Quantum Field Theory II, Exercise Set 5.

FS 08/09 *Due: 31.03.09*

1. Decay width of the ρ field into Goldstone bosons.

In the exercise set 2, problem 3, we considered a complex field φ whose potential was minimized at some $\varphi_0 \neq 0$, and we showed that angular oscillations around this minimum are massless, while radial oscillations acquire a mass. Our model therefore contains a Goldstone boson σ and a massive real scalar field ρ. We also saw that the Lagrangian contains a $\rho - \sigma - \sigma$ vertex which here we will write as $\mathrm{i}\lambda$. In this exercise we propose to compute the decay width Γ of the massive particle into two Goldstone bosons.

In order to do this, let us recall the relativistic Breit-Wigner formula for the cross section σ in the region of a resonance,

$$\sigma \propto \left| \frac{1}{p^2 - m^2 + \mathrm{i}m\Gamma} \right|^2 , \tag{1}$$

where p and m are the 4-momentum and the mass of the unstable particle, respectively, and Γ is the decay rate, or width, of the particle.[1] Since, in our case, this is the only decay mode, the inverse of Γ corresponds to the lifetime of ρ.
Recall that the two-point function for a scalar field, ρ, is given by

$$\langle \rho(p)\rho(-p) \rangle = \frac{\mathrm{i}}{p^2 - m_0^2 - M^2(p^2)} , \tag{2}$$

$-\mathrm{i}M^2(p^2)$ being the sum of all 1PI insertions into the propagator of ρ.

i) Separate $M^2(p^2)$ into its real and imaginary part and show that the real part produces a displacement in the pole of the propagator along the real axis and therefore a shift in the particle's mass m, while the imaginary part leads to

$$\Gamma = -\frac{1}{m}\mathrm{Im}M^2(m^2) . \tag{3}$$

Hint: In order to prove the second part, recall that the cross section is proportional to the square of (2). Compare this with the Breit-Wigner formula.

ii) Compute the 1-loop contribution to the 1PI two-point function, namely the "bubble diagram" with the massless Goldstone field propagating in the loop. Perform the integral in d dimensions (eventually, you will set $d = 4 - 2\varepsilon$) using the Feynman parametrization

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[Ax + B(1-x)]^2} . \tag{4}$$

After introducing this, complete the square in $l = (k + p \ldots)$ in the denominator, which should end up looking like $(l^2 - \Delta)$. At this point perform a Wick rotation to Euclidean space and introduce spherical coordinates. After this is done, you should find it useful to consider a replacement of the form $t = \Delta/(\Delta + l^2)$. It is also useful to remember that

$$\int_0^1 \mathrm{d}y y^{\alpha-1}(1-y)^{\beta-1} = B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta) \tag{5}$$

[1] For a derivation, see for example 'Introduction to Nuclear Physics' by W.N. Cottingham & D.A. Greenwood, Appendix D.

and that

$$\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} = 1 + 2\varepsilon + \dots . \tag{6}$$

Finally, pay attention to the logarithm of negative quantities.

2. Chiral symmetry of two flavour QCD

The fermionic part of the QCD Lagrangian with only two quarks is

$$\mathcal{L}_q = \mathrm{i}\overline{u}\gamma^\mu \nabla_\mu u + \mathrm{i}\overline{d}\gamma^\mu \nabla_\mu d\,, \tag{7}$$

where ∇_μ is the covariant derivative of the color gauge-group. This Lagrangian has two *global* SU(2) symmetries. It is invariant under

i) SU(2) vector isospin transformations: $q \to \mathrm{e}^{-\mathrm{i}\frac{\alpha_i}{2}\sigma_i} q$,

ii) SU(2) axial isospin transformations: $q \to \mathrm{e}^{-\mathrm{i}\frac{\beta_i}{2}\sigma_i\gamma_5} q$,

where $q = \begin{pmatrix} u \\ d \end{pmatrix}$, and $\alpha_i,\, \beta_i \in \mathbb{R}$, $i = 1,\, 2,\, 3$.

a) Show that the Noether currents corresponding to these symmetries are

$$j_i^\mu = \overline{q}\gamma^\mu \frac{\sigma_i}{2} q \quad \text{and} \quad j_{i,\,5}^\mu = \overline{q}\gamma^\mu\gamma_5 \frac{\sigma_i}{2} q\,, \tag{8}$$

respectively.

These currents have corresponding charges

$$Q_i = \int_{t=\text{const}} \mathrm{d}x^3 j_i^0(x)\,, \quad \text{and} \quad Q_{i,5} = \int_{t=\text{const}} \mathrm{d}x^3 j_{i,5}^0(x)\,. \tag{9}$$

We define the chiral charges as

$$Q_{i,R} := \frac{1}{2}(Q_i + Q_{i,5}) \quad \text{and} \quad Q_{i,L} := \frac{1}{2}(Q_i - Q_{i,5})\,. \tag{10}$$

b) Show that these chiral charges satisfy the commutation relations of a su(2)×su(2) Lie algebra, i.e.,

$$[Q_{i,R},\, Q_{j,R}] = \mathrm{i}\varepsilon_{ijk} Q_{k,R}\,,$$
$$[Q_{i,L},\, Q_{j,L}] = \mathrm{i}\varepsilon_{ijk} Q_{k,L}\,,$$
$$[Q_{i,L},\, Q_{j,R}] = 0\,.$$

As argued in the lecture, the QCD ground states contain a $\overline{q}q$ condensate which breaks this SU(2)×SU(2) symmetry, i.e.,

$$\langle \Omega_{\mathrm{QCD}} |\, \overline{q}q \,| \Omega_{\mathrm{QCD}} \rangle \neq 0\,. \tag{11}$$

Note that only the axial isospin symmetry is broken *spontaneously*, while the vector isospin symmetry remains unbroken.

c) Argue that the pions are the corresponding Goldstone bosons.

However, the observed pion triplet has, in contradiction to the Goldstone theorem, a small mass. This is because the quarks are massive, what we have ignored so far. Adding to (7) the mass term

$$\mathcal{L}_m = m_u \overline{u}u + m_d \overline{d}d = \frac{1}{2}(m_u + m_d)\,\overline{q}q + \frac{1}{2}(m_u - m_d)\,\overline{q}\sigma_3 q\,, \tag{12}$$

will break the SU(2)×SU(2) symmetry *explicitly.*

e) Verify that $\overline{q}q$ is invariant under SU(2) isospin, but not under SU(2) axial isospin.

Assuming that $m_u \simeq m_d$ the theory is almost invariant under SU(2) isospin. Moreover, if the masses are small the axial isospin symmetry is approximate. This suggests that the vacuum vectors $|\Omega_{\rm QCD}\rangle$ align in such a way as to break the approximate SU(2) axial symmetry, while preserving the SU(2) isospin symmetry. One argues that the breaking of an approximate continous symmetry, yields almost massless Goldstone bosons. In our case, these are the lightest pseudo-scalar mesons, the three pions.[2]

3. Completing the lecture notes: on the Källen-Lehmann representation

In this exercise we verify equation (3.119) of the lecture notes. We consider a canonical scalar field theory, meaning that there is no field strength renormalisation, i.e.,

$$[\pi(\mathbf{x},\,0),\,\varphi(\mathbf{y},\,0)] = -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\,. \tag{13}$$

Recall the Källen-Lehmann representation of the 2-point function

$$\langle\Omega|\varphi(x)\varphi(y)|\Omega\rangle = \mathrm{i}\int_0^\infty \mathrm{d}\mu(m^2)\Delta_m^+(x-y)\,, \tag{14}$$

where $\mathrm{i}\Delta_m^+$ is the 2-point function of a free scalar field with mass m,

$$\mathrm{i}\Delta_m^+ = \frac{1}{(2\pi)^3}\int \mathrm{d}^4p\,\delta(p^2-m^2)\theta(p_0)\mathrm{e}^{-\mathrm{i}p(x-y)}\,.$$

Show that

$$\int \mathrm{d}\mu(m^2) = 1\,.$$

Hint: Write (13) *as* $\langle\Omega|[\dot{\varphi}(\mathbf{x},\,t)\,,\varphi(\mathbf{y},\,0)]|\Omega\rangle\big|_{t=0} = -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})$*, then use* (14)*.*

[2] For a detailed treatment we refer to 'The Quantum Theory of Fields', $2^{\rm nd}$ volume, chapter 19, by S. Weinberg.