Quantum Mechanics II
Exercise 7

Exercise 7.1 Scattering cross section in the first Born approximation

a) Calculate the differential scattering cross section in the first Born approximation for the spherical-box potential

$$V(\mathbf{r}) = V(r) = -V_0 \Theta(r_0 - r). \tag{1}$$

b) Show that at low energy, $kr_0 \to 0$, the scattering becomes isotropic and find the total scattering cross section σ.

c) Argue that in the low-energy limit the first Born approximation is only valid for sufficiently shallow potentials, i.e., $V_0 \ll \hbar^2/mr_0^2$.

Exercise 7.2 Low energy resonances and the Breit-Wigner formula

In the following we are considering a particle that scatters at a potential $V(\mathbf{r})$ in three space dimensions. Assume that the potential is spherically symmetric, i.e., $V(\mathbf{r}) = V(r)$ and that there exists a radius $R > 0$ ($R < \infty$) such that $V(r) = 0$ for all $\mathbf{r} \in \{\mathbf{r} : r > R\}$. We have seen in the discussion of 2-body problems with Coulomb interaction (also spherically symmetric) that a separation ansatz with spherical harmonics leads to the ordinary differential equation

$$\left(-\partial_r^2 + \frac{l(l+1)}{r^2} + \Phi(r)\right) rR_l(k,r) = k^2 rR_l(k,r), \tag{2}$$

with

$$\Phi(r) := \frac{2m}{\hbar^2}V(r), \;\; k := \frac{\sqrt{2mE}}{\hbar}. \tag{3}$$

Note that the energies E are taken from the positive, continuous spectrum since we are only interested in scattering states. For $r > R$ the partial wave expansion leads to the wave function

$$R_l^{>}(k,r) = \frac{1}{2}\left(h_l^*(kr) + e^{2i\delta_l(k)}h_l(kr)\right), \tag{4}$$

where $h_l(x) = j_l(x) + in_l(x)$ denotes the l-th spherical Hankel function. Let $R_l^{<}(k,r)$ be the corresponding wave function inside the ball $\{\mathbf{r} : r < R\}$ and define

$$\alpha_l := \partial_r \log R_l^{<}|_{r=R}\,. \tag{5}$$

The goal of this exercise is to determine the behavior of the partial cross sections $\sigma_l(E)$ close to their maxima E_r (the *resonance energies*) in case of low-energy scattering, i.e., $kR \ll 1$.

a) Use the continuity of the total wave function to show that

$$\cot\delta_l = \left.\frac{k\partial_x n_l(x) - \alpha_l n_l(x)}{k\partial_x j_l(x) - \alpha_l j_l(x)}\right|_{x=kR}. \tag{6}$$

b) Express the total cross section σ_l of the l-th partial wave,

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l, \tag{7}$$

as a function of $\cot\delta_l$. When is σ_l maximal in case of low-energy scattering? The corresponding energy eigenvalues E_r are the low lying *resonance energies*.
Hint: Use the asymptotic expressions

$$j_l(x) \approx \frac{x^l}{(2l+1)!!}, \; n_l(x) \approx -\frac{(2l-1)!!}{x^{l+1}}, \quad \text{for } x \to 0, \tag{8}$$

with $(2l \pm 1)!! := 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2l \pm 1)$.

c) Show that the Breit-Wigner formula

$$\sigma_l(E) \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E-E_r)^2 + (\Gamma/2)^2} \tag{9}$$

approximates $\sigma_l(E)$ around a resonance energy E_r.