Gamma(x_1, \dots x_n, g, m, \epsilon) \qquad (8.66)$$

If we compare (8.66) and (8.60), note that in (8.66) Z_ϕ appears with inverse power.

Proof of equation (8.65)

$$\Gamma_R(\phi_c, g_R, m_R) = -\langle \phi_{c,R}, J\rangle + W_R[J, g_R, m_R]$$
$$= -\langle Z_\phi^{1/2} \phi_{c,R}, J'\rangle + W[J', g, m]$$

$$Z_\phi^{1/2} \phi_{c,R} = i\,\frac{\delta W}{\delta J'}$$
$$= \Gamma_{bare}[Z_\phi^{1/2} \phi_{c,R}, g, m]$$

q. e. d.

8.6.2 The renormalized S-matrix

The S-matrix is obtained by evaluating the residues of the poles of Green-functions in momentum space. This follows from the LSZ reduction formula.

The Green-functions in momentum-space are obtained by Fourier transformations

$$\tilde{G}(k_1, k_2, .. k_n) = \prod_{i=1}^{n}\left(\int d^4x_i\, e^{-ik_ix_i}\right)\langle 0| T\,\varphi(x_1) \dots \varphi(x_n)|0\rangle \quad (8.67)$$

this contains the overall momentum conserving δ-function factor. $G(k_1, .. k_n)$ is obtained by dropping this factor.

The LSZ-theorem states that

$$\langle 0, out| \{k_i\}_n, in\rangle = \left[i\,(2\pi)^{3/2} R^{1/2}\right]^{-n} \prod_{i=1}^{n} (k_i^2 - m^2)\, \tilde{G}_n(k_1, .. k_n)\Big|_{k_i^2 = m^2}$$

where

$$R(k_i^2) = (2\pi)^3 \left[-i\,(k_i^2 - m^2)\, G_2(p^2)\right] \quad \begin{matrix}(8.68)\\(8.69)\end{matrix}$$

Then

$$R_R = \lim_{p^2 \to m_p^2} \left[-i (p^2 - m_p^2)\, G_{2,R}(p^2) \right]$$

($G_{2,R}(p^2) = \Delta_R(p^2)$ in IZ notation)

Then clearly

$$R_R = Z_\phi^{-1} R \qquad (8.70)$$

The multiplicative renormalization constant momentum independent, therefore the pole position of G_2 does not change because of renormalization

The S-matrix, therefore is defined in the same way in terms of renormalized and unrenormalized Green functions and residues, since Z_ϕ cancels in the ratio

$$S(p_i) = \prod_i^n \left[i (2\pi)^{3/2} R_R^{1/2} \right]^{-n} \prod (p_i^2 - m_R^2)\, \tilde{G}_n(p_1 \ldots p_n) \Big|_{p_i^2 = m_R^2} \qquad (8.71)$$

The S-matrix is the same with renormalized and unrenormalized Green-functions! Therefore it must be the same for every renormalization scheme and any choice of the scale parameters unavoidably introduced in the regularization scheme and the renormalization prescriptions. In the case of MS ($\overline{MS}$) scheme we have only one such scale: μ, appearing together with the coupling constant

$$g \to g_R\, \mu^\epsilon$$

8.6.3 The value of the renormalized mass and renormalized coupling

In the $\overline{MS}$-scheme e.g. $\Gamma_2(m_p^2)=0$

$$0 = m_p^2 - m_R^2 + \frac{\bar{g}_R^2}{2}\Big[m_R^2 - \frac{1}{6} m_p^2 + $$
$$+ m_R^2 \int_0^1 dx \, [1 - x(1-x) \frac{m_p^2}{m_R^2} \ln\Big[(1-x(1-x)\frac{m_p^2}{m_R^2}) \frac{m_R^2}{\mu^2}\Big] \quad (8.72)$$

This defines the relation between the pole mass m_p^2 and the renormalized $\overline{MS}$ mass m_R^2

$$m_R^2 = F(m_p^2, \mu^2, g_R^2) = m_R^2(\mu^2, g_R(m_p, \mu)) \quad (8.73)$$

$\uparrow$ measured mass

Similarly for some measured physical quantity we obtain

$$\mathcal{O}_{measured} \simeq \mathcal{O}^{(LO)}(m_R^2, g_R^2, \{p_i\}) +$$
$$+ \mathcal{O}^{NLO}(m_R^2, \mu^2, g_R^2, \{p_i\}) \quad (8.74)$$

The numerically obtained value of $\mathcal{O}_{measured}$ allows to calculate g_R^2 in the regularization scheme and choice of the μ parameter in the $\overline{MS}$-scheme to next-to-leading order (NLO) accuracy

$$g^{(NLO)}_{R,\overline{MS}} = g_R^{NLO}(\mathcal{O}_{measured}, m_p, \mu) =: g_R^{(\overline{MS},NLO)}(\mu) =: g_R(\mu) \quad (8.75)$$

where for simplicity of the notation we suppressed the dependence on $\mathcal{O}_{measured}$ and $\overline{MS}$ and NLO.

Testing the validity of the theory:

If we use the same renormalization scheme and same renormalization prescription, the numerical value of $g_R^{(NLO)}$ must be the same for any choice of the measured physical quantity (of course within the accuracy of the different measurement and the accuracy of the NLO calculation)

The precision of the theoretical evaluation must match the precision of the measurements.

8.6.4 Renormalization group for S-matrix elements (or physical quantities)

In the $\overline{MS}$ scheme e.g.

$$\mu \frac{d}{d\mu} S(\{k_i\}, g, m)\Big|_{\text{at fixed } g, m} = 0 \tag{8.76}$$

$$\mu \frac{d}{d\mu} S(\{k_i\}, g_R, m_R, \mu)\Big|_{\text{at fixed } g, m} = 0 \tag{8.77}$$

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(g_R, m_R) \frac{\partial}{\partial g_R}\Big|_{\mu, m_R} - \gamma_m(g_R, m_R)\left(m_R \frac{\partial}{\partial m_R}\right)\Big|_{\mu, g_R}\right) * S(\{k_i\}, g_R(\mu, m_R), m_R(g_R, \mu)) = 0$$

(8.78)

where

$$\beta(g_R, m_R) = \mu \frac{\partial}{\partial\mu} g_R(\mu) \Big|_{g, m} \qquad (8.79)$$

$$\gamma_m(g_R, m_R) = -\frac{1}{2 m_R^2} \mu \frac{\partial}{\partial\mu} m_R^2(\mu) \Big|_{g, m} \qquad (8.80)$$

The advantage of equation (8.78) in comparison with (8.77) that all derivatives with fixed bare quantities are absorbed into the physics independent universal functions β and γ_m

To make these equation useful we have to calculate β and γ_m.

Equation (8.78) is known as <u>renormalization group equation</u>.

Why group? Gell-Mann and Low invented the name. Shifting the value of μ is something then the group of dilatation symmetry? Indeed it is related to the violation of dilation symmetry.

$$g_R(\mu) = g \mu^{-\epsilon} Z_g^{-1} \Big|_{g, m \text{ fixed}} \qquad (8.81)$$

$$\beta(g_R, m_R) = g \mu \frac{\partial}{\partial\mu} (\mu^{\epsilon} Z_g)^{-1} \qquad (8.82)$$

$$\gamma_m(g_R, m_R) = \frac{m^2}{2 m_R^2} \mu \frac{\partial}{\partial\mu} Z_m^{-1} \Big|_{g, m \text{ fixed}} \qquad (8.83)$$

To evaluate this derivatives is particularly simple if Z_g and Z_m are independent from m_R and μ, which is the case in the MS ($\overline{MS}$) schemes.

8.7 Calculating the β-function in MS-scheme

$$g = g_R \mu^{\epsilon} Z_g \quad \text{therefore} \tag{8.84}$$

$$\epsilon\, g_R Z_g + \mu \frac{\partial}{\partial \mu}(g_R Z_g)\Big|_{g \text{ fixed}} = 0 \tag{8.85}$$

and

$$g_R Z_g = g_R \left(1 + \sum_{n=1}^{\infty} z_{g,n}(g_R) \frac{1}{\epsilon^n}\right) \tag{8.86}$$

(see equation (8.371))

We introduce the notation

$$a_n(g_R) = z_{g,n}(g_R) \cdot g_R \tag{8.87}$$

Then we can write

$$\epsilon\left(g_R + \sum_{n=1}^{\infty} a_n \epsilon^{-n}\right) = \underbrace{-\mu \frac{\partial g_R}{\partial \mu}\Big|_g}_{\text{this is the } \beta\text{-function}} \left\{\left(1 + \sum_{n=1}^{\infty} \frac{da_n}{dg_R} \epsilon^{-n}\right\}\right. \tag{8.88}$$

In the left-hand side and the right-hand side we have power series in ϵ. All coefficient must match.

Assuming that $\beta(g,\epsilon)$ is analytic in ϵ we can write it also in power series

$$\beta(g_R,\epsilon) = \sum_{i=0}^{M} \beta_i \epsilon^i \tag{8.89}$$

We can easily see by inserting 8.89 into (8.88) that $\beta_i = 0$ if $i > 1$

$$\beta_1 = -g_R \quad , \quad \beta_0 = -a_1 + g_R \frac{da_1}{dg_R} \tag{8.59}$$

therefore

$$\beta(g_R, \epsilon) = -\epsilon g_R - a_1 + g_R \frac{da_1}{dg_R} \tag{8.60}$$

It is remarkable that the higher pole terms in the expansion of Z_g can be obtained via the recursion relations

$$g_R^2 \frac{d}{dg_R}(a_{n+1}/g_R) = \left[-a_1(g_R) + g_R \frac{da_1}{dg_R}\right] \frac{da_n}{dg_R} \tag{8.61}$$

The whole Z_g is completely determined by the coefficient of the single pole in Z_g. But beyond one loop it is must difficult to calculate the single pole term. Therefore these recursion relations can only be used for testing purposes

We obtain similar properties also for γ_m.

In $\phi^3_{d=6}$ theory (see eq. 8.57)

$$Z_g = 1 - \frac{3}{8}\bar{g}_R^2 \frac{1}{\epsilon} \qquad a_1(g_R) = -\frac{3}{8}\frac{1}{(4\pi)^3} g_R^3 \tag{8.62}$$

therefore

$$\beta(g_R, \epsilon) = -\epsilon g_R + \frac{3}{8}\frac{g_R^3}{(4\pi)^3} - \frac{9}{8}\frac{g_R^3}{(4\pi)^3}$$

$$= -\epsilon g_R - \frac{3}{4}\frac{g_R^3}{(4\pi)^3} \tag{8.63}$$

<u>Excercise</u>: show that in $\varphi^4_{d=4}$ theory

$$\beta(g_R, \epsilon) = -\epsilon g_R + 3\frac{g_R^2}{(4\pi)^2} \tag{8.64}$$

8.8 Running couplings, Landau-pole, asymptotic freedom, fixed points

If we know the β-function then we can calculate the value of the renormalized coupling at any scale μ provided we know its value at some fixed scale μ_0.

Let us first introduce the logarithmic mass

$$t = \ln \mu^2 \qquad (8.65)$$

then $$\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R)$$

can be written as

$$\frac{d}{dt} g_R^2(t) = g_R \beta^{(3)}(g_R) = -\frac{3}{4} \frac{(g_R^2)^2}{(4\pi)^3} \qquad (8.66)$$

or $$\frac{d}{dt} \lambda_R(t) = \frac{1}{2} \beta^{(4)}(\lambda_R) = \frac{3}{2} \frac{\lambda_R^2}{(4\pi)^2} \qquad (8.67)$$

where $\beta^{(3)}, \beta^{(4)}$ denote the β-functions of the $\phi^3_{d=6}$ and $\phi^\lambda_{d=4}$ theories, respectively.

We can easily integrate the differential equations using

$$\int \frac{dg_R^2}{(g_R^2)^2} = -\frac{3}{4} \frac{1}{(4\pi)^3} \int dt \qquad (8.68)$$

or

$$\int \frac{d\lambda_R}{\lambda_R^2} = +\frac{3}{2} \frac{1}{(4\pi)^2} \int dt \qquad (8.69)$$

we obtain

$$\lambda_R(\mu^2) = \frac{\lambda_R(\mu_0^2)}{1 - \frac{3}{2}\frac{\lambda_R^2(\mu_0)}{(4\pi)^2}\ln\frac{\mu^2}{\mu_0^2}} \qquad (8.70)$$

and

$$g_R^2(\mu^2) = \frac{g_R^2(\mu_0^2)}{1 + \frac{3}{4}\frac{g_R^2(\mu_0^2)}{(4\pi)^3}\ln\frac{\mu^2}{\mu_0^2}}$$

Let us consider the high-energy behaviour: $\mu^2 \gg \mu_0^2$

$\lambda_R(\mu^2)$ becomes infinite where the denominator vanishes. The running coupling has a pole the so called Landau-pole at the scale

$$\mu_{LP}^2 = \mu_0^2 \, e^{2(4\pi)^2/3\lambda_R(\mu_0^2)} \qquad (8.71)$$

This theory is meaningful only as effective field theory. At the scale of the position of the Landau pole it has to be modified.

It was conjectured by Wilson and proved by J. Fröhlich that non-perturbatively $\lambda\phi^4$ theory is equivalent with field theory. $\lambda\phi^4$ theory flows to the trivial theory.

More interestingly if the β-function is negative

$$\lim_{\mu^2\to\infty} g_R^2(\mu^2) \sim \lim_{\mu^2\to 0} \frac{4(4\pi)^3}{3\, g_R(\mu_0^2)} \frac{1}{\ln\frac{\mu^2}{\mu_0^2}} \to 0 \qquad (8.72)$$

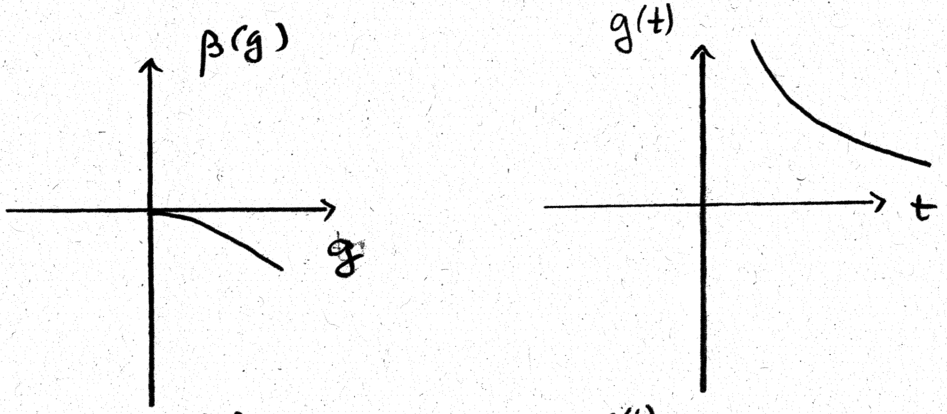
In $\varphi^3_{d=6}$ theory at large momentum scales the running coupling constant vanishes logarithmically

$$g_R^2(\mu^2) \xrightarrow[\mu^2 \to \infty]{} \sim \frac{c}{\ln \mu^2/\mu_0^2} \qquad (8.73)$$

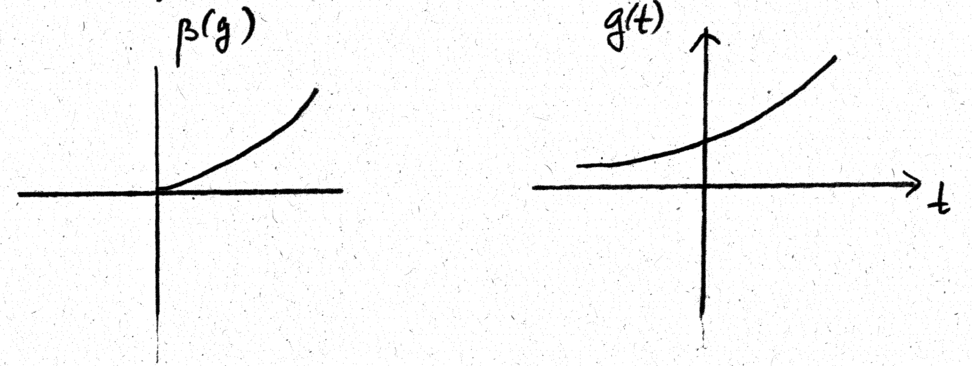
This is the consequence of negative β function at one-loop.
Theories with negative β-functions we call <u>asymptotically free theories</u>

These result have been obtained at one-loop.
The result however are uneffected by the corrections of higher loops since at sufficiently small g_R the higher loop corrections are suppressed by powers of the coupling.

In general we can have three distinct behaviours of running coupling



asymptotic freedom



infrared freedom

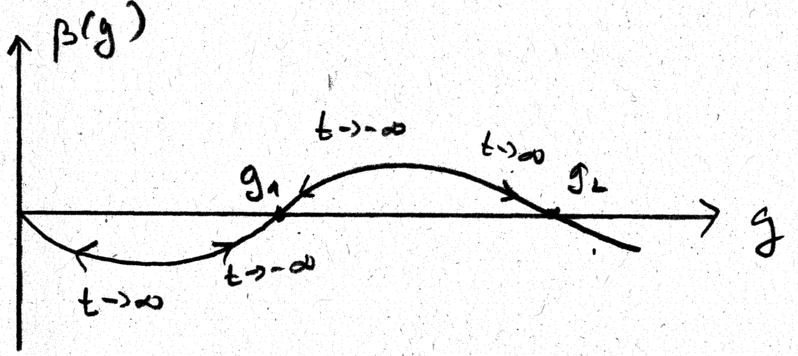


Figure fixed point

Imagine that the $\beta(g)$ has zeros at some values of $g = g_i^*$. Then depending on the first derivative of β with respect g around g^* we either have infrared or ultraviolet fixed points.

In the example given on the figure above if

$$g_0 < g_R < g_1 \quad , \quad g_0 = 0$$

we have an ultraviolet fixed point at $g_0 = 0$ and a region of attraction of $g_0 = 0$, while region

$$g_1 < g_R < g_2$$

is an attraction of the infrared fixed point g_1.

A $\beta(g)$ as given in „Figure fixed point" can not be obtained in perturbation theory.

8.9 Renormalization of gauge theories at one-loop

We already have learned the Lagrangian of QCD. See equation (5.5.11.).

The renormalizability and unitarity of gauge theories are one of the main results of perturbative field theory. The renormalization procedure described in the previous sections in the case of scalar field theories applies straightforwardly to gauge theories. The classical Lagrangians are modified by adding gauge fixing and ghost terms. The complication in the case of gauge theories with respect to scalar theories the allowed propagation and mixture of physical (quarks, transverse gluons) and unphysical (ghosts, longitidinal and time-like gluons). The one-loop renormalization of gauge theories in dimensional regularization indicate that the gauge symmetries of the classical Lagrangian are respected also by the renormalization constant (Ward-identities) and imply the gauge-invariance of the physical S-matrix. The also imply the unitarity of the physical S-matrix. The effects of the unphysical gluon-degrees of freedom introduced by gauge fixing are cancelled by the contributions from ghost particles. These features which can easily be seen at one loop can be extended to any fixed order of perturbative gauge theories. In the case of chial gauge theories the gauge symmetry may fail. Renormalizable theories are require the cancellation of the so called axial anomalies. The latter in turn requires relations between the charges of the matter fields. The gauge theory of the electroweak interactions is a chiral gauge theory and in the case of one quark and lepton doublet and one quark and lepton singlet the axial anomalies cancel.

In this section we shall discuss only vector gauge theories via their physical prototype QCD. Vector gauge theories are free from axial anomalies.

8.9.1 The renormalized Lagrangian of QCD

$$\mathcal{L}_{eff}(A_{\mu,ren}, \psi_{ren}, \bar\psi_{ren}, \xi_{ren}, m_R, g_R, \alpha_R) =$$

$$= \bar\psi^{(f)}_{ren}\, i \not{D}\, \psi^{(f)}_{ren} - m^{(f)}_{ren}\, \bar\psi^{(f)}_{ren}\psi^{(f)}_{ren} - \frac{1}{4} F^a_{\mu\nu,ren} F^{\mu\nu,a}_{ren}$$
$$- \frac{1}{2\alpha_R} \partial_\mu A^a_{ren} \partial^\mu A^a_{ren} + \partial^\mu \bar\xi^a_{ren} (\delta^{ab}\partial_\mu - g_R(\mu)\mu^\epsilon f^{abc}) \xi^c_{ren} \qquad (8.74)$$

$$(\not{D})_{ij} = \gamma^\mu (\delta_{ij}\partial_\mu + i g_R \mu^\epsilon T^a_{ij} A^a_{\mu,ren})$$

$\mathcal{L}_{eff}$: is the classical Lagrangian of non-Abelian gauge theory plus the gauge fixing term plus the corresponding Fadeev-Popov term, all in terms of renormalized fields, coupling constant and masses

$$\mathcal{L}_R(g_R, m_R, \alpha_R) = \mathcal{L}_{eff}(g_R, m_R, \alpha_R) + \mathcal{L}_{CT}(g_R, m_R, \alpha_R) \qquad (8.75)$$

where $\mathcal{L}_{CT}$ is obtained by the usual decomposition of the non-renormalized fields, couplings and masses in terms of renormalization factors and renormalized fields, masses and couplings

$$\left.\begin{aligned} &A_\mu = Z_A^{1/2} A_{\mu,ren}, \quad \psi = Z_\psi^{1/2}\psi_{ren}; \quad \xi = Z_\xi^{1/2}\xi_{ren}, \\ &g = g_R \mu^\epsilon, \quad m = Z_m m_R, \quad \alpha = Z_\alpha \alpha_R \end{aligned}\right\} (8.76)$$

$$\begin{aligned}
\mathcal{L}_{CT} = {} & \bar\psi_{ren}\,(i\partial\!\!\!/)\,\psi_{ren}\,(Z_\psi - 1) - g_R\mu^\epsilon\,\bar\psi_{ren}T^a A\!\!\!/^a_{ren}\psi_{ren}\,(Z_1 - 1) \\
& - m_R\,\bar\psi_{ren}\,\psi_{ren}\,(Z_m Z_\psi - 1) - \frac{1}{2\alpha}\left(\partial_\mu A^a_{ren}\right)\left(\partial^\mu A^a_{ren}\right)\left(Z_{gf} - 1\right) \\
& - \frac{1}{4}\left(\partial_\mu A^a_{\nu,ren} - \partial_\nu A^a_{\mu,ren}\right)\left(\partial^\mu A^{\nu,a}_{ren} - \partial^\nu A^{\mu,a}_{ren}\right)\left(Z_A - 1\right) \\
& + \frac{1}{2} g_R\mu^\epsilon f^{abc} A^{\nu,a}_{ren} A^{\lambda,b}_{ren}\left(\partial_\nu A^c_{\lambda,ren} - \partial_\lambda A^c_{\nu,ren}\right)\left(Z_1' - 1\right) \\
& - \frac{1}{2} g_R^2\mu^{2\epsilon} f^{abc} f^{ab'c'} A^{\nu,b}_{ren} A^{\lambda,c}_{ren} A^{b'}_{\nu,ren} A^{c'}_{\lambda,ren}\left(Z_4 - 1\right) \\
& - \bar\xi^a_{ren}\,\partial_\mu\partial^\mu\,\xi^a_{,ren}\left(Z_\xi - 1\right) + g_R\mu^\epsilon f^{abc}\,\bar\xi^a_{ren}\,\partial_\mu A^{b,\mu}_{ren}\,\xi^c_{ren}\left(Z_1'' - 1\right)
\end{aligned} \qquad (8.77)$$

where

$$\left.\begin{aligned}
& Z_1 = Z_\psi Z_A^{1/2} Z_g\,, \quad Z_1' = Z_A^{3/2} Z_g\,, \quad \underline{Z_1'' = Z_\xi Z_A^{1/2} Z_g} \\
& \qquad Z_4 = Z_A^2 Z_g^2\,, \quad Z_{gf} = Z_\alpha^{-1} Z_A
\end{aligned}\right\} \quad (8.78)$$

Since dimensional regularization does not break the gauge symmetry of the Lagrangian all interaction vertices are proportional to the <u>same</u> renormalized coupling. This in turn requires relations between the renormalization constants

$$Z_1/Z_\psi = Z_1'/Z_A = Z_1''/Z_\xi = \left(Z_4/Z_A\right)^{1/2} \qquad (8.79)$$

In addition one obtains the transversality condition

$$Z_\alpha = Z_A \qquad (8.80)$$

(no counter term for the gauge-fixing term at one-loop)

Slavnov - Taylor identities

The identities of equation (8.75) are called Slavnov-Taylor identities. It gives relations between the renormalization counter terms of proper vertices. Because of these identities we can use the same $g_R(\mu)$ renormalized coupling constant for all the different interaction vertices of the renormalized perturbation series. Because of these identities the gauge structure of the renormalized Lagrangian is the same as of the classical Lagrangian and therefore it also preserves unitarity.

8.9.2 Renormalization group for QCD

The renormalization group equations can be derived for gauge theories in just the same way as scalar theories.

In particular, observables, such as the S-matrix obey exactly the same equations (see equation (8.76)-(8.78)) as before.

One can derive renormalization group equations also for 1PI Green-functions. They differ slightly from the renormalization equation of observables since the renormalized Green functions are not the same with renormalized and unrenormalized parameters. The difference is coming from the renormalization factors of the external fields (see equation 8.66).
therefore we obtain

$$\mu \frac{d}{d\mu} \Gamma_n = 0$$

and therefore

$$\left[\left(\mu \frac{\partial}{\partial \mu}\right)\Big|_{g_R, m_R} + \beta(g_R) \frac{\partial}{\partial g_R}\Big|_{\mu, m_R} - \gamma_m(g_R)\left(m_R \frac{\partial}{\partial m_R}\right)\Big|_{\mu, g_R} - n\,\gamma_\phi(g_R) \right] \Gamma_{ren,n} = 0 \qquad (8.81)$$

where

$$\gamma_\phi(g_R) = \frac{1}{2}\left(\mu \frac{d}{d\mu}\right)(\ln Z_\phi) = \frac{1}{2}\beta(g_R)\frac{\partial}{\partial g_R}\ln Z_\phi \qquad (8.82)$$

The finiteness of $\Gamma_{ren,n}$ implies that $\gamma_\phi(g_R)$ is finite as $\epsilon \to 0$. The only finite term comes from the $-\epsilon g$ term in $\beta(g)$ times the $1/\epsilon$ term in Z_ϕ. All the other terms must be cancelled. ~~Therefore~~

$$\gamma_\phi(g_R) = -\frac{1}{2} g_R \frac{\partial}{\partial g_R}(\text{single pole term in } Z_\phi) \qquad (8.83)$$

In $\phi^3_{d=6}$ theory

$$\gamma_\phi(g_R) = \frac{1}{12}\frac{g^2}{(4\pi)^3} + O(g^4) \qquad (8.84)$$

Equation (8.81) is the renormalization group equation for 1PI Green-functions in scalar theory. $\gamma_\phi(g_R)$ is called the anomalous dimension of the field ϕ. The name is following from the role of γ_ϕ in equation (8.81). The solution of (8.81) indicates that if we change the scale of the momentum it can be absorbed into a scale change into the running coupling and mass and an overall scale change.

$$\Gamma_{ren,n}\left(e^{-2t}\frac{k_i k_j}{\mu^2}, \frac{\bar{m}^2(t)}{\mu^2}, \bar{g}(t)\right) e^{[(4-n)t + \int^t \gamma_\phi(t')dt']}$$

$$= \Gamma_{ren}\left(\frac{k_i k_j}{\mu^2}, \frac{m_{R}^2}{\mu^2}, g_R\right) \qquad (8.85)$$

is the solution of the renormalization group equation (8.81). From this we see that in addition to the physical dimension $(4-n)$, the 1PI Green function picks up an „anomalous" scale factor when we change the scale μ. $(\bar{m}^2(t) = \bar{m}(m_R, t)$, $\bar{m}(m_R, 0) = m_R$;

In gauge theory the new feature is that the Green-function also depend on the gauge parameter since 1PI Green-functions in general are not gauge invariant.

We define the

$$\beta_\alpha(g_R, \alpha_R) = \mu\frac{\partial}{\partial\mu}\alpha_R\Big|_{g,m} \qquad (8.86)$$

as the „β-function" of the gauge parameter we obtain the RG-equation

$$\left[\left(\mu\frac{\partial}{\partial\mu}\right) + \beta(g_R)\frac{\partial}{\partial g_R} - \gamma_m(g_R)\, m_R\frac{\partial}{\partial m_R} + \beta_\alpha(g_R, \alpha_R)\frac{\partial}{\partial\alpha_R}\right.$$

$$\left. - n_A\gamma_A - n_f\gamma_\psi - n_{gh}\gamma_{gh}\right]\Gamma_{R,n}(g_R, m_R, \alpha_R) = 0 \qquad (8.87)$$

where n_A, n_f, n_{gh} denotes the number of external gauge-bosons, quarks and FP ghosts.

$\beta(g_R)$ and $\gamma_m(g_R)$ are independet of α_R since they appear in the RG-equation of the S-matrix.

8.9.3 The β-function of QCD

Many ways to calculate it. Straightforward method:

i) calculate the ghost-gluon vertex Z_1''

ii) calculate the ghost wave-function renormalization factor Z_ξ

iii) calculate the gluon renormalization factor Z_A.

$$Z_g = Z_1'' / (Z_\xi Z_A^{1/2})$$

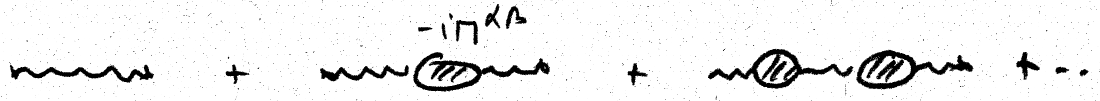
Alternative method: calculate Z_1, Z_ψ and $Z_A^{1/2}$ ...

Tricky methods (e.g. using background gauge).

Calculation of Z_A

$$D_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + D_{\mu\alpha}(k)\left(-i\Pi_{\alpha\beta}(k)\right) D^{(0)}_{\beta\nu}(k)$$

recursion relation relates $D_{\mu\nu}$ to $\Pi_{\alpha\beta}$ self energy



Solution:

$$D^{-1}_{\mu\nu}(k) = D^{(0)^{-1}}_{\mu\nu}(k) - \Pi_{\mu\nu}(k) \qquad (8.88)$$

Decompose tensors always into longitudinal and transverse components

$$A_{\mu\nu}(p) = A_\perp\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + A_{\parallel}\frac{p_\mu p_\nu}{p^2} \qquad (8.89)$$

$$A^{-1}_{\mu\nu}(p) = A_\perp^{-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + A_{\parallel}^{-1}\frac{p^\mu p^\nu}{p^2}$$

Recall properties from QED.

As a result of Ward identities (gauge invariance)

$$\Pi_{\mu\nu}(k)k^{\nu} = 0 \qquad \Pi_{\parallel}(k^2) = 0 \qquad (8.90)$$

therefore the longitudinal part of the photon propagator is the same as in free theory

$$-iD_{\mu\nu}(k^2) = \frac{1}{p^2(1-\Pi(p^2))}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + a_0 \frac{p_\mu p_\nu}{(p^2)^2} \qquad (8.91)$$

$$D^{\mu\nu}_{ren\,\perp} = -iD_{ren,\perp}(k^2; \mu^2)\left[g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] - ia(\mu)\frac{k_\mu k_\nu}{(k^2)^2} \qquad (8.92)$$

In the (MS)-scheme Z_A constructed such that

$$D_{ren,\perp}(k^2; \mu^2) = \frac{1}{p^2(1-\Pi(p^2))} + (Z_A^{-1} - 1)\frac{1}{p^2} \qquad (8.93)$$

where

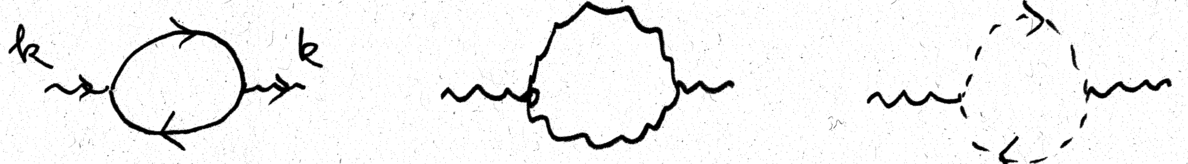
$$\Pi_{\mu\nu}(k^2) = [p^2 g_{\mu\nu} - p_\mu p_\nu]\,\Pi(p^2)$$

In the case of the gluon propagator we have similar decomposition in terms of transverse and longitudinal pieces. It is not trivial to prove but remains true that

$$\Pi_{\mu\nu}(k)k^{\mu} = 0 \qquad (8.94)$$

The counter term has to cancel the divergent contributions in (8.93). This defines Z_A.

The gluon-self energy receives contributions from three diagrams



$$\Pi^{(a)}_{\mu\nu}(k) \qquad \Pi^{(b)}_{\mu\nu}(k) \qquad \Pi^{c}_{\mu\nu}(k) \qquad (8.95)$$

The first diagram is the same like in QED apart from the color and flavor factor $T_f n_f$

$$\Pi^{(a)}_{\mu\nu}(k) = T_f n_f \Pi^{(QED)}_{\mu\nu}$$

We introduce the notation

$$\Gamma_K(\epsilon) = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{\epsilon} \simeq \frac{1}{\epsilon}$$

then $$\Pi^{(a)}(k^2) = -2T_f n_f \frac{g_R^2}{(4\pi)^{d/2}}\left(-\frac{\mu^2}{k^2}\right)^{\epsilon}\frac{1}{\epsilon} \qquad (8.96)$$

The second and the third diagrams do not give transverse contribution. Their sum is however, transverse. It is enough to calculate

$$\Pi^{(b)\,\mu}_{\ \mu}(k^2) + \Pi^{(c)\,\mu}_{\ \mu}(k^2) = k^2(d-1)\,\Pi^{(b+c)}(k^2)$$

(8.97)

$$D_\perp(k^2) \approx \frac{1}{k^2}\left(1+\Pi(k^2)\right) + \frac{1}{k^2}\left(Z_A - 1\right)$$

$$\Pi(k^2)_{\text{divergent}} + (Z_A - 1) = 0 \qquad (8.98)$$

$$\Pi^{(a+b+c)}(k^2)_{\text{divergent}} = \frac{g_R^2}{(4\pi)^2}\left[-\frac{1}{2}\left(a-\frac{13}{3}\right)C_A - \frac{2}{3}n_f\right]\frac{1}{\epsilon}$$

$$Z_A = 1 - \frac{\alpha_s}{4\pi\epsilon}\left[\frac{1}{2}\left(a-\frac{13}{3}\right)C_A + \frac{2}{3}n_f\right] \qquad (8.99)$$

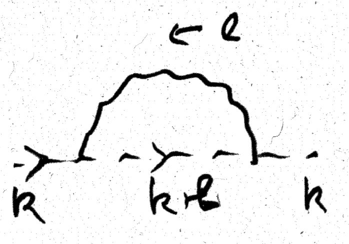
$$\gamma_A = \frac{\alpha_s}{4\pi}\left[\left(a-\frac{13}{3}\right)C_A + \frac{2}{3}n_f\right] \qquad (8.100)$$

Calculation of Z_{gh}

$$-iG(p) \approx \frac{1}{p^2}\left(1+\Sigma(p^2)\right) + \frac{1}{p^2}Z_{gh} \qquad (8.101)$$

$$\underline{Z_{gh} = 1 + C_A\,\frac{3-a}{4}\,\frac{\alpha_s}{4\pi\epsilon}} \qquad (8.102)$$

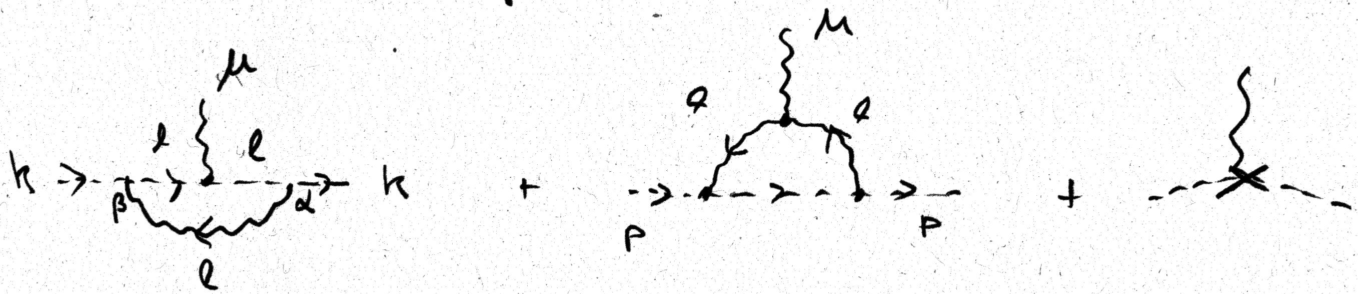
We need to calculate only the diagram



(8.103)

Calculation of ghost-gluon vertex and Z_1''

We have two diagrams



We can set all the external four momenta zero

$$\Gamma_{gh}^{(a)\mu} = -i\, \frac{C_A}{2}\, g_R^2 \mu^{2\epsilon} \int \frac{d^d l}{(4\pi)^d}\, \frac{l^\mu k^\alpha l^\beta}{(l^2)^3} \left(g_{\alpha\beta} - (1-a)\frac{l^\alpha l^\beta}{l^2} \right)$$

$$= -i\, \frac{C_A}{2}\, g_R^2 \mu^{2\epsilon}\, \frac{1}{4}\, k^\mu \int \frac{d^d k}{(2\pi)^d}\, \frac{1}{(k^2)^2}$$

$$= \frac{1}{8}\, C_A\, a\, \frac{g_R^2}{(4\pi)^2} \cdot \frac{1}{\epsilon} \qquad (8.104)$$

$$\Gamma_{gh}^{(b)\ \mu} = \frac{3}{8}\, C_A\, a\, \frac{g_s^2}{(4\pi)^2}\, \frac{1}{\epsilon} \qquad (8.105)$$

$$\Gamma_{gh}^{\mu} = \frac{1}{2}\, C_A\, a\, \frac{g_s^2}{(4\pi)^2}\, \frac{1}{\epsilon} \qquad (8.106)$$

$$Z_1'' = 1 - \frac{1}{2}\, C_A \cdot a\, \frac{g_s^2}{(4\pi)^2}\, \frac{1}{\epsilon} \qquad (8.107)$$

Now we can calculate Z_g

$$Z_g = Z_1''/Z_3 Z_A^{1/2}$$

$$= \left(1 - \frac{1}{2} C_A a \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\right)\left(1 - C_A \frac{3-a}{4}\frac{\alpha_s}{4\pi\epsilon}\right)$$
$$\left(1 + \frac{\alpha_s}{4\pi\epsilon}\left(\frac{1}{4}\left(a - \frac{13}{3}\right)C_A + \frac{1}{3} n_f\right)\right)$$

$$Z_g = 1 - \left(\frac{11}{6} C_A - \frac{1}{3} n_f\right)\frac{g_s^2}{(4\pi)^2}\frac{1}{\epsilon} \qquad (8.108)$$

$$a_g^{(1)} = -\left(\frac{11}{6} C_A - \frac{1}{3} n_f\right)\frac{g_s^3}{(4\pi)^2} \qquad (8.109)$$

$$\boxed{\begin{aligned}\beta_{QCD}(g) &= -a_g^{(1)} + g\frac{d}{dg} a_g^{(1)} = 2 a_g^{(1)} \\ &= -\left(\frac{11}{3} C_A - \frac{2}{3} n_f\right)\frac{g_s^2}{(4\pi)^2}\frac{1}{\epsilon}\end{aligned}} \qquad (8.110)$$

We obtained that QCD is asymptotically free provided

$$\frac{11}{3} C_A > \frac{2}{3} n_f \qquad (8.111)$$

$$11 > \frac{2}{3} n_f \qquad (8.112)$$

8.9.4 Renormalization and broken symmetry, chiral gauge theories

The renormalizability of gauge theories with broken symmetry can be proven using the idea of parameter continuation.

We start with parameter values of the Lagrangian such that they do not produce spontaneous symmetry breaking. Then one has to show that no ultraviolet divergences are generated when we continue these parameters to region where spontaneous symmetry breaking can happen. The unitarity can be shown to remain valid again using analytic continuation and observing that the masses of unphysical particles are gauge-dependent and in specific gauge choice R_ξ-gauge one can send these masses to infinity

9. Applications to particle physics

9.1 Dimensionless observable depending on a single energy scale

Naive scaling would suggest that if a dimensionless physical observable R depends on a single energy scale then $R = \text{const}$, its value is independent of Q. This result is not true in a renormalizable quantum field theory. When we calculate R as a perturbation theory in the renormalized coupling $\alpha_s = g^2/4\pi$, the perturbation theory requires regularization to carry out renormalization, a well defined algorithm to remove ultra-violet divergences. This procedure requires the introduction a second mass scale μ. Therefore in general R depends on Q^2/μ^2.

However R is an arbitrary parameter. The value of R cannot depend on μ. Holding the bare parameter fixed the renormalized coupling aquire μ-dependence. The μ-dependence of R is expressed by the renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} R\left(Q^2/\mu^2, \alpha_s \right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right] R\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = 0 \tag{9.1}$$

Let us introduce the notations

$$\beta(\alpha_s) = \mu^2 \left. \frac{\partial \alpha_s}{\partial \mu^2} \right|_{g \,\text{bare fixed}}, \qquad t = \ln \frac{Q^2}{\mu^2} \tag{9.2}$$

Equation (9.1) can be rewritten as

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right] R(e^t, \alpha_s) = 0 \qquad (9.3)$$

We can find the solution to this equation by introducing the new function $\alpha_s(Q^2)$ called running mass with the help of the implicit equation

$$t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)} \quad , \quad \alpha_s(\mu^2) = \alpha_s \qquad (9.4)$$

Differentiating (9.3) with t and α_s we obtain

$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2)) \quad , \quad \frac{\partial \alpha_s(Q^2)}{\partial \alpha_s} = \frac{\beta(\alpha_s(Q^2))}{\beta(\alpha_s)} \qquad (9.5)$$

We can easily see that $R(1, \alpha_s(Q^2))$ is a solution of Eq. (9.3). The dimensionless physical quantity depends on Q^2, the single scale of the problem only through the running coupling constant $\alpha_s(Q^2)$.

If we calculate R in fixed order of perturbation theory at a given $Q^2 = Q_0^2 = \mu^2$ value, the renormalization group solution predicts the value of R at other scales.

We have calculated already the $\beta(\alpha_s)$ for QCD

$$\beta(\alpha_s) = -b\alpha_s^2\left(1 + b'\alpha_s + b''\alpha_s^2 + O(\alpha_s^3)\right) \qquad (9.6)$$

$$b = \frac{11 N_c - 2n_f}{12\pi} = \frac{33 - 2n_f}{12\pi} \qquad (9.7)$$

$$b' = \frac{17 N_c^2 - 5 N_c n_f - 3\frac{N_c^2-1}{2N_c} n_f}{2\pi(11 N_c - 2n_f)} = \frac{153 - 19 n_f}{2\pi(33 - 2n_f)} \qquad (9.8)$$

(b'' is also known).

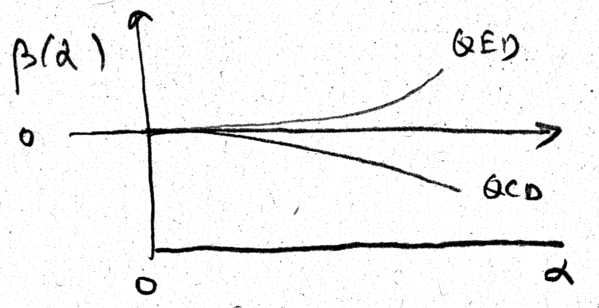
We have extracted the value of b from the one loop calculation of the coupling renormalization constant, which in turn required calculation of one-loop corrections to one vertex function and self-energy corrections for gluons and ghosts or quarks.

We obtain the value valid for QED by setting $N_c = 0$

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 \qquad b_{QED} = -\frac{1}{3\pi} \tag{9.9}$$

Note that particle and antiparticles are separately counted like active light flavour.

QCD is asymptotically free, QED is not



QCD is asymptotically free if $n_f \leq 16$.

The solution of equation $\frac{\partial \alpha_s}{\partial t} = -b\alpha_s^2$ (see eq. 9.5)

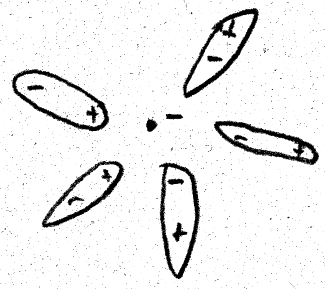
$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b \ln Q^2/\mu^2} \tag{9.10}$$

If $Q^2 > \mu^2$ $\alpha_s(Q^2) < \alpha_s(\mu^2)$. As $Q^2 \to \infty$ perturbation theory remains valid and therefore

$$\lim_{Q^2 \to} \alpha_s(Q^2) \to 0 \tag{9.11}$$

9.2 Intuitive interpretation of asymptotic freedom

There is no simple intuitive explanation of the property of asymptotic freedom in QCD. This is in contrast to QED where we interpret the result that the electric charge is larger at smaller distances by (the intuitive) vacuum polarization



which screens the electric charge at large distances. Since the vacuum can be polarized we introduce the dielectric constant $\epsilon_v(q^2)$.

But we can obtain an intuitive picture in QED using magnetic polarization of the vacuum aswell

$$\mu_v(q^2)\,\epsilon_v(q^2) = 1 \qquad (9.12)$$

$$\alpha(q^2) = \frac{\alpha(\Lambda_{uv}^2)}{\epsilon(q^2)} \qquad (9.13)$$

$$\frac{1}{\epsilon_v(q^2)} = 1 + b\,\alpha_{uv} \ln\frac{\Lambda^2}{q^2}$$

agnetic permeability:

$$\mu_v(q^2) = \frac{1}{\epsilon_v(q^2)} > 1 \quad \mu_v(q^2) = 1 + \chi_v(q^2) \qquad (9.14)$$

magnetic susceptibility:

$$\chi_v = b\,\alpha_{uv} \ln\frac{\Lambda^2}{q^2} \qquad (9.15)$$

In QED $\epsilon_v > 1$. The dielectric constant leads to screening. If $\epsilon_v < 1$ we have „anti screening" we interpret it as the „leak out" of the charge into the surrounding vacuum.

This „leak out" of the charge can be better understood via the magnetic properties of the vacuum.

The magnetic property can be recalled by summarizing the result of free, non-relativistic electron gas.

There are two sources of magnetic susceptibility: Pauli mechanism, due to the spin magnetic momentum and the Landau mechanism due to the couplize of the magnetic field to the orbital motion of the electrons
The Pauli effect gives paramagnetic, the Landau effect gives diamagnetic susceptibilities

$$\left.\begin{array}{lll} \chi_{Landau} < 0 & & \text{diamagnetic} \\ \chi_{Pauli} > 0 & & \text{paramagnetic} \end{array}\right\} \quad (9.16)$$

In free electron gas $\chi_{Landau} = -\frac{1}{3}\chi_{Pauli}$

$$\chi = \chi_{Landau} + \chi_{pauli} = \frac{2}{3}\chi_{pauli} > 0 \qquad (9.17)$$

The free electron gas is paramagnetic.

In relativistic field theory we can have particles with different spin, therefore the Pauli-mechanism gives spin-dependent result. In addition the fermionic vacuum energy, including the energy due to an external magnetic field is negative, whereas for bosons it is positive. This gives $(-1)^{2s}$ factor with respect to the non-relativistic electron gas.
The relativistic electron gas is paramagnetic for the negative vacuum energy, so it gives $\chi < 0$ and therefore it is diamagnetic.

In general we obtain

$$b = \frac{(-1)^{2s}}{2\pi} \left[(2s)^2 - 1/3 \right] \qquad (9.18)$$

For spin 0, 1/2 and 1 we obtain

$$b = -\frac{1}{6\pi}, \; -\frac{1}{3\pi}, \; \frac{11}{6\pi} \qquad \text{for } S = 0, 1/2, 1 \qquad (9.19)$$

For QCD we have to insert also appropriate color factors $C_A/2$ and $n_f/2$ for gluons and quarks and we obtain

$$b = \frac{1}{12\pi} (33 - 2n_f) \qquad (9.20)$$

9.3 The Λ-parameter

Renormalized quantum field theory tells us how the coupling constant varies with the scale, not the absolute value itself. The latter has to be obtained from experiment. In QED we define the coupling constant at zero mass scale (Thomson limit) with on-shell electrons. The experiment gives us the famous $\alpha \sim 1/137\ldots$ value.

In QCD the perturbatively calculated coupling at zero energy diverges. We need some other fixed scale to characterize its magnitude. For example we can specify the value of $\alpha_s(\mu)$ in the $\overline{MS}$ scheme at $\mu = M_Z$. The precision data obtained at LEP tells us that

$$\alpha_s^{(\overline{MS})}(\mu = M_Z) \approx 0.119 \qquad (9.21)$$

An alternative approach, convenient for many purposes, is to introduce a dimensionful parameter directly into the definition of $\alpha_s(Q^2)$

$$\ln\frac{Q^2}{\Lambda^2} = -\int_{\alpha_s(Q^2)}^{\infty}\frac{dx}{\beta(x)} \qquad (9.22)$$

$$\alpha_s(Q^2) = \frac{1}{b\ln Q^2/\Lambda^2} \qquad \text{for } \beta = -b\alpha_s^2 \qquad (9.23)$$

$$\frac{1}{\alpha_s(Q^2)} + b'\ln\left(\frac{b'\alpha_s(Q^2)}{1+b'\alpha_s(Q^2)}\right) = b\ln\frac{Q^2}{\Lambda^2} \qquad (9.24)$$

$$\text{for } \beta = -b\alpha_s^2(1+b'\alpha_s)$$

If we measure $\alpha_s(Q^2)$ at a certain scale we can infer the value of Λ

Since the analytic relation between $\alpha_s(Q^2)$ and Λ depends on n_F, the numerical value of Λ depend on n_F. In addition using (9.23) or (9.24) we get Λ in leading order or next to leading order accuracy.

For the same value of $\alpha_s(Q^2)$ the Λ_{LO} and Λ_{NLO} are related by

$$\Lambda_{LO} \simeq \left(\frac{b}{b'}\right)^{\frac{b'}{2b}}\Lambda_{NLO} \qquad (n_f=5) \qquad (9.25)$$

The dependence of Λ on the choice of the accuracy of calculating $\alpha_s(Q^2)$ and on the number of flavours disfavours the use of Λ. However if we calculate the bound state masses in QCD (assuming massless quarks) we obtain

$$m_h = \Lambda_{QCD}\, C_h \qquad (9.26)$$

since Λ_{QCD} is the only scale of the theory.

9.4 Choosing the appropriate renormalized coupling

We have found that the theoretical prediction for a physical observable of zero dimension is that

$$R(Q^2/\mu^2, \alpha_s) = R(1, \alpha_s(Q^2)) \qquad (9.27)$$

Assume that in perturbation theory

$$R = 1 + R_1\alpha_s + R_2\alpha_s^2 + \dots \qquad (9.28)$$

If we have calculated only the R_1 using $\alpha_s(Q^2)$ we obtain logarithmic terms to all orders using $\alpha_s(\mu^2)$

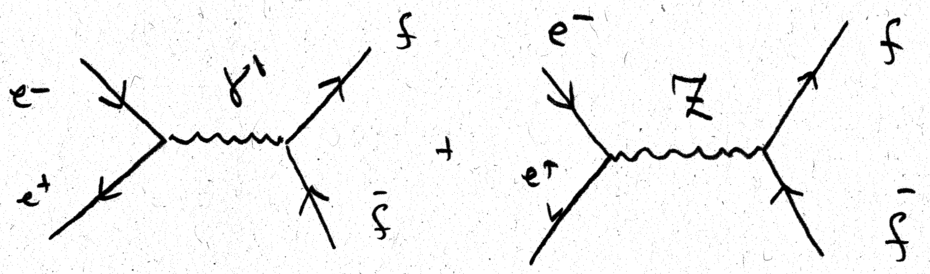
$$R = 1 + R_1\,\alpha_s(Q^2) = 1 + R_1 \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b \ln \frac{Q^2}{\mu^2}}$$

$$= 1 + R_1\,\alpha_s(\mu^2) \sum_{n=0}^{\infty} (-1)^n \alpha_s^n(\mu^2)\, b^n \ln^n \frac{Q^2}{\mu^2} \qquad (9.29)$$

Thus order by order in perturbation theory there are logarithmic terms of type $\alpha_s^n b^n \ln^n \frac{Q^2}{\mu^2}$ which are automatically resummed by using the running coupling. Higher order terms when expanded give terms fewer logarithms per power of α_s

9.5 Electron - positron annihilation into hadrons

9.5.1 The total hadronic cross-section



In e^+e^- annihilation in the Standard Model we may have either a virtual photon or a Z-boson in the s-channel. This is the case at LEP, with E ~ 200 GeV

At lower energies the contribution of the Z-boson is suppressed by powers of (Q^2/M_Z^2) and in many circumstances it is negligible.

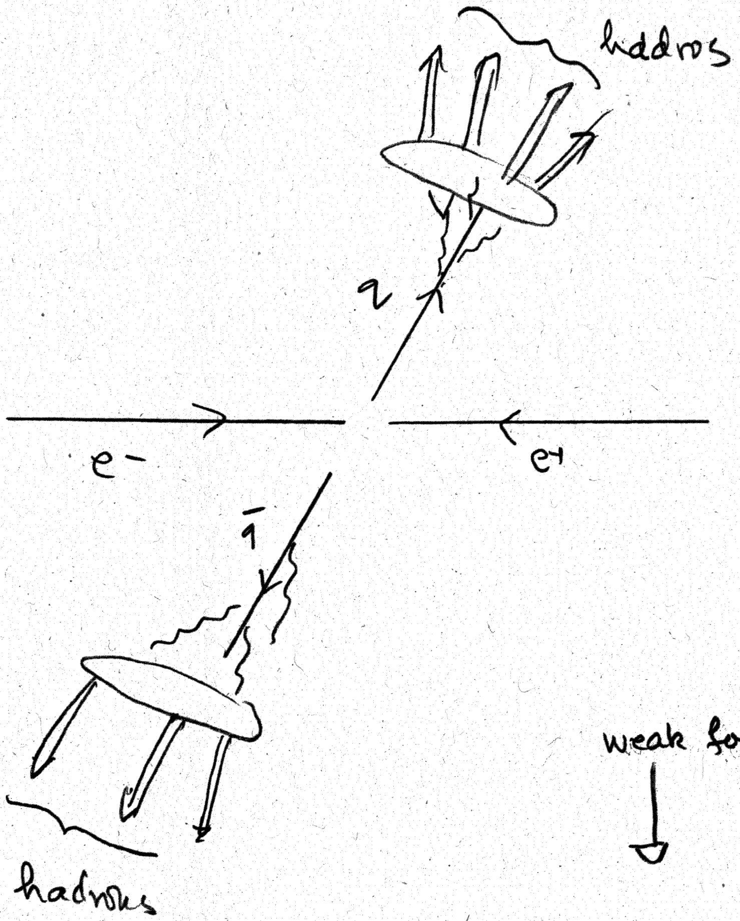
Then we can calculate in perturbative theory the dimensionless quantity

$$R_{QCD}^{LO} = \frac{\Sigma_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_q Q_q^2 \qquad (9.30)$$

We assume that

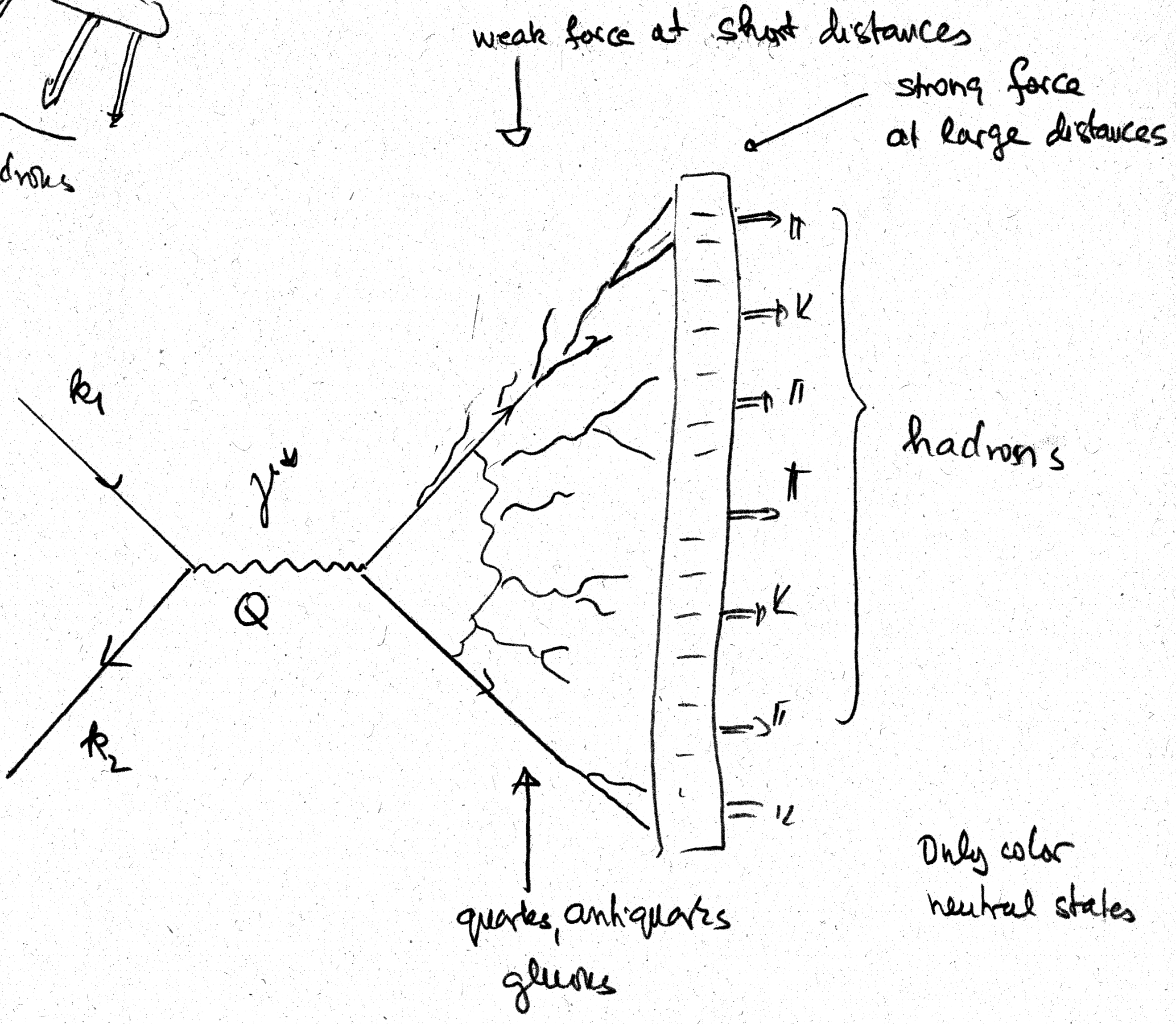
$$R_{QCD}^{LO} \simeq R_{hadrons} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \qquad (9.31)$$

We may assume that $R_{QCD} \simeq R_{hadrons} + \mathcal{O}\left(\frac{m^4}{Q^4}\right)$ (9.32)



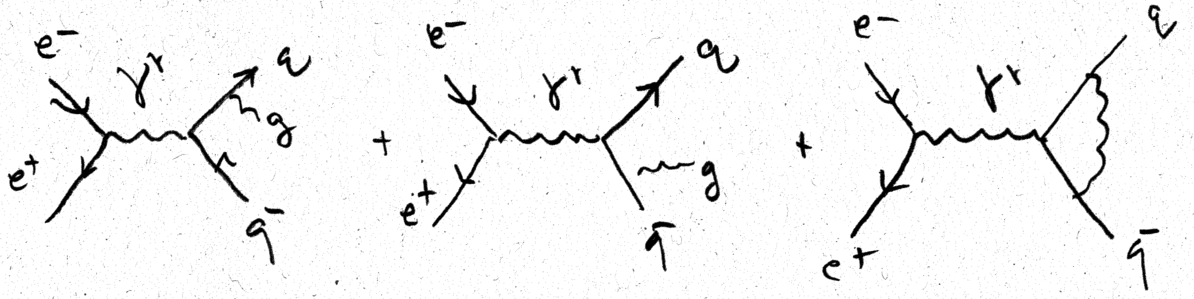
2 jets of hadrons

$Q \gg \Lambda_{QCD}$



$$U(t_f, t_i) = \underbrace{U(t_f, t)}_{1}\,\underbrace{U(t, t_i)}_{\text{perturbative QCD}} \qquad \begin{matrix} t_f \to \infty \\ t_i \to -\infty \end{matrix}$$

$$\approx U(t_f, t_i)_{\text{perturbative QCD}}$$



$$\mathcal{M}_2 = \mathcal{M}_2^0 + \mathcal{M}_{2V}^{(1)}$$

$$|\mathcal{M}_2|^2 \approx |\mathcal{M}^0|^2 + 2\,\mathrm{Re}\,\mathcal{M}_2^{(0)}\mathcal{M}_2^{*(1)}$$

$$|\mathcal{M}_3|^2 \approx |\mathcal{M}_3^{(0)}|^2$$

$$\sigma_{TOT} = \frac{1}{2s}\left(\int |M_2|^2 d\Phi_2 + \int |M_3|^2 d\Phi_3\right) \qquad (9.33)$$

$$R = \frac{\sigma_{TOT}(e^+e^- \to q\bar{q}, q\bar{q}g)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2 \left\{1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right\} \qquad (9.34)$$

If we choose $\alpha(Q^2=s) = \alpha_s$ as expansion parameter

$$R = \left(3\sum_q Q_q^2\right)\left\{1 + \frac{\alpha_s}{\pi} + \sum_{n=2}^{\infty} C_n \frac{\alpha(s)^n}{\pi^n}\right\} \qquad (9.35)$$

In the $\overline{MS}$-scheme

$$C_2 = \left(\frac{2}{3}\zeta(3) - \frac{11}{12}\right)n_f + \left(\frac{365}{24} - 11\zeta(3)\right) \approx 1.986 - 0.115 n_f$$

$$C_3 \approx -6.637 - 1.200 n_f - 0.005 n_f^2 - 1.24 \frac{(\sum_q Q_q)^2}{3\sum Q_q^2} \qquad (9.36)$$

$\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$ is the Riemann ζ function. (9.37)

We set equal the renormalization scale μ the single physical scale of the problem

$$\sqrt{s} = \sqrt{Q^2} = \mu$$

and therefore in equation (9.35) the expansion parameter of the perturbative series is $\alpha_s(Q^2=s)$. Of course we could use an arbitrary renormalization scale. Then we have to replace $\alpha_s(Q^2=s)$ with $\alpha_s(\mu^2)$ using the relation

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\alpha_s(\mu^2)\, b \ln\frac{Q^2}{\mu^2}} = \alpha_s(\mu) - \alpha_s^2(\mu^2)\, b \ln\frac{Q^2}{\mu} + \frac{1}{2}\alpha_s^3(\mu^2)\, b^2 \ln\frac{Q^2}{\mu^2} + \dots \qquad (9.38)$$

Then the coefficient of the perturbation series in $\alpha_s(\mu)$ will depend on $t = \ln Q^2/\mu^2$. In the case $Q^2/\mu^2 \sim \mu^2/\Lambda^2_{QCD}$ the logarithms are

$$\alpha_s(\mu^2)\, b \ln\frac{Q^2}{\mu^2} = \frac{\ln\frac{Q^2}{\mu^2}}{\ln\frac{\mu^2}{\Lambda^2_{QCD}}} \sim O(1) \qquad (9.39)$$

For example at LEP-200, for calculating the total cross-section we naively may choose $\mu = 6\,GeV$ then

$$Q/\mu = \frac{200}{6} \approx 33 \quad \text{and} \quad \frac{6}{0.2} \approx 30 \qquad (9.40)$$

We can see that this choice of the renormalization scale for this observable $R_{e^+e^-}(\sqrt{s}=200\,GeV)$ is not allowed since calculating higher order corrections we obtain $O(1)$ corrections! The renormalization group tells us that the perturbative series can be improved by choosing a better expansion parameter which resums all the leading logarithmic corrections appearing by the „wrong" choice of the renormalization scale.

9.5.2 Running quark mass and the decay width $\Gamma(H \to b\bar{b})$

In 9.5.1 discussing $R_{e^+e^-}(Q^2)$ we assumed massless quarks. We shall now investigate effects coming from finite quark mass values.

First note that in the QCD Lagrangian the quark mass is just another parameter like α_s. Let us assume for simplicity that we have one quark only with renormalized mass m_R. I recall that for the coupling g_R we used the notation g_s. In this section we use m_q for m_R. If we need the bare parameter we denote it with g and m.

Second we note that in the MS($\overline{\text{MS}}$) scheme the residuum of the poles are independent from m and the anomalous dimension γ_m of the mass term (see equations (8.76) - (8.83)) as well as the β function are only functions of the renormalized coupling $\alpha_s = g_s^2/4\pi$.

In this scheme the renormalization group has the form

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} - \gamma_m(\alpha_s)\, m_q \frac{\partial}{\partial m_q}\right] R\left(Q^2/\mu^2, \alpha_s, m_q/Q\right) = 0 \qquad (9.41)$$

where

$$\gamma_m(\alpha_s)\, m_q = -\frac{1}{m_q^2}\, \mu^2 \frac{\partial m_q}{\partial \mu^2}\Bigg|_{m,\, g \text{ fixed}} \qquad (9.42)$$

The $\gamma_m(\alpha_s)$ can be obtained by calculating the self-energy corrections for quarks in perturbation theory with the result

$$\gamma_m(\alpha_s) = c\,\alpha_s(1 + c'\alpha_s + \dots) \tag{9.43}$$

$$c = \frac{1}{\pi} \qquad c' = \frac{303 - 10 n_f}{72\pi} \tag{9.44}$$

(Excercise: calculate c and prove that $c = 1/\pi$).

Since R is a dimensionless function changing the scales of the dimensionful parameters leaves R invariant

$$\mu^2 \to e^t \mu^2 \;,\; m_q^2 \to e^t m_q^2 \;,\; Q^2 \to e^t Q^2$$

therefore

$$\left[Q^2 \frac{\partial}{\partial Q^2} + \mu^2 \frac{\partial}{\partial \mu^2} + m_q^2 \frac{\partial}{\partial m_q^2} \right] R\left(\frac{Q^2}{\mu^2}, \alpha_s, m_q/Q\right) = 0 \tag{9.45}$$

Subtracting (9.41) from (9.45) we obtain

$$\left[Q^2 \frac{\partial}{\partial Q^2} - \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} + \left(\frac{1}{2} + \gamma_m(\alpha_s)\right) m_q \frac{\partial}{\partial m_q^2} \right] R\left(\frac{Q^2}{\mu^2}, \alpha_s, m_q/Q\right) = 0 \tag{9.46}$$

With equation (9.4) we defined the running coupling $\alpha_s(Q^2)$, similarly we define now the running mass

$$Q^2 \frac{\partial m_q}{\partial Q^2} = -\gamma_m(\alpha_s(Q^2))\, m_q(Q^2) \tag{9.47}$$

$$\ln \frac{m(Q^2)}{m_q(\mu^2)} = -\int_{\mu^2}^{Q^2} \frac{dQ'^2}{Q'^2} \gamma_m(\alpha_s(Q'^2)) \tag{9.48}$$

$$m_q(Q^2) = m_q(\mu^2)\, e^{-\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha_s \frac{\gamma_m(\alpha_s)}{\beta(\alpha_s)}} \qquad (9.49)$$

where we have used that $\frac{dQ^2}{Q^2} = \frac{d\alpha_s}{\beta(\alpha_s)}$

Now we can see that all the scale dependence of R can be included in the running of the mass and the coupling.

If $R\left(\frac{Q^2}{\mu^2}, \alpha_s, \frac{m_q}{Q}\right)$ is a solution of the renormalization group equation then it has the functional form $R(1, \alpha_s(Q^2), m_q(Q^2)/Q^2$.

Using $\gamma_m(\alpha_s) = c\,\alpha_s$ and $\beta(\alpha_s) = -b\,\alpha_s^2$ we get

$$m_q(Q^2) = \hat{m}_q [\alpha_s(Q^2)]^{b/c} \qquad (9.50a)$$

where $\hat{m}$ is a renormalization group invariant parameter similarly to the Λ_{QCD} parameter.

(Excercise: prove that in next-to-leading order in α_s

$$m_q(Q^2) = \hat{m}_q [\alpha_s(Q^2)]^{b/c} \left[1 + \frac{c(c'-b')}{b} \alpha_s(Q^2) + \mathcal{O}(\alpha_s^2)\right]$$

9.50b)

Since in QCD $b/c > 0$

$$\lim_{Q \to \infty} m_q(Q) \to 0 \qquad (9.51)$$

Let us assume that R is regular at $m=0$
(This is true for $R_{e^+e^-}$ as a result of the Kinoshita-Lee theorem which says that all the singularities at $m=0$ cancel).

In such a case we can expand R in power series of m/Q

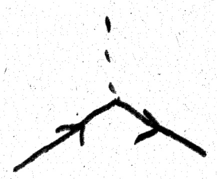
$$R_{e^+e^-}(1, \alpha_s(Q^2), m_q/Q) \approx R_{e^+e^-}(1, \alpha_s(Q^2), 0) + \frac{m_q(Q^2)}{Q} R'_{e^+e^-}(1, \alpha_s(Q^2), 0) + \frac{m_q^2(Q^2)}{Q^2} R''_{e^+e^-}(1, \alpha_s(Q^2), 0) + \dots \qquad (9.52)$$

In asymptotically free theory the anomalous dimension of the mass term leads to a variation of the mass with Q which varies like a power of $1/\ln Q$. This gives further suppression when $Q \gg m(Q^2)$.

The partial decay width of the Higgs boson for the decay

$$H \to b + \bar{b}$$

are given by the vertex



$\frac{m_b}{2 M_W} g_W$

$$\Gamma(H \to b\bar{b}) \sim c\, m_b^2\, m_H \frac{g_W^2}{M_W^2} \left(1 + \mathcal{O}\left(\frac{m_b^2}{m_H^2}\right) + \mathcal{O}(\alpha_s) + \dots\right) \qquad (9.53)$$

The QCD corrections modify this vertex. We can neglect the m_b^2/m_H^2 corrections. The QCD corrections tells us that for the $\frac{m_b}{2M_W} g_W$ coupling we have to use the running $m_b(Q)$ at $Q = m_H$.

This is a large correction in comparison to the naive leading order correction using the physical value of $m_b \approx 5$ GeV.

In the mass range

$$115 \text{ GeV} < m_H < 140 \text{ GeV} \qquad (9.54)$$

the dominant decay mode of the Higgs-boson is the decay $H \to b + \bar{b}$

$$\Gamma_{tot}(H) \approx \Gamma(H \to b + \bar{b}) \qquad (9.55)$$

In calculating the next-to-leading order QCD corrections we obtain

$$\Gamma(H \to b+\bar{b})_{NLO} = \Gamma(H \to b+\bar{b})_{LO} \left(\frac{m_b(m_H)}{m_b(m_b)}\right)^{2c/b} \left(1 + O\left(\frac{\alpha_s}{\pi}\right)\right) \qquad (9.56)$$

Assuming $m_H = 120$ GeV $m_b = 5$ GeV, $\Lambda_{QCD} = 0.2$ GeV

$$\left(\frac{m_b(m_H)}{m_b(m_b)}\right)^{2c/b} = \left(\frac{\alpha_s(m_H)}{\alpha_s(m_b)}\right)^{2c/b} \approx 0.5 \qquad (9.57)$$

The full NLO evaluation shows that $1 + O(\frac{\alpha_s}{\pi}) \approx 1.05$ therefore the dominant correction comes from the running of the bottom-quark mass. This correction enhances by a factor of 2 the physics signal events in the 2γ channel

$$\sigma(p+p \to H+X \to \gamma+\gamma+X) = \sigma(p+p \to H+X)\, BR(H \to \gamma\gamma)$$

where

$$BR = \frac{\Gamma(H\to\gamma\gamma)}{\Gamma(H)} \approx \frac{\Gamma(H\to\gamma\gamma)}{\Gamma(H\to b\bar{b})}, \qquad (BR)_{NLO} \approx \frac{1}{2}(BR)_{LO} \qquad (9.58)$$

This clearly shows the importance of QCD corrections for LHC physics.

9.6 Decoupling of heavy fermions, active and passive flavour

Now consider the case of a heavy quark with

$$m_q \gg Q$$

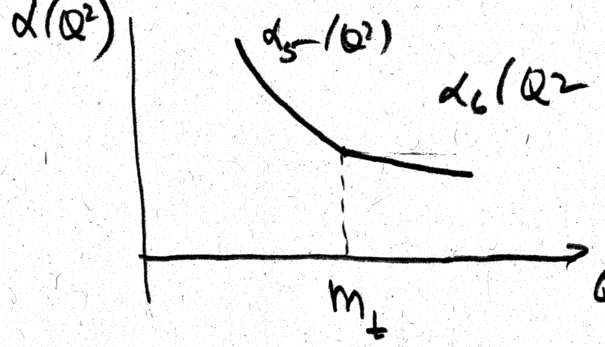
The top quark ($m_t \sim 172$ GeV) and the bottom quark ($m_b \sim 5$ GeV) can be considered already heavy and still $Q \gg \Lambda_{QCD}$.

In this case one can show that the effects of the heavy quarks on cross-sections are suppressed by inverse powers of m and can therefore be ignored for $Q \ll m$ (in theories without spontaneous symmetry breaking). This property is referred as the Appelquist-Carazzone theorem.
Therefore the contributions to $\alpha_s(Q^2)$ from heavy quarks is also decoupled. In the coefficient of the β-function we use n_f with a value of quarks which correspond to active flavors.

The running coupling constant $\alpha_s(Q^2)$ runs with $n_f = 4$ up to the bottom mass then runs with $n_f = 5$ above the bottom mass scale up to the top mass.

But it remains the question how to relate $\alpha_s(Q^2)$ in the two domains $Q \gg m_q$, $Q \ll m_q$.
Technically this is achived by matching the full theoretical prediction below and above the threshold scale with differently defined renormalized running $\alpha_s(Q^2)$.

Let us denote with $\alpha_5(Q^2)$ the running coupling of QCD defined in terms of 5 quarks and denote the renormalized coupling in terms of six quark $\alpha_6(Q^2)$.



We require that

$$\alpha_6(m_t) = \alpha_5(m_t)$$

therefore

$$\frac{1}{\alpha_6(Q^2)} - b_6 \ln\frac{Q^2}{m_t^2} = \frac{1}{\alpha_5(Q^2)} - b_5 \ln\frac{Q^2}{m_t^2} = \frac{1}{\alpha_s(m_t^2)}$$

$$b_5 - b_6 = \frac{2}{12\pi} = \frac{1}{6\pi} \quad \text{and hence}$$

$$\alpha_6(Q^2) = \alpha_5(Q^2)\left[1 - \frac{1}{6\pi}\alpha_5(Q^2)\ln\frac{Q^2}{m_t^2}\right]^{-1}$$

We might wonder why did we choose for the matching $Q = m_t$. What matters the consitency of the calculation performed with 6-quarks, versus 5 quarks in the full theory

We need to calculate the coefficient in the general expression

$$\alpha_6(Q^2) = \alpha_5(Q^2) + \sum_{k=1}^{\infty} C_k\left(\ln\frac{Q^2}{m^2}\right)\alpha_5(Q^2)^{k+1}$$

In the $\overline{MS}$-scheme with explicit calculation one finds

$$C_1(x) = \frac{1}{6\pi}x$$

This is in next-to-leading order equivalent to perform the matching at $Q = m_t$

9.7 QCD perturbation theory and soft and collinear infrared singularities

QCD perturbation is plagued by infrared singularities. If a physical observables receives contributions from configurations corresponding to soft gluon emission or a collinear splitting of quarks, antiquarks or gluons into a pair of gluon and quark, gluon and antiquark or two gluons

$$q \to q + g, \quad \bar{q} \to \bar{q} + g, \quad g \to g + g$$

the perturbative result is divergent. A careful study the infrared structure of the corresponding matrix element and cross-sections revealed important cancellation and factorization theorems. In many important cases the singularities cancel or factorize to process independent universal factors. We call these factors process independent since they can be associated by the individual quarks and gluons participating in the scattering process. This property allows to apply perturbative QCD to all hard scattering processes provided we introduce a few phenomenological function corresponding to the quark and gluon densities within the proton in the infinite momentum frame at a certain fixed factorization scale. The techniques to carry out these studies resembles strongly on the renormalization theory of the ultra-violet divergences.

But because of lack of time we do not describe this physics in this lecture note.

10. Chiral anomaly

10.1 Introduction

In quantum field theory we talk about <u>anomalies</u> in connection of symmetries. It may happen that the symmetry of the classical theory can not be maintained in the corresponding quantum theory. In this case we call the symmetry of the classical theory anomalous.

We can have <u>anomalous global symmetry</u>. In this case the quantum theory is less symmetric: certain forbidden transition are not forbidden. But the „anomalous" quantum theory remain still renormalizable if the interaction terms in the Lagrangian have dimension 4 or less. Therefore the anomalies may have interesting physical consequences.

But we can have also <u>anomalous local symmetry</u>. In <u>pure gauge</u> theory this is <u>not possible</u>. But in the case when the gauge theory is coupled to matter field certain <u>gauge theories may become anomalous</u>. This is <u>disastorous</u> for the quantum theory since the local gauge symmetry at the quantum level is a necessary condition for renormalizability.

We call a gauge theory <u>vector gauge theory</u> if the left-handed and right handed component of the spin 1/2 fermion fields are treated in the same way. In QCD the color charge of the left-handed quarks are the same as the right handed quarks. Therefore <u>QCD is a vector gauge theory</u>, and is <u>anomaly free</u>.

We call a gauge theory chiral gauge theory if the left-handed spin-1/2 Weyl spinors treated differently from their right-handed counter parts. The electroweak sector of the Standard Model is chiral gauge theory since the left-handed quarks and leptons are in the doublet representation of the SU(2) weak isospin gauge group while the right handed fields are singlets under the transformation of the weak isospin group. We may say that the left-handed quark and lepton fields are „charged" under the weak isospin while the singlet fields are nutral, that is they do not have direct weak isospin interactions.

For chiral gauge theories the anomalies do not cancel automatically. But they cancel provided the individual spin 1/2 fermion contributions cooperate to cancel. This happens in the Standard Model with individual families of quark and leptons. We have three families. Each of them consists of 15 fermions. The first family is : $[u_L^a, d_L^a, u_R^a, d_R^a; e_L, e_R, \nu_L]$ $a=1,2,3$.
First we discuss global chiral anomaly in QCD and its relation to the so called $U(1)_A$ problem and the θ-term. Next we consider the conditions of chiral anomaly cancellation in gauge theories on the example of the Standard Model

We shall define chirality, chiral symmetry and calculate the triangle anomaly using simple triangle Feynman diagrams.

In quantum field theory scale and conformal symmetry is also anomalous as a result of renormalization.

10.2 Chiral fermions

Consider the Lagrangian of free massless fermions

$$\mathcal{L}(\psi) = i\bar{\psi}_j \not{\partial} \psi_j = i\bar{\psi}\not{\partial}\psi \qquad j=1,..N \qquad (10.1)$$

This Lagrangian is symmetric $U(N) = SU(N)\times U(1)$

$$\delta\psi_j = i\epsilon^a T^a_{jk}\psi_k \qquad \delta\bar{\psi}_j = -i\bar{\psi}_{\ell}(\epsilon_a T^a)_{\ell j}$$
$$\delta\bar{\psi}\psi = 0 \quad , \quad (T^a)^\dagger = T^a \qquad (10.2)$$

The Noether current is

$$J^a_\mu = \bar{\psi}\gamma_\mu T^a\psi = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_i} T^a_{ij}\psi_j \qquad (10.3)$$

But the Lagrangian actually has an $U(N)_L\times U(N)_R$ symmetry.

Consider the projection

$$P_{L,R} = (1\pm\gamma_5)/2 \qquad \psi_L = P_L\psi \;,\; \psi_R = P_R\psi \qquad (10.4)$$
$$\bar{\psi}_L = \bar{\psi}P_R \qquad \bar{\psi}_R = \bar{\psi}P_L$$

then
$$\mathcal{L} = i\bar{\psi}_L\not{\partial}\psi_L + i\bar{\psi}_R i\not{\partial}\psi_R$$

If we separately transform ψ_L and ψ_R with two $U(N)$ groups the Lagrangian remains invariant.
The corresponding generators are T^a_L and T^a_R.

For massless fermions chirality = helicity

Let us consider free fermions along the 3-axis. Then the Dirac equation says ($p^0 = p^3$, $p^1 = p^2 = 0$)

$$\gamma^0 \psi = \gamma^3 \psi$$

The helicity operator $\frac{\vec{\Sigma}\vec{p}}{|\vec{p}|} = \Sigma^3 = \frac{i}{4}[\gamma^1, \gamma^2] = \frac{i}{2}\gamma^1\gamma^2$

$$\Sigma^3 \psi_L = \frac{i}{2}\gamma^1\gamma^2 \psi_L = \frac{i}{2}\gamma^0\gamma^0\gamma^1\gamma^2\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^0\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^3\psi_L$$

$$= -\frac{1}{2}\gamma_5\psi_L = -\frac{1}{2}\psi_L \qquad (10.5)$$

$\psi_{L(R)}$ describes negative (positive) helicity states

The symmetry under which ψ_L and ψ_R fields transform differently is called chiral symmetry

Replacing the free theory of N massless fermions with gauge theory the global $U(N)\times U(N)$ symmetry is still a symmetry of the interacting theory.

In QCD it is good approximation neglect u and d-quark masses since $m_u, m_d \ll \Lambda_{QCD}$.

The QCD Lagrangian has approximate accidental global $U(2)\times U(2) = SU(2)_L \times U(1)_L \times SU(2)_R \times U(1)_R$ chiral symmetry which rotates the (u_L^a, d_L^a) and u_R^a, d_R^a components into each other at fixed color.

10.3 Vector and axial currents

Since the Lagrangian of N non-interacting free fermions is symmetric under global $SU(N)_L \times SU(N)_R \times U(1)_L \times U(1)_R$ symmetry we can construct the corresponding conserved currents, according to the Noether theorem

$$J_L^{a,\mu} = \bar{\psi}_L \gamma^\mu T^a \psi_L \quad , \quad J_R^{a,\mu} = \bar{\psi}_R \gamma^\mu T^a \psi_R \qquad (10.6)$$

For many purposes it is convenient to use instead the vector and axial-vector currents

$$J_V^{a,\mu} = \bar{\psi} \gamma^\mu T^a \psi \quad , \quad J_5^{a,\mu} = \bar{\psi} \gamma^\mu \gamma_5 T^a \psi \qquad (10.7)$$

The corresponding vector and axialvector charges Q^a and Q_5^a generate the transformation

$$\delta\psi = i(\epsilon^a - i\gamma_5 \epsilon_5^a) T^a \psi \qquad (10.8)$$

where

$$\epsilon^a = \epsilon_R^a + \epsilon_L^a , \quad \epsilon_5^a = \epsilon_R^a - \epsilon_L^a \qquad (10.9)$$

In the case of $N=3$ we get $SU(3)$ generators $T^a = \frac{1}{2}\lambda^a$, in the case of $N=2$ we get $SU(2)$ generators $T^a = \frac{1}{2}\sigma^a$. In the case of QCD, if $m_q = 0$, we get accidental $U(6)_L \times U(6)_R$ global symmetry. If we take into account that $m_u, m_d \ll \Lambda_{QCD}$ we get $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$ symmetry, which can be also written as $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$ symmetry. $SU(2)_V$ gives the standard isospin symmetry (introduced by Heisenberg for nuclear physics) $U(1)_V$ gives the conservation of baryon number. The $U(1)_A$ symmetry in the quantum theory turns out to be anomalous! The axial $SU(2)_A$ symmetry spontaneously broken and the $\pi^\pm, \pi^0$ give the corresponding Goldstone-bosons.

10.4 Calculating the triangle anomaly

Consider a theory of a single massles fermions coupled to photon:

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - e \bar{\psi} \gamma^\mu \psi A_\mu - \frac{1}{4} F_{\mu\nu}^2 \qquad (10.10)$$

This Lagrangian manifestly invariant under the transformations

$$\psi \to e^{i\theta} \psi \, , \quad \psi \to e^{i\theta\gamma^5} \psi \qquad (10.11)$$

The corresponding currents (see 10.3 and 10.5) are conserved

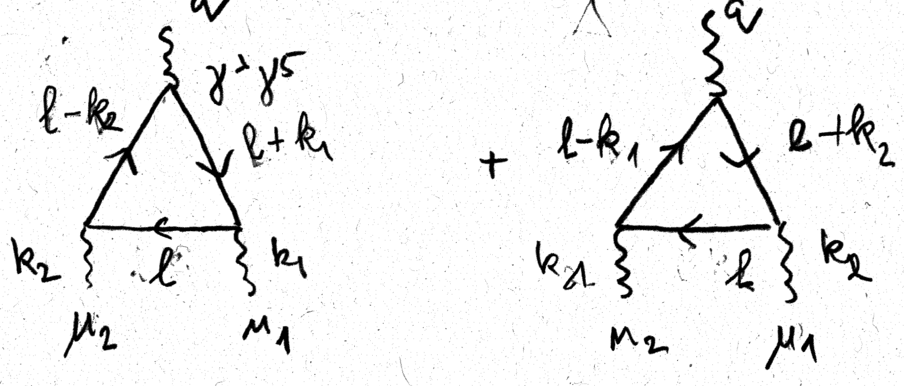
$$\partial_\mu J^\mu(x) = \partial_\mu J^\mu_5(x) = 0 \qquad (10.12)$$

Actually these conservation laws follow from the classical equations of motions.

We want to test that conservation of the axial current by calculating the leading order diagrams to the matrix elements

$$\langle k_1 k_2 | J_\lambda^5(0) | 0 \rangle \qquad (10.13)$$

We need to calculate the sum of the two diagrams below



, $\quad q = k_1 + k_2 \qquad (10.14)$

In the case of the first diagram we obtain (we use 10.22 !)

$$e^2 \int \frac{d^d\ell}{(2\pi)^d} \, \mathrm{tr}\Big[\gamma^5 \frac{1}{\not\ell - \not k_2} \gamma^{\mu_2} \frac{1}{\not\ell} \gamma^{\mu_1} + \gamma^5 \gamma^{\mu_2} \frac{1}{\not\ell} \gamma^{\mu_1} \frac{1}{\not\ell + \not k_1}$$

$$- 2 \gamma^5 \not\ell_{D-4} \frac{1}{\not\ell - \not k_2} \gamma^{\mu_2} \frac{1}{\not\ell} \gamma^{\mu_1} \frac{1}{\not\ell + \not k_1} \Big] \qquad (10.15)$$

The first two term becomes antisymmetric for the exchange $(\mu_1, k_1) \leftrightarrow (\mu_2, k_2)$ after shifting the loop momentum in the first term as $\ell = \ell + k_2$. Therefore it will be cancelled by similar contributions from the second diagram.

We need to keep only the terms containing $\ell_{(D-4)}$. But doing the loop integral the rank 1 and rank 3 tensor integrals in $\ell_{\perp\mu}$ vanish. In calculating the trace we need to retain $\gamma^{\mu_1}, \gamma^{\mu_2}$ and k_1, k_2 to get a non-vanishing value, therefore the numerator is quadratic in $\not\ell_{D-4}$

$$\mathcal{M}^{(1)} = e^2 \int \frac{d^d\ell}{(2\pi)^d} (-2) \, \frac{\mathrm{tr}\left(\gamma^5 \not\ell_{D-4} (-\not k_2) \gamma^{\mu_2} \not\ell_{D-4} \gamma^{\mu_1} \not k_1 \right)}{(\ell - k_2)^2 \, \ell^2 \, (\ell + k_1)^2} \qquad (10.16)$$

After Feynman parametrization $\mathrm{tr}\left(\gamma^5 (-\not k_2) \gamma^{\mu_2} \gamma^{\mu_1} \not k_1\right) \ell_{D-4}^2$

$$= i \, \epsilon^{\mu_1 \mu_2 k_1 k_2} \times$$

$$\times \; e^2 \int \frac{d^d\ell}{(2\pi)^d} (-2) \frac{\ell^2_{(D-4)}}{(\ell^2 - \Delta)^3} \, dx \, dy \, \delta(1-x-y)$$

This is a logarithmically divergent integral which can be easily evaluated to $O(1)$ in $(4-d)/2$

$$\langle k_1 k_2 | J_5^{\lambda}(0) | 0 \rangle = \epsilon^*_{\mu_1}(k_1)\, \epsilon^*_{\mu_2}(k_2)\, \mathcal{M}^{\mu_1 \mu_2 \lambda}(k_1, k_2) \tag{10.17}$$

The first diagram gives

$$\mathcal{M}_{(1)}^{\mu_1 \mu_2 \lambda}(k_1, k_2) = (-1)(-ie)^2 \int \frac{d^d l}{(2\pi)^d} \operatorname{tr}\Big[\gamma^{\lambda} \gamma^5 \frac{i(\not{l} - \not{k}_2)}{(l-k_2)^2} \gamma^{\mu_2} \frac{i\not{l}}{l^2}$$
$$\gamma^{\mu_1} \frac{i(\not{l} + \not{k}_1)}{(l+k_1)^2} \Big]$$

$$\mathcal{M}_2^{\mu_1 \mu_2 \lambda}(k_1, k_2) = \mathcal{M}_{(1)}^{\mu_2 \mu_1 \lambda}(k_2, k_1) \tag{10.18}$$

We had to introduce dimensional regularization since the individual diagrams linearly divergent. Naive calculation in four dimension would give vanishing result for the sum of the two diagrams if we assume that the loop momentum can be shifted. But in the case of linearly or stronger divergent integrals we can not shift the integration variables since

$$\int_{-\Lambda}^{\Lambda} dx \left(f(x+a) - f(x) \right) = \int_{-\Lambda}^{\Lambda} dx \left(a f'(x) + \dots \right) =$$
$$= a \left(f(\Lambda) - f(-\Lambda) \right) \neq 0$$

After dimensional regularization we can shift the integrals. But there is a <u>subtlety with γ^5</u>
In d-dimension as suggested 't Hooft and Veltman we can use for the definition of γ^5

$$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \tag{10.19}$$

in any d-dimension. This definition gives anti commutation of γ^5 for γ^μ with $\mu = 0,1,2,3$

but higher values of μ it has commutation relations

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{for} \quad \mu = 0,1,2,3$$
$$[\gamma^5, \gamma^\mu] = 0 \quad \text{if} \quad \mu > 3 \qquad (10.20)$$

In general d-dimension this definition of γ_5 allows for extension to d-dimension with a γ_5 which in $d=4$ is the usual γ_5 but otherwise it is not the discrete automorfism for $d = 6, 8, ..$ dimensions.

Since dimensional regularization preserve vector gauge symmetries we must have

$$k_1^{\mu_1} M_{\lambda \mu_1 \mu_2}(k_1, k_2) = 0$$
$$k_2^{\mu_2} M_{\lambda \mu_1 \mu_2}(k_1, k_2) = 0 \qquad (10.21)$$

But because of keeping γ_5 its four-dimensional definitions, we have to check wether the axial current is conserved or not

$$(k_1 + k_2)^\lambda M_{\lambda \mu_1 \mu_2}(k_1, k_2) = ?$$

We have to split ℓ^μ into ℓ_4^μ and $\ell_{(D-4)}^\mu$ components

therefore

$$\not{q}\gamma^5 = (\not{\ell} + \not{k}_1)\gamma^5 + \gamma^5(\not{\ell} - \not{k}_2) - 2\gamma^5 \not{\ell}_{D-4} \qquad (10.22)$$

We can easily see that to $O(\epsilon)$ the integral is

$$\int d^d\ell \, \frac{\ell^2_{D-4}}{(\ell^6)} = \frac{d-4}{d} \int d^d\ell \, \frac{1}{\ell^4} = \frac{d-4}{d} \, \Omega_4 \int d\ell_E \, \ell_E^{d-5}$$

$$= i \frac{1}{2\epsilon} 2\pi^2 \qquad (10.23)$$

Collecting the other factors $\frac{1}{(2\pi)^4}(-4)$, we obtain

$$q_\lambda \mathcal{M}^{\epsilon_1^* \epsilon_2^* \lambda}(k_1, k_2) = + \frac{e^2}{2\pi^2} \epsilon^{\mu_1 \mu_2 \lambda \sigma} k_{1\lambda} k_{2\sigma} \epsilon_1^* \epsilon_2^* \qquad (10.24)$$

$$= - \frac{e^2}{16\pi^2} \langle k_1 k_2 | \epsilon^{\mu_1 \mu_2 \lambda \sigma} F_{\mu_1 \lambda} F_{\mu_2 \sigma}(0) | 0 \rangle$$

This suggests the operator relation

$$\partial_\mu J^\mu_5 = + \frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\lambda\sigma} F_{\alpha\beta} F_{\lambda\sigma} \qquad (10.25)$$

In QCD

$$\partial_\mu J^{\mu a}_5 = - \frac{g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F^c_{\alpha\beta} F^d_{\lambda\sigma} \, \mathrm{tr}(\tau^a t^c t^a) \qquad (10.26)$$

For axial vector isospin current $\mathrm{tr}\, \tau^a = 0$ therefore

$$\mathrm{tr}(\tau^a t^c t^d) = \mathrm{tr}\, \tau^a \, \mathrm{tr}(t^c t^d) = 0 \qquad (10.27)$$

There is no anomaly for the axial isospin current, however, there is an anomaly of the isosinglet current

$$\partial_\mu J^{\mu 5} = - \frac{g^2 n_f}{32\pi^2} \epsilon^{\alpha\beta\lambda\sigma} F^c_{\alpha\beta} F^c_{\lambda\sigma} \qquad (10.28)$$

10.5 The $U(1)_A$ problem and the Θ-term

To understand spontaneous chiral symmetry breaking was a problem in QCD, since no particle could be associated with the Goldstone-boson of the spontaneous breakdown of $U(1)_A$ (no light isospin singlet pseudoscalar meson in the hadron spectrum).

Finally 't Hooft discovered that in QCD we need to add the term

$$\mathcal{L}_\Theta = \frac{\Theta}{64\pi^2} g^2 \epsilon_{\mu\nu\lambda\sigma} F_A^{\mu\nu} F_A^{\lambda\sigma} \qquad A = 1,2,..8 \qquad (10.29)$$

to the QCD Lagrangian. This terms had previously been neglected, because it is a total divergence. But using semiclassical instanton techniques 't Hooft showed that it could not be neglected. The reason is that gauge fields relevant in the quantum theory do not fall fast enough at infinity to justify the neglect of the surface term.

This can be easily seen. Consider path integral in the Euclidean space $Z = \int D(A) e^{-S(A)}$

$$S(A) = \int d^4x \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \qquad (10.30)$$

We might which to evaluate Z in steepest descent approximation, in which case we have to find the extrema of $S(A)$ with finite action. This implies that at infinity $|x| = \infty$ must vanish faster than $1/|x|^2$ and at infinity A must be a pure gauge $A = g d g^+$ $g(x) \in$ Gauge group
Configurations for which this is true called instantons.

It can be easily seen that $\int \mathcal{L}_\theta \, d^4x$ gets non-vanishing contributions from instantons.
Choose the gauge group SU(2) and parametrize it as $g = x_4 + i\vec{x}\vec{\sigma}$. Since $g^+ g = 1$ and $\det g = 1$ we obtain

$$x_4^2 + |\vec{x}|^2 = 1 \qquad (10.31)$$

We learn the known fact that the group manifold of SU(2) is S_3. Thus, in an instanton, the gauge potential at infinity

$$A \xrightarrow[|x| \to \infty]{} g\,dg^+ + O(1/x^2) \qquad (10.32)$$

defines a map

$$S^3 \to S^3 . \qquad (10.33)$$

Mathematically maps of S^n into a manifold are classified by the homotopy group $\Pi_n(M)$. In particular

$$\Pi_n(S^4) = Z \quad \text{(the set of all integers)} \qquad (10.34)$$

In the case of S^1 we see that the map

$$\varphi \xrightarrow[r \to \infty]{} v e^{im\theta} \qquad (10.35)$$

is a map of $S^1 \to S^1$ with m integer. This map wraps one circle around, the other wraps m times.

We need to use differential forms and we find

$$A = A_\mu dx^\mu \qquad A^2 = \tfrac{1}{2}[A_\mu, A_\nu] dx^\mu dx^\nu$$
$$F = \tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu \qquad (10.36)$$
$$F = dA + A^2$$

$$\mathrm{Tr}\, \epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} = \mathrm{Tr}\, F^2 \qquad d\, \mathrm{tr}\, F^2 = 0 \qquad (10.37)$$

Since $d\,\mathrm{Tr}\,F^2=0$, $\mathrm{tr}\,F^2$ is exact. Indeed it is easy to see that (by writing out indices)

$$\mathrm{tr}\,F^2 = d\,\mathrm{tr}\left(A\,dA + \tfrac{2}{3}A^3\right) \qquad (10.38)$$

Therefore

$$\frac{\theta}{64\pi^2}g^2 \int \mathrm{Tr}\,F^2 d^4x = \frac{\theta g^2}{64\pi^2}\int_{S_3} \mathrm{tr}\left(A\,dA + \tfrac{2}{3}A^3\right) =$$

$$= \frac{\theta g^2}{64\pi^2}\int_{S_3} \mathrm{tr}\left(AF - \tfrac{1}{3}A^3\right) = -\frac{\theta g^2}{192\pi^2}\int_{S^3} \mathrm{tr}\left(g\,dg^+\right)^3 \qquad (10.39)$$

where we used the fact that F vanishes at infinity. This shows clearly that the $\int \mathcal{L}_\theta$ depends only on the homotopy of the map $S^3 \to S^3$ defined by g and is a topological quantity. The quantity

$\int_{S^3} \mathrm{tr}\left(g\,dg^+\right)^3$ is called Pontryagin index

It is easy to see that it is equal to $-64\pi^2 n$ for SU(3) therefore

$$\int \mathcal{L}_\theta\, d^4x \Big|_{\text{instantons}} = \theta g^2 n \neq 0 \qquad n = \pm 0, 1, 2 \ldots \qquad (10.40)$$

The presence this so called θ-term in the Lagrangian has important physical consequences. It can give T-violating effect. Actually it generates electric dipole moment for the neutron. Since experimentally no dipole moment was found we have an upper limit on its value

$$|\theta| < 10^{-8} \qquad (10.41)$$

The presence of the θ-term in the Lagrangian leads to the violation of the $U(1)_A$ global symmetry in a peculiar way. It is related to the axial anomaly which says that the isospin singlet axial current in QCD is not conserved.

$$\partial_\mu J_5^\mu(x) = -\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\lambda\sigma}F_a^{\mu\nu}F_a^{\lambda\sigma} \tag{10.42}$$

The problem is that when the axial current couples to two gluons through quark loops this coupling in lowest order is given by two triangle diagrams which are linearly divergent. The regularization breaks current conservation and an explicit calculation leads the result given by (10.23) where

(10.43)

Because of the divergence of the axial current involves the term appearing in $\mathcal{L}_\theta$, the associate $U(1)_A$ symmetry transformation changes a similar term of the Lagrangian, it changes θ. It follows from a generalization of the Noether theorem

$$\epsilon_5\,\partial_\mu J_5^\mu(x) = \delta_{\epsilon_5}\mathcal{L} \tag{10.44}$$

The divergence of the current of a symmetry transformation is equal to the corresponding change in the Lagrangian. If the transformation is a symmetry the current is conserved.

$$\delta\mathcal{L}(\phi) = \frac{\delta\mathcal{L}}{\delta\varphi_j}\delta\varphi_j + \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\varphi_j}\delta\,\partial_\mu\varphi_j = \partial_\mu\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\varphi_j}\delta\varphi_j\right]}_{J_\mu}$$

10.6 Anomaly cancellation in the Standard Model

The electroweak sector of the Standard Model consists of a gauge sector of group $SU(2)_L \times U(1)_Y$

$$\mathcal{L} = -\frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_W \epsilon^{ijk} W^j_\mu W^k_\nu \qquad (10.45)$$

$$B_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$

g_W is the $SU(2)_L$ weak gauge coupling

The coupling of the gauge fields to spin-1/2 fermionic matter is implemented using covariant derivatives

$$D^\mu = \delta_{rs} \partial^\mu + i g_W (T^i W^{i\mu})_{rs} + i Y \delta_{rs} g'_W B^\mu \qquad (10.46)$$

where g'_W is the $U(1)$ coupling. The matrices T_i are a representation of the $SU(2)_L$ weak isospin algebra and the $U(1)$ charge is called the weak hypercharge.

In order to specify the coupling to matter we have to specify the $SU(2)_L \times U(1)_Y$ quantum numbers of the matter fields.

We note that

$$[T^i, T^j] = i \epsilon^{ijk} T^k \qquad \epsilon^{123} = 1 \quad , \quad T^i = \frac{1}{2}\sigma^i_{\text{Pauli}}$$

$$W^\pm_\mu = (W^1_\mu \mp i W^2_\mu)/\sqrt{2} \qquad T^\pm = T^1 \pm i T^2 \qquad (10.47)$$

$$W \cdot T = W^3_\mu T^3 + \frac{1}{\sqrt{2}} W^+ T^- + W^- T^+$$

$$\langle k_1 \mu_1 i_1 , k_2 \mu_2 i_2 | \partial_\mu J^{\mu, i}_5 (0) | 0 \rangle = \frac{g^2}{8\pi^2}$$

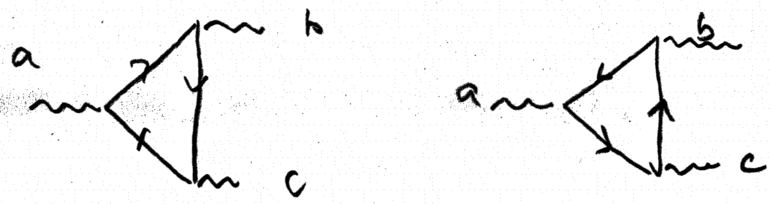
Fermion multiplets

			T_L^3	Y_L	T_R^3	Y_R	Q_f
u	c	t	$1/2$	$1/6$	0	$2/3$	$2/3$
d	s	b	$-1/2$	$1/6$	0	$-1/3$	$-1/3$
ν_e	ν_μ	ν_τ	$1/2$	$-1/2$	$-$	0	0
e^-	μ^-	τ	$-1/2$	$-1/2$	0	1	-1

$$\mathcal{L} = \bar\psi_L i(\not\partial + i g_w T \not W + i g'_w Y_L \not B)\psi_L + \qquad (10.48)$$
$$\bar\psi_R i(\not\partial + i g'_w Y_L \not B)\psi_R$$

$$\psi_L = \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L \qquad (10.49)$$

$$\psi_R = e^-_R, \mu^-_R, \tau^-_R, u_R, \dots t_R$$



$$A^{abc} = \mathrm{Tr}\, R^a\{R^b, R^c\} - \mathrm{Tr}(L^a\{L^b, L^c\}) \qquad (10.50)$$

sum over all possible fermion triangles is proportional to A.

Some useful relations

$$\mathrm{Tr}\, T^a\{T^b, T^c\} = 0$$ no three index invariant matrix for $SU(2)$

$$\mathrm{Tr}(Y_R^3 - Y_L^3) = -3\, \mathrm{Tr}[Y_L (T^3)^2] \qquad (10.51)$$

There is no massless charged field

$$\mathrm{Tr}(Q_R^3 - Q_L^3) \equiv -\mathrm{Tr}[(T^3 + Y_L)^3 - Y_R^3] = 0 \qquad (10.52)$$

We note that $\mathrm{Tr}\,(T^3)^3 = 0$ and $\mathrm{Tr}(T^3 Y_L^2) = 0$ since isospin is block diagonal with fixed value of Y_L.

With these identities we can demonstrate the cancellation of the anomalies

1) $$\mathrm{Tr}(T^a\{T^b, T^c\}) = 0 \quad , \quad \mathrm{Tr}\{T^a Y_L^2\} = 0 \qquad (10.53)$$

These traces vanish separately for leptons and quarks

2) $$\mathrm{Tr}\,(Y_L\{T^a, T^b\}) = \mathrm{Tr}(Q_L\{T^a, T^b\}) = 0 \qquad (10.54)$$

This follows if $$\mathrm{Tr}\,(Q_L) = 0 \qquad (10.55)$$

Indeed

$$3(Q_u + Q_d) + Q_e = 0 \qquad (10.56)$$

The charges of quarks and gluons cancel for a single family.

Comment

We did not prove (10.50) which states that the anomaly is contained in the symmetric part of the traces.

The Feynman diagrams give

$$\mathrm{Tr}\, T^a T^b T^c \cdot \mathrm{Diag}\,1 + \mathrm{Tr}\, T^a T^c T^b \cdot \mathrm{Diag}\,2$$

$$= \frac{1}{2}\mathrm{Tr}\, T^a[T^b, T^c](\mathrm{Diag}\,1 - \mathrm{Diag}\,2) +$$

$$\frac{1}{2}\mathrm{Tr}\, T^a\{T^b, T^c\}(\mathrm{Diag}\,1 + \mathrm{Diag}\,2)$$

I leave as an excercise to show that the first contribution does not have anomalies. For solution see Weinberg QFT II, Chapter 22.3 (page 373-374).

10.A.1 Summary of formulae using differential forms

Instead of a vectorfield we use a 1-form $dA = A_\mu dx^\mu$.
We consider dx^μ Grassmanian

$$dA = \partial_\nu A_\mu \, dx^\nu dx^\mu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu dx^\nu \qquad (A.1)$$

In general a 2-form and a p-form are

$$H_2 = \frac{1}{2!} H_{\mu_1\mu_2} dx^{\mu_1} dx^{\mu_2} \,, \quad H_p = \frac{1}{p!} H_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p} \qquad (A.2)$$

$$dd = 0 \qquad dF = ddA = 0 \qquad (A.3)$$

Closed is not necessarily globally exact

p-form $\alpha_{(p)}$ is closed if $d\alpha_{(p)} = 0$.
p-form $\alpha_{(p)}$ is exact if $\alpha_{(p)} = d\,\beta_{(p-1)}$

Poincaré lemma: a closed form is locally exact
if $dH_{(p)} = 0$ then $H_{(p)} = dK_{(p-1)}$
locally but not globally.

Example if $\mathrm{rot}\,\vec{v} = 0$ then $\vec{v} = \mathrm{grad}\,\phi$ locally.

Stokes-theorem, Gauss law

$$\int_M dH = \int_{\partial M} H$$

Example: $\int_S \mathrm{rot}\,\vec{A}\, d\vec{f} = \oint_L \vec{A}\, d\vec{s}$

H is a p-form and ∂M is the boundary of a $(p+1)$ dimensional manifold

10.A.2 Using forms for non-Abelian gauge-fields

Manipulating anomalies and instanton contributions, it is very convenient to use 1-form and 2-form notations for A^a_μ and $F^a_{\mu\nu}$

$$A = A_\mu dx^\nu \qquad F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2 \qquad (A.4)$$

The gauge transformation of the gauge field becomes

$$A \rightarrow UAU^+ + UdU^+ \qquad (A.5)$$

where U is a ϕ form. It is easy to prove that

$$F \rightarrow UAU^+ \qquad (A.6)$$

Here A_μ is an element of the Li-algebra $A = A^a T^a$

Then we can prove that

$$D = d + A \,, \qquad D^2 = (dA) + A^2 = F \qquad (A.7)$$

$$\mathrm{tr}\, F^2 = d\, \mathrm{tr}\left(A dA + \frac{2}{3} A^3\right) \quad \text{and} \quad d\, \mathrm{tr}\, F^2 = 0 \qquad (A.8)$$

We note that $\mathrm{tr}\, A^4 = 0$ because of the Jacobi identity of the structure constants f^{abc}.

10.A.3 "Shifting the loop momentum" method

Let us consider (10.18), shift the loop momentum with $l+k_1 = l'$ and multiply it with $k_{1\mu_1}$

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2)k_{1\mu} = ie^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{tr}\,\gamma^\lambda\gamma^5\Big(\frac{1}{\not l-\not q}\gamma^{\mu_2}\frac{1}{\not l+\not k_1}\not k_1\frac{1}{\not l} + \frac{1}{\not l-\not q}\not k_1\frac{1}{\not l-\not k_2}\gamma^{\mu_2}\frac{1}{\not l}\Big) \qquad (A.9)$$

we use the identity $\not k_1 = \not l-(\not l-\not k_1)$ in the first term and $\not k_1 = (\not l-\not k_2)-(\not l-\not q)$ in the second term

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2)k_{1\mu} = ie^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{tr}\,\gamma^\lambda\gamma^5\Big(\frac{1}{\not l-\not q}\gamma^{\mu_2}\frac{1}{\not l-\not k_1} - \frac{1}{\not l-\not k_2}\gamma^{\mu_2}\frac{1}{\not l}\Big) \qquad (A.10)$$

Shifting $l \to l'+k_1$ we see that the first term cancels the second term. This is true if we evaluate the integral in d-dimension. Note that we did not use yet any special properties of γ^5. But we can proceed also in 4-dimension. We have linear divergences, therefore we can not make shifts.

We can write, however,

$$\int\frac{d^4l}{(2\pi)^4}\big(f(l+k_1)-f(l)\big) = k_1^\mu\int\frac{d^4l}{(2\pi)^4}\Big(\frac{\partial f(l)}{\partial l^\mu}+\dots\Big) \qquad (A.11)$$

and use Gauss-theorem after Wick rotation

$$\lim_{l_E\to\infty} k_{1E}^\mu\, i\int\frac{d\Omega_4}{(2\pi)^4}\, l_E^3\, n_\mu\, f(l_E), \qquad n_\mu = \frac{l_E^\mu}{l_E} \qquad (A.12)$$

We note that

$$f(\not{D}_x^2/M^2)\, e^{-ik(x-y)}\Big|_{y=x} = f\left([-i\not{k} + \not{D}_x]^2/M^2\right) \qquad (A.45)$$

Rescaling k as $k'M$ we can write

$$\mathcal{A}(x) = -2M^4 \int \frac{d^4k}{(2\pi)^4}\, Tr\{\gamma_5 \tau\, f([i\not{k} + \not{D}_x/M]^2)\} \qquad (A.46)$$

We also note that

$$(-i\not{k} + \not{D}_x/M)^2 = -k^2 - 2i\, k\cdot D_x + \left(\frac{\not{D}_x}{M}\right)^2 \qquad (A.47)$$

In the limit $M \to \infty$ we need to pick up the leading behavior. Since $Tr(\gamma_5 \cdots)$ is not vanishing only if we have at least four γ matrices we can write

$$f\left(-k^2 - 2i\, k\cdot D_x + \left(\frac{\not{D}_x}{M}\right)^2\right) \simeq f(-k^2) + \ldots + \frac{1}{2} f''(-k^2)\left(\frac{\not{D}_x^2}{M^2}\right)^2 + \ldots$$

and therefore

$$\mathcal{A}(x) = -\int \frac{d^4k}{(2\pi)^4}\, f''(-k^2)\, Tr\{\gamma_5 \tau\, \not{D}_x^4\} \qquad (A.48)$$

Evaluating in Euclidean space the k integration

$$\int \frac{d^4k}{(2\pi)^4}\, f''(+k^2) = i\Omega_4 \int_0^\infty \frac{1}{2} k_E^2\, dk_E^2\, f''(+k_E^2) = i\pi^2 \int_0^\infty ds\, s\, f''(s)$$

$$= -i\pi^2 \int_0^\infty ds\, f'(s) = i\pi^2 \qquad (A.49)$$

where we used partial integration and we used equations (A.42) and (A.43).

10.A.4 The Fujikawa method of calculating the axial anomaly

Let us consider again the functional integral with electrons coupled to external background field A_μ

$$D_\mu = \partial_\mu + ie A_\mu \qquad (A.28)$$

$$Z = \int D\psi D\bar{\psi}\, e^{iS}, \quad S = \int d^4x\, \bar{\psi}(x)\, i\not{D}\, \psi(x) \qquad (A.29)$$

$$\left.\begin{aligned} U_5(x) &= \exp(i\gamma_5 \alpha(x)) \qquad \psi'(x) = U_5(x)\psi(x) \\ \bar{U}_5(x) &= \gamma_0 \exp(-i\gamma_5 \alpha(x))\gamma_0 \end{aligned}\right\} \qquad (A.30)$$

$$\bar{\psi}(x) = \psi^{+}\gamma_0 \qquad \bar{\psi}' = \bar{\psi}(x)\bar{U}_5(x)$$

$$\left.\begin{aligned} \bar{U}_5(x) &= U_5(x) \quad \text{and} \quad \bar{U}U = U^2 \\ \bar{U}_5\gamma_0 U_5 &= \gamma_0 \qquad U_5 \text{ is pseudo-unitary} \end{aligned}\right\} \qquad (A.31)$$

Because of this property

$$D\psi' D\bar{\psi}' = [\det U_5]^{-2}\, D\psi D\bar{\psi} \qquad (A.32)$$

The measure not necessarily invariant. We have to check it!
In case of $U(x) = \exp(i\alpha(x))$, $\det U \det U^{+} = 1$ the measure is invariant.

Now we can calculate

$$\lim_{l_E\to\infty} l_E^3 f(l_E) = tr\left(\gamma^\lambda \gamma^5 \frac{1}{\not{l}_E \not{k}_E} \gamma^{\mu_2} \frac{1}{\not{l}_E}\right) l_E^3 = 4i \epsilon^{\alpha\mu_2\beta\lambda} k_{2\alpha} n_\beta \qquad (A.13)$$

where in the last step we performed back the four momenta to Euclidean space

We can also use that $\int \frac{d\Omega_4}{2\pi^2} n^\mu n^\nu = \frac{1}{4}\eta^{\mu\nu}$ (A.14)

therefore

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2)\,k_{1\mu_1} = \frac{i}{8\pi^2}\epsilon^{\lambda\mu_2\alpha\beta} k_{1\alpha} k_{2\beta} \qquad (A.15)$$

This is a disaster since the vector current does not conserve.

Clearly depending on the chosen parametrization of the loop momenta we get different answers. If we would have a shift with k_2^μ we could obtain conserved current. There are infinitely many ways to parametrize the loop momenta and for every parametrization we get a different answer.

Is it possible to choose a parametrization such that

$$\mathcal{M}^{\mu_1\mu_2\lambda} k_{1\mu_1} = \mathcal{M}^{\mu_1\mu_2\lambda} k_{2\mu_2} = 0 \quad ?$$

The answer is yes.

Let us consider

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,a) = ie^2 \int \frac{d^4\ell}{(2\pi)^4}\, tr\Big[\gamma^\lambda\gamma^5 \frac{1}{\not\ell+\not a-\not q}\gamma^{\mu_2}\frac{1}{\not\ell+\not a-\not k_1}\gamma^{\mu_1}\frac{1}{\not\ell+\not a}$$

$$+ \quad \mu_1 \leftrightarrow \mu_2,\ k_1 \leftrightarrow k_2\Big] \qquad (A.16)$$

Carrying out similar manipulation as before we obtain for the difference

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,a) - \mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2\,0) =$$

$$e^2 \int \frac{d^4\ell_E}{(2\pi)^4}\left[F(\ell+a) - F(\ell)\right] =$$

$$= \frac{i}{8\pi^2} e^2 \in^{\mu_1\mu_2\lambda\alpha} a_\alpha + (\mu_1 k_1 \leftrightarrow \mu_2 k_2) \qquad (A.17)$$

Here we used that

$$\lim_{|\ell_E|\to\infty} \ell_E^3 \frac{tr\,\gamma^\lambda\gamma^5 \not\ell_E\gamma^\nu\not\ell_E\gamma^\mu\not\ell_E}{\ell_E^6} = -4i \in^{\lambda\nu\mu\alpha} n_\alpha \qquad (A.18)$$

Imposing the conservation of the vector currents with using

$$a^\mu = \alpha(k_1+k_2)^\mu + \beta(k_1-k_2)^\mu \qquad (A.19)$$

we find that

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,a) = \mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,0) + \frac{i\beta}{4\pi^2}\in^{\mu_1\mu_2\lambda\tau}(k_1-k_2)_\tau$$

But we already have obtained that (A.20)

$$\mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,0)\, k_{1\mu_1} = \frac{i}{8\pi^2}\in^{\mu_1\mu_2\lambda\tau} k_{1\mu_1} k_{2\tau} \qquad (A.21)$$

This tells us that we have to choose $\beta = -1/2$.

We can also see that, α remains unconstrained.

We can interpret this result that the Feynman rules do not suffice in determining

$$\langle 0 | T\, J_5^{\lambda}(0)\, J^{\mu_1}(x_1)\, J^{\mu_2}(x_2) | 0 \rangle$$

They have to supplemented by the requirement that the vector current is conserved. The reason is that the two Feynman diagrams separately give linearly divergent integrals. Their values are defined by the behaviour at infinity. The linear divergence cancel in the sum of the two diagrams but the finite answer depends on the parametrization of the loop momenta since shifting the loop momentum is not allowed. But this ambigous answer is uniquely fixed allowing for a shift (with our initial parametrization) in the form

$$a_0^{\mu} = \alpha (k_1 + k_2)^{\mu} - \tfrac{1}{2}(k_1 - k_2^{\mu}) \qquad \text{(A.22)}$$

~~Now~~ ~~We~~ now we have to calculate

$$-q_{\lambda}\, \mathcal{M}^{\mu_1 \mu_2 \lambda}(k_1, k_2, a_0)$$

the answer is

$$-q_{\lambda}\, \mathcal{M}^{\mu_1 \mu_2 \lambda}(k_1, k_2, a_0)$$
$$= q_{\lambda}\, \mathcal{M}^{\mu_1 \mu_2 \lambda}(k_1, k_2, 0) + \frac{i e^2}{4\pi^2}\, \epsilon^{\mu_1 \mu_2 \alpha \beta}\, k_{1\alpha} k_{2\beta}$$

(A.23)

We can easily prove that

$$q_\lambda \mathcal{M}^{\mu_1\mu_2\lambda}(k_1,k_2,0) = ie^2 \int \frac{d^4 \ell}{(2\pi)^4} \mathrm{tr}\Big[\gamma^5\Big(\frac{1}{\not\ell-\not q}\gamma^{\mu_1}\frac{1}{\not\ell-\not k_1}\gamma^{\mu_2} - \frac{1}{\not\ell-\not k_2}\gamma^{\mu_2}\frac{1}{\not\ell}\gamma^{\mu_1}\Big)\Big) + (\mu_1\, k_1 \leftrightarrow k_2)\Big]$$

$$= \frac{i}{4\pi^2}\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta} \tag{A.24}$$

and therefore

$$q_\lambda \mathcal{M}^{\mu_1\mu_2\lambda}\big(k_1,k_2,a=\alpha(k_1+k_2)-\tfrac{1}{2}(k_1-k_2)\big)$$

$$= \frac{i}{2\pi^2}\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta} \tag{A.25}$$

The axial current is not conserved and again the answer is independent from α in the parametrization of the shift.

In summary, classically

$$\partial_\mu J_5^\mu = 0 \tag{A.26}$$

but in quantum theory

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{A.27}$$

$\partial_\mu J_5^\mu$ is not zero and as an operator is capable to produce two photons.

The γ_5 defined by 't Hooft and Veltman is a good definition since it reproduces the anomaly using dimensional regularization.

Let us specialize to the case of infinitesimal transformations

$$\left.\begin{array}{l}\psi(x) \to \psi'(x) = (1 + i\alpha(x)\gamma^5)\psi(x) \\ \bar\psi(x) \to \bar\psi'(x) = \bar\psi(x)(1 + i\alpha(x)\gamma^5)\end{array}\right\} \quad (A.33)$$

$$S = \int d^4x\, \bar\psi' i \not{D} \psi' = \int d^4x\, \bar\psi(i\not{D}\psi) - \partial_\mu \alpha(x)\, \bar\psi \gamma^\mu \gamma^5 \psi(x)$$

$$= \int d^4x \left[\bar\psi(i\not{D}\psi) + \alpha(x)\, \partial_\mu\left(\bar\psi \gamma^\mu \gamma^5 \psi(x)\right)\right] \quad (A.34)$$

Let us calculate the change in the measure

$$\mathcal{D}\psi\, \mathcal{D}\bar\psi = (\mathrm{Det}\, \mathcal{U})^{-2}\, \mathcal{D}\psi \mathcal{D}\bar\psi$$

$$[\mathcal{U} - 1]_{nx,ny} = i\alpha(x)[\gamma_5 t]_{nm}\, \delta^4(x-y) \quad (A.35)$$

Using the famous identity

$$\mathrm{Det}\, M = \exp \mathrm{Tr} \ln M \quad (A.36)$$

$$(\det \mathcal{U})^{-2} = \exp -2\, \mathrm{Tr} \ln((\delta_{nm} + i\alpha(x)[\gamma_5 t]_{nm})\delta(x-y)) \quad (A.37)$$

$$\mathcal{D}\psi\, \mathcal{D}\bar\psi = \mathcal{D}\psi'\, \mathcal{D}\bar\psi'\, e^{i\int d^4x\, \alpha(x) A(x)} \quad (A.38)$$

$$A(x) = -2(\mathrm{Tr}\, \gamma_5 t)\, \delta(x-x) \quad (A.39)$$

since in functional determinant the space-time coordinates are considered as „continuum indicees."

Therefore the full transformation of S formally is

$$Z = \int D\psi D\bar\psi \, e^{iS + i\int d^4x \, \alpha(x) \left(\partial_\mu J_5^\mu(x) + A(x) \right)} \tag{A.40}$$

Since this is a change of variables, it follows that

$$\partial_\mu J_5^\mu(x) = -A(x) \tag{A.41}$$

But $A(x)$ is not well defined does not look this resull very good since $\delta(x-x)$ is infinite an $\mathrm{tr}\,\gamma^5 = 0$.

We can add this manipulation mathematical meaning if we regularize the $\delta(x-x)$.

We want to regularize it in a gauge invariant manner

Recall that $\delta(x-y) = \int dk \, e^{ikx}$

but $f\left(\frac{1}{M^2}\frac{d}{dx^2}\right)\delta(x-y) = \int dk \, f\left(\frac{1}{M^2}k^2\right) e^{ikx}$

$$\lim_{M\to\infty} f\left(\frac{1}{M^2}\frac{d}{dx^2}\right)\delta(x-y) = \int dk \, f(0) \, e^{ikx} = \int dk \, e^{ikx}$$

provided $f(\infty) = 0$ and $f(0) = 1$ (A.42)

we require smooth behaviour also

$s f'(s) = 0$ at $s=0$ and $s=\infty$ (A.43)

We want to have gauge invariant regularization: we choose $s = \left(\frac{D_x}{M}\right)^2$

$$A(x) = -2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left\{ \gamma_5 \, t \, f\left(\left[ik + D_x \right]^2 / M^2 \right) \right\} \tag{A.44}$$

Using (A.49) in (A.48) we obtain

$$\mathcal{A}(x) = -i \frac{1}{16\pi^2} \mathrm{Tr}\{\gamma_5 \mathcal{T} \not{D}_x^4\} \qquad (A.50)$$

We can proceed with the manipulation of $\not{D}_x^4$.

$$\not{D}_x^2 = \frac{1}{4}[D_x^\mu, D_x^\nu][\gamma^\mu, \gamma^\nu] + \frac{1}{4}\{D_x^\mu, D_x^\nu\}\{\gamma^\mu, \gamma^\nu\}$$

$$= D_x^2 + \frac{1}{4} F_{\mu\nu} [\gamma^\mu, \gamma^\nu] \qquad (A.51)$$

Since in the trace with γ^5 we need four γ-matrices

$$\not{D}_x^4 \sim \frac{1}{16} F_{\mu\nu} F_{\alpha\beta} [\gamma^\mu, \gamma^\nu][\gamma^\alpha, \gamma^\beta] \qquad (A.52)$$

Inserting (A.52) into (A.51) we get

$$\mathcal{A}(x) = -i \frac{1}{(16\pi)^2} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta} \mathcal{T}) \,\mathrm{Tr}\, \gamma^5 [\gamma^\mu, \gamma^\nu][\gamma^\alpha, \gamma^\beta]$$

$$(A.53)$$

Using

$$\mathrm{Tr}\, \gamma^5 [\gamma^\mu, \gamma^\nu][\gamma^\sigma, \gamma^\beta] = 16 i \epsilon^{\mu\nu\alpha\beta}$$

we get for the anomaly

$$\mathcal{A}(x) = \frac{1}{16\pi^2} \mathrm{Tr}(\tilde{F}_{\mu\nu} F^{\mu\nu} \mathcal{T}) = \frac{1}{16\pi^2} \tilde{F}^a_{\mu\nu} F^{\mu\nu, b}\, \mathrm{Tr}(t^a t^b \mathcal{T}) \qquad (A.54)$$

where $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$

$$F_{\mu\nu} = F^a_{\mu\nu}\, t^a \qquad (A.55)$$

and therefore

$$\langle \partial_\mu J_5^\mu \rangle_A = -\frac{1}{16\pi^2} \tilde{F}^a_{\mu\nu} F^{\mu\nu, b}\, \mathrm{Tr}(t^a t^b \mathcal{T}) \qquad (A.56)$$

In the trace we include the full fermionic multiplet. In the case of QCD with chiral $SU(2)_A$ transformation

$$\mathcal{U}(x) = \exp(i\gamma_5 \alpha^i \tau^i) \qquad (A.54)$$

The π^0 is connected to the third component of the axial current*

$$\langle 0 | J_5^{\mu,3}(0) | \pi^0, k \rangle = i f_\pi k^\mu \qquad (A.55)$$

and we choose $\alpha_{1,2} = 0$.

$$\delta u = i\alpha_3 \gamma_5 u \qquad \delta d = -i\alpha_3 \gamma_5 d \qquad (A.56)$$

In QCD with $J_{5}^{\mu,3}(0)\, J_{\mu_1}^{a_1}(x)\, J_{\mu_2}^{a_2}(y)$

$$\mathrm{Tr}(t^{a_1} t^{a_2} \tau_3) = \mathrm{Tr}\, t^{a_1} t^{a_2}\, \mathrm{Tr}\, \tau_3 = 0 \qquad (A.57)$$

There is no anomaly.

In the case of $\pi^0 \to \gamma + \gamma$ decay we have

$$\mathrm{Tr}(Q^2 \tau_3) = N_c \left(\tfrac{2}{3}\right)^2 + N_c \left(-\tfrac{1}{3}\right)^2 = N_c/3 \qquad (A.58)$$

$$\langle \gamma\gamma | \pi^0 \rangle = \int \langle 0 | T\, \varphi_\pi(0)\, J_{\mu_1}(x_1)\, J_{\mu_2}(x_2) | 0 \rangle\, m_\pi^2\, d^4x_1\, d^4x_2\, e^{i(k_1 x_1 + k_2 x_2)}$$

$$= \frac{1}{f_\pi} \langle 0 | \partial_\mu J^3_{5,\mu}(0) | \gamma_1 k_1\, \gamma_2 k_2 \rangle$$

$$= \frac{e^2}{f_\pi} A = \frac{e^2}{f_\pi} \frac{N_c}{3} \frac{1}{16\pi^2} \tilde{F}_{\mu\nu}(k_1) F^{\mu\nu}(k_2) \qquad (A.59)$$

$$F_{\mu\nu}(k_1) = \epsilon_\mu^{\nu}(k_1)\, k_{1\nu} - \epsilon_\nu^{i}(k_1)\, k_{1\mu} \qquad (A.60)$$

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* We note that f_π is determined experimentally for the decay with $\Gamma(\pi^0 \to \mu\nu) \sim G_F f_\pi^2$

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{N_c^2 \alpha^2 m_\pi^3}{144\, \pi^3 f_\pi^2} = \left(\frac{N_c}{3}\right)^2 1.11 \times 10^{16}\, s^{-1} \qquad (A.61)$$

The observed rate is $\Gamma(\pi^0 \to \gamma\gamma) = (1.19 \pm 0.08)\, 10^{16} s^{-1}$

It is in good agreement if $N_c = 3$. Without anomaly it was predicted $\Gamma(\pi^0 \to \gamma\gamma)$ is three order of magnitude using Gell-Mann Levy σ-model, which incorporates the property that π^0 is a Goldstone-boson.

Comments

1. $N_c = 3$ according to the data
2. Bardeen-theorem: the anomaly does not get radiative corrections therefore the prediction must be as precise as the accuracy of the relation m_u^2/m_π^2 or $m_u^2/\Lambda_{QCD}^2 \sim (2-5)\,\%$.

We can firmly conclude that quarks must come in three color! *

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* Similar conclusion we had from $R_{e^+e^-} = \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$