Quantum Information Theory
Serie 1

Exercise 1.1 Trace distance

The trace distance (or L_1-distance) between two probability distributions P_X and Q_X over a discrete alphabet $\mathcal{X}$ is defined as

$$\delta(P_X, Q_X) = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)|. \tag{1}$$

The trace distance may also be written as

$$\delta(P_X, Q_X) = \max_{\mathcal{S} \subseteq \mathcal{X}} |P_X[\mathcal{S}] - Q_X[\mathcal{S}]|, \tag{2}$$

where we maximize over all events $S \subseteq \mathcal{X}$ and the probability of an event is given by $P_X[\mathcal{S}] = \sum_{x \in \mathcal{S}} P_X(x)$.

a) Show that $\delta(\cdot, \cdot)$ is a good measure of distance by proving that $0 \leq \delta(P_X, Q_X) \leq 1$ and the triangle inequality $\delta(P_X, R_X) \leq \delta(P_X, Q_X) + \delta(Q_X, R_X)$ for arbitrary probability distributions P_X, Q_X and R_X.

b) Show that definitions (2) and (1) are equivalent and use (2) to give a physical interpretation of the trace distance.

Exercise 1.2 Weak Law of Large Numbers

Let A be a positive random variable with expectation value $\langle A \rangle$ and let $P[A \geq \varepsilon]$ denote the probability of an event $\{A \geq \varepsilon\}$.

a) Prove Markov's inequality

$$P[A \geq \varepsilon] \leq \frac{\langle A \rangle}{\varepsilon}. \tag{3}$$

b) Use Markov's inequality to prove the weak law of large numbers for i.i.d. X_i:

$$\lim_{n \to \infty} P\left[\left(\frac{1}{n}\sum_i X_i - \mu\right)^2 \geq \varepsilon\right] = 0 \quad \text{for any } \varepsilon > 0,\ \mu = \langle X_i \rangle. \tag{4}$$

Exercise 1.3 Is a Quantum Theory of Information really necessary?

Consider the following game played by N players $P_i, i = 1 \ldots . N$: Each player P_i receives an input $r_i \in \mathbb{R}$ not known to the rest of the players, with the constraint that $S = \sum_i r_i \in \mathbb{Z}$. The goal is to determine S mod 2 correctly. However, each player P_i is only allowed to communicate one classical/quantum bit to his neighbor P_{i+1}. P_N is supposed to output the result. Before the game starts, all players are free to agree on an optimal strategy.
In class you have seen that the players always succeed when they are allowed to transmit one quantum bit, but what about the classical case?

a) Show that, for N=2, the players can win the game with the transmission of classical bits.

b) Show that the players cannot succeed classically for $N > 2$. (*Hint: Consider the case N=3.*)

c)* Show that, for $N = 3$, P_1 generally has to transmit infinitely many classical bits when we allow P_2 only to transmit a single classical bit. What does this tell us about the simulatability of quantum mechanics?