Quantum Information Theory
Serie 6

Exercise 6.1 The Hilbert-Schmidt inner product and entanglement

Suppose R and Q are two quantum systems with the same Hilbert space. Let $|i_R\rangle$ and $|i_Q\rangle$ be two orthonormal basis sets for R and Q. Let A be an operator on R and B an operator on Q. Define $m = \sum_i |i_R\rangle|i_Q\rangle$.

a) Show that $A \otimes \mathbb{1}|m\rangle = \mathbb{1} \otimes A^T|m\rangle$.

b) Use a) to conclude that $tr(A^T B) = \langle m|A \otimes B|m\rangle$.

Exercise 6.2 Fidelity and Uhlmann's Theorem

Given two states ρ and σ on $\mathcal{H}_\mathrm{A}$ with fixed basis $\{|i\rangle_\mathrm{A}\}_i$ and a reference Hilbert space $\mathcal{H}_\mathrm{B}$ with fixed basis $\{|i\rangle_\mathrm{B}\}_i$, which is a copy of $\mathcal{H}_\mathrm{A}$, Uhlmann's theorem claims that the fidelity can be written as

$$F(\rho,\sigma) = \max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle| \,, \tag{1}$$

where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ on $\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$. Let us introduce a state $|\psi\rangle$ as:

$$|\psi\rangle = (\sqrt{\rho} \otimes U_\mathrm{B})\,|\gamma\rangle, \qquad |\gamma\rangle = \sum_i |i\rangle_\mathrm{A} \otimes |i\rangle_\mathrm{B}, \tag{2}$$

where U_B is any unitary on $\mathcal{H}_\mathrm{B}$.

a) Show that $|\psi\rangle$ is a purification of ρ.

b) Argue why every purification of ρ can be written in this form.

c) Use the construction presented in the proof of Uhlmann's theorem to calculate the fidelity between $\sigma' = \mathbb{1}_2/2$ and $\rho' = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ in the 2-dimensional Hilbert space with computational basis. (*Hint: Use Exercise 6.1 and Lemma 4.1.2 from the lecture notes*)

d) Give an expression for the fidelity between any pure state and the completely mixed state $\mathbb{1}_\mathrm{n}/n$ in the n-dimensional Hilbert space.

Exercise 6.3 Depolarizing channel

We are given two two-dimensional Hilbert spaces $\mathcal{H}_\mathrm{A}$ and $\mathcal{H}_\mathrm{B}$ and a CPM $\mathcal{E}_p : \mathcal{S}(\mathcal{H}_\mathrm{A}) \to \mathcal{S}(\mathcal{H}_\mathrm{B})$ defined as

$$\mathcal{E}_p : \rho \mapsto \frac{p}{2}\mathbb{1} + (1-p)\rho. \tag{3}$$

a) Find an operator-sum representation for $\mathcal{E}_p$. Note that $\rho \in \mathcal{S}(\mathcal{H}_\mathrm{A})$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{X}), \quad \vec{r} \in \mathbb{R}^3, \quad \vec{r}\cdot\vec{X} = r_x X + r_y Y + r_z Z, \tag{4}$$

where X, Y and Z are Pauli matrices.

b) What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? What is the physical interpretation of this?

c) A probability distribution $P_A(0) = q$, $P_A(1) = 1-q$ can be encoded in a quantum state on $\mathcal{H}_\mathrm{A}$ as $\rho = q|0\rangle\langle 0|_\mathrm{A} + (1-q)|1\rangle\langle 1|_\mathrm{A}$. Calculate $\mathcal{E}(\rho)$ and the conditional probabilities $P_{B|A}$ as well as P_B, which are defined accordingly on $\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$.

d) Maximize the mutual information over q to find the classical channel capacity of the depolarizing channel.