Quantum Field Theory II, Exercise Set $\sharp$ 3.

FS 08/09 *Due: 18.03.09*

1. Harmonic oscillator

The aim of this exercise is to find the well-known eigenvalues and eigenfunctions of one-dimensional harmonic oscillator using the path integral formalism. The Hamiltonian is given by $H := \frac{1}{2}(P^2 + \omega^2 Q^2)$.

a) Show that the Euclidean action of the classical trajectory $\{q_c(\cdot)\}$ is given by

$$S^{\mathrm{E}}[q_c(\cdot)] := \int_0^t \mathrm{d}s \frac{1}{2}\left(\dot{q}_c(s)^2 + \omega^2 q_c(s)^2\right) = \frac{\omega}{2\sinh(\omega t)}[(q_a^2 + q_b^2)\cosh(\omega t) - 2q_a q_b]\,, \qquad (1)$$

where $q_c(0) = q_a$ and $q_c(t) = q_b$ are the starting and ending point of the trajectory.

Denote by $|\Omega\rangle$ the ground state vector and with $\Omega(q) = \langle q\,|\,\Omega\rangle$ the corresponding wavefunction. As a warm-up we derive a formula for $\Omega(q)$.

b) The spectral decomposition of H is given by

$$\mathrm{e}^{-tH} = \sum_{n=0}^{\infty} \mathrm{e}^{-tE_n}|\psi_n\rangle\langle\psi_n|\,,$$

where $|\psi_n\rangle$ is a complete orthonormal set of eigenvectors with eigenvalues E_n. Note that $|\Omega\rangle = |\psi_0\rangle$. Show that

$$|\Omega\rangle = \lim_{t\to\infty}\frac{1}{Z_t}\mathrm{e}^{-tH}|q\rangle\,,$$

where Z_t is an appropriate normalisation factor. Here $|q\rangle = \delta(q - q')$ is a generalized eigenvector of Q.

c) Using the Feynman-Kac formula show that

$$\Omega(q) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}}\exp\left(\frac{-\omega q^2}{2}\right)\,. \qquad (2)$$

Hint: Write $q(s) = q_c(s) + \xi(s)$, where q_c is a solution to the classical Euler-Lagrange equation. Show that $S^{\mathrm{E}}[q(\cdot)] = S^{\mathrm{E}}[q_c(\cdot)] + S^{\mathrm{E}}[\xi(\cdot)]$.

Next we derive the formula

$$\langle q_b|\mathrm{e}^{-tH}|q_a\rangle = \left(\frac{\omega}{2\pi\sinh\omega t}\right)^{\frac{1}{2}}\exp\left\{-\frac{\omega}{2\sinh\omega t}[({q_b}^2 + q_a^2)\cosh\omega t - 2q_b q_a]\right\}\,. \qquad (3)$$

d) Show that, formally,

$$\langle q_b|\mathrm{e}^{-tH}|q_a\rangle = \mathcal{N}\det(A)^{-\frac{1}{2}}\mathrm{e}^{-S^{\mathrm{E}}[q_c]}\,, \qquad (4)$$

where $A := -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \omega^2$ is a self-adjoint differential operator acting on the space of 'path functions' equal to zero at $s = 0$ and $s = t$. $\mathcal{N}$ is a (divergent) normalisation factor, which will be determined later on.

e) Using the fact that the determinant of a self-adjoint operator is formally given by the product of its eigenvalues show that

$$\det(A) = \prod_{n=1}^{\infty} \left(\frac{n^2\pi^2}{t^2} + \omega^2 \right) = K(t) \left(\frac{\sinh(\omega t)}{\omega t} \right) , \tag{5}$$

where $K(t)$ is some (divergent) normalisation factor independent of ω.

Hint: Use Euler's formula $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$ *to handle the infinit product.*

Combining d) and e) we have derived equation (3) up to some finite normalisation, which can be determined, for instance, by taking the limit $\omega \to 0$. Here is another way. From () we obtain

$$\lim_{t\to\infty} \langle q_b | \mathrm{e}^{-tH} | q_a \rangle = \lim_{t\to\infty} \sum_{n=1}^{\infty} \langle q_b \,|\, \psi_n \rangle \, \langle \psi_n \,|\, q_a \rangle \mathrm{e}^{-E_n t} = \lim_{t\to\infty} \langle q_b \,|\, \Omega \rangle \, \langle \Omega \,|\, q_a \rangle \mathrm{e}^{-E_0 t} \,. \tag{6}$$

f) Using this argument and (2) derive the result (3).

g) By expanding $\sinh(\omega t)^{-\frac{1}{2}}$ in (3) and comparing the result with

$$\langle q_b | \mathrm{e}^{-tH} | q_a \rangle = \sum_{n=1}^{\infty} \langle q_b \,|\, \psi_n \rangle \, \langle \psi_n \,|\, q_a \rangle \mathrm{e}^{-E_n t} \,,$$

find the spectrum of the hamonic oscillator.

Performing the Wick rotation to real times in (3) finally yields the propagator for the harmonic oscillator.

2. Completing the lecture notes

Derive equation (2.47) from (2.45) and (2.46) in the lecture notes.

3. φ^4 theory: renormalization and β-function

Let us start from the 'bare' Lagrangian of the φ^4 theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 \,. \tag{7}$$

In QFT I, the renormalized field

$$\varphi_R = Z^{-1/2} \varphi$$

was introduced, so that the two-point Green's function in terms of φ_R is finite. Here we also introduce renormalized mass and coupling and impose a set of conditions on the 1PI 2- and 4-point functions in order to find an explicit form for the counterterms.

In analogy to QED, let us express the renormalized quantites as a power series in λ_R. To lowest order we have

$$\begin{aligned} m &= m_R + \lambda_R \delta m \,, \\ \lambda &= \lambda_R + \lambda_R^2 \delta \lambda \,, \\ Z &= 1 + \lambda_R \delta Z \,. \end{aligned}$$

a) Rewrite the Lagrangian (7) in terms of the renormalized fields, masses and couplings. Retain only terms up to one order higher in λ_R than those appearing in (7), i.e., up to $\mathcal{O}(\lambda_R)$ in $(\partial_\mu\varphi)^2$ and φ^2 and up to $\mathcal{O}(\lambda_R^2)$ in φ^4.

Separate the terms that give the propagator, the φ_R^4 interaction and the counterterms for the 2- and 4- points functions. What are the corresponding Feynman rules? Why did we choose $Z \sim 1+$ *higher order*?

We now need to fix a set of conditions in order to obtain the renormalization parameters. In particular, we require that

i) the renormalized coupling is the magnitude of the scattering amplitude at some specific value of the Mandelstam variables, s_0, t_0, u_0, i.e.,

$$\Gamma_4(s_0, t_0, u_0) = -i\lambda_R\,.$$

Different choices for s_0, t_0, u_0 are possible; yet, as we will se in the lectures, up to two loops the β function should not depend on this choice;

ii) the square of the renormalized mass m_R^2 is the pole of the propagator, i.e.,

$$\Gamma_2(k^2 = m_R^2) = 0\,;$$

iii) the residue at the pole is one, i.e.,

$$\frac{\partial}{\partial k^2}\Gamma_2(k^2)|_{k^2=m_R^2} = 0\,.$$

b) Find the 1PI 2-point function $\Gamma_2(k^2)$ up to $\mathcal{O}(\lambda_R)$, including the counterterms. Then use iii) and ii) to fix δZ and δm respectively.

c) Find the 1PI 4-point function Γ_4 up to $\mathcal{O}(\lambda_R^2)$ (for zero external momentum), including the counterterms. Then use i) to fix $\delta\lambda$.

Hint: In the computation of Γ_4 you will need to evaluate a divergent integral. First of all, you will need to think about which the right symmetry factor is – either think about how many ways you can connect the external lines and swap the internal ones or work all the way from equation (3.37) in the script with an $\mathcal{O}(\lambda_R^2)$ expansion of the exponentials on the RHS. Then, after going to Euclidean space, introduce a momentum cutoff Λ and go to spherical coordinates. To make your calculation simpler, consider the two limiting cases $k \to 0$ and $k \gg (p_1 + p_2)$; when does the integral diverge? Since we are only interested in the divergent part, which simplification can we introduce?
Hint: Recall that $\int d\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$, where d is the number of space-time dimensions and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Finally, introduce some change of variables in order to evaluate the integral. Since we are only interested in the diverging terms, you can denote the finite parts by a "+ finite".

d) From QFT I, recall that the β-function is defined by

$$\beta := \Lambda\frac{\partial}{\partial\Lambda}\lambda\,.$$

Find the β-function for φ^4 theory?