
Exercise 7.1 The Choi-Jamiolkowski Isomorphism

The Choi-Jamiolkowski Isomorphism can be used to determine whether a given mapping is a CPM. Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \ \mapsto \ (1-\alpha)\,\frac{\mathbb{1}_2}{2} + \alpha\,\Big(\frac{\mathbb{1}_2}{2} + X\rho Z + Z\rho X\Big), \qquad 0 \leq \alpha \leq 1. \tag{1}$$

a) Use the Bloch representation to determine for what range of α these mappings are positive. What happens to the Bloch sphere?

b) Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:

$$\sigma_\alpha = (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

For what range of α is the mapping a CPM?

*c) Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.

Exercise 7.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that works fine at least in low dimensions.

a) Show that the *transpose* is a positive operation, and that it is basis-dependent.

b) Let $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a separable state, and let Λ_A be a positive operator on $\mathcal{H}_A$. Show that $\Lambda_A \otimes \mathbb{1}_B$ maps ρ on a positive operator.

The task of characterizing the sets of separable states then reduces to finding a suitable positive map that distinguishes between separable and entangled states.

c) Show that the *transpose* is a probable candidate by testing it on a Werner state (impure singlet)

$$W = x|\psi^-\rangle\langle\psi^-| + (1-x)\,\mathbb{1},$$

with $x \in [0,1]$.

Remark: Indeed, it can be shown that the PPT (*positive partial transpose*) criterion is necessary and sufficient for systems of dimension 2×2 and 2×3.

d) Show that although the partial transpose is basis-dependent, the corresponding eigenvalues are independent under local basis-transformations.