Quantum Mechanics II
Exercise 2

Exercise 2.1 Degenerate Perturbation Theory I

In the last task of Exercise 1, we have seen that the formalism of (time-independent) perturbation theory breaks down if some energy eigenvalues of the unperturbed system are degenerate. In order to treat such a system perturbatively, all singularities due to the vanishing denominators $(E_n^{(0)} - E_m^{(0)})$ in the degenerate subspace need to be removed before expanding the eigenenergies and eigenstates in perturbative series. As an illustrative example, we examine a perturbed two-level system where one level is degenerate,

$$H = \begin{pmatrix} E & \lambda & 0 \\ \lambda & E & \lambda \\ 0 & \lambda & E' \end{pmatrix}, \tag{1}$$

with $E \neq E'$ and $\lambda \ll |E - E'|$.

a) Apply perturbation theory through order $\mathcal{O}(\lambda^2)$ to compute the energy eigenvalues.

b) Let us apply perturbation theory in an unorthodox manner: write $H = H_0' + \lambda V'$ with

$$H_0' = \begin{pmatrix} E & \lambda & 0 \\ \lambda & E & 0 \\ 0 & 0 & E' \end{pmatrix}, \quad V' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

How does this approach differ from the straight forward Ansatz used in *a*), i.e., $H_0 = H\big|_{\lambda=0}$?

c) Using part *a*) solve the following equation,

$$x^3 - x^2 - 2\lambda^2 x + \lambda^2 = 0, \tag{3}$$

as a perturbative series in λ up to second order.

Exercise 2.2 Degenerate Perturbation Theory II

Now we assume that the perturbation does not mix the two degenerate states,

$$H = \begin{pmatrix} E & 0 & 0 \\ 0 & E & \lambda \\ 0 & \lambda & E' \end{pmatrix}, \tag{4}$$

with $E \neq E'$ and $\lambda \ll |E - E'|$.

a) Find the exact solution.

b) Try to apply degenerate perturbation theory up to NLO (as discussed in class).

c) If $\langle n^{(0)}|V|m^{(0)}\rangle = 0$ for degenerate states $|n^{(0)}\rangle$ and $|m^{(0)}\rangle$ one needs to take into account second order corrections in order to remove the degeneracy. Repeating the analysis which lead to the formula

$$E_n^{(1)}\delta_{nm} = \langle m^{(0)}|V|n^{(0)}\rangle, \quad \text{for } m \in D_n, \tag{5}$$

where D_n denotes the degenerate subspace, show that the energy corrections are given by the formula

$$E_n^{(2)}\delta_{mn} = \sum_{k\notin D_n} \frac{\langle m^{(0)}|V|k^{(0)}\rangle\langle k^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}. \tag{6}$$

d) Find the energy eigenvalues of (4) in NNLO and compare the result to the exact solution.

Exercise 2.3 Variational Method

a) Consider the Hamiltonian

$$H = \frac{p^2}{2} + V(x), \quad V(x) = \begin{cases} \infty, & |x| > 1 \\ 0, & |x| < 1. \end{cases} \tag{7}$$

Apply the variational method for $\Psi_\lambda(x) = 1 - \lambda x^2$. Explain why this ansatz does not yield a satisfactory result.

b) Once again, we have a look at the perturbed harmonic oscillator from Exercise 1.1,

$$H = \frac{1}{2}(p^2 + x^2) + gx^4. \tag{8}$$

Find an upper bound for the ground state energy using the ground state $|0\rangle$ of the unperturbed system as a trial state. Compare your result to the one obtained in first order perturbation theory.
Now, apply the variational method with $\Psi_\lambda(x) = \langle x|0\rangle(1 + \lambda x^2)$.