Exercise 4.1 Trace distance

The trace distance between two states given by density matrices $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is defined as

$$\delta(\rho, \sigma) = \frac{1}{2} \operatorname{tr} |\rho - \sigma| \,. \tag{1}$$

Alternatively we may write

$$\delta(\rho, \sigma) = \max_P \operatorname{tr}(P(\rho - \sigma)) \,, \tag{2}$$

where we maximize over all projectors P onto a subspace of $\mathcal{H}$.

The following lemma can be used to show equivalence of the two definitions:

Lemma 1. *Given two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, there exist two positive operators $S, R \in \mathcal{P}(\mathcal{H})$ with orthogonal support such that $\rho - \sigma = R - S$.*

a) Prove Lemma 1.

b) Show that (1) and (2) are equivalent.

Exercise 4.2 Trace distance of pure states

Find a simple expression for the trace distance of two pure states $\delta(|\phi\rangle, |\psi\rangle)$.

Exercise 4.3 Purification

A decomposition of a state $\rho_{\mathrm{A}} \in \mathcal{S}(\mathcal{H}_{\mathrm{A}})$ is a—non-unique—convex combination of pure states $\rho_{\mathrm{A}}^x = |a_x\rangle\langle a_x|$ such that $\rho_{\mathrm{A}} = \sum_x \lambda_x \rho_{\mathrm{A}}^x$.

a) Show that $|\Psi\rangle = \sum_x \sqrt{\lambda_x}\, |a_x\rangle_{\mathrm{A}} \otimes |b_x\rangle_{\mathrm{B}}$ is a purification for *any* orthonormal basis $\{\, |b_x\rangle_{\mathrm{B}} \,\}_x$ of $\mathcal{H}_{\mathrm{B}}$.

b) Show that any two purifications are related by a *local* unitary transformation on the purifying system.

*b) For ρ_{A} as defined above, and any purification $|\Phi\rangle$ of ρ_{A} on $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$, find an orthogonal measurement $\{\, M_{\mathrm{B}}^x \,\}_x$ on $\mathcal{H}_{\mathrm{B}}$, such that

$$\lambda_x = \operatorname{tr}(|\Phi\rangle\langle\Phi|(\mathbb{1}_{\mathrm{A}} \otimes M_{\mathrm{B}}^x)) \quad \text{and} \quad \rho_{\mathrm{A}}^x = \frac{\operatorname{tr}_{\mathrm{B}}(|\Phi\rangle\langle\Phi|(\mathbb{1}_{\mathrm{A}} \otimes M_{\mathrm{B}}^x))}{\lambda_x}. \tag{3}$$

In this picture λ_x is the probability of measuring x and ρ_{A}^x is the state after such a measurement.