ETH
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Quantum Mechanics II
Exercise 9

FS 09
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Exercise 9.1 Non-interacting spin-$1/2$-particles in a harmonic trap

We consider N non-interacting spin-1/2-particles in a one-dimensional harmonic trap (frequency ω).

(i) Calculate the ground-state energy for N particles which is given as

$$E_{gs} = \langle \Phi(N)|\mathcal{H}|\Phi(N)\rangle \tag{1}$$

with $\mathcal{H}$ the Hamiltonian of the non-interacting particles and $|\Phi(N)\rangle$ the N-particle wave-funciton.
The Fermi energy ε_F is defined as the energy, up to which all the states are filled. What is the Fermi energy for N particles?

(ii) Instead of the particle number, we can also fix the Fermi energy and ask how many particles can be found in the trap. Plot the N-ε_F-diagram. How does the picture change qualitatively when we introduce a weak Coulomb repulsion within the system?

Exercise 9.2 Non-interacting spin-$1/2$-particles in a box

We now want to consider N non-interacting spin-1/2-particles in a box of dimension d ($V = L^d$).

(i) What are the single-particle eigenenergies ε_n of this system?

(i) For systems with a quasi-continuous spectrum, it is often easiest to write expectation values such as given in Eq. (1) as an energy integral,

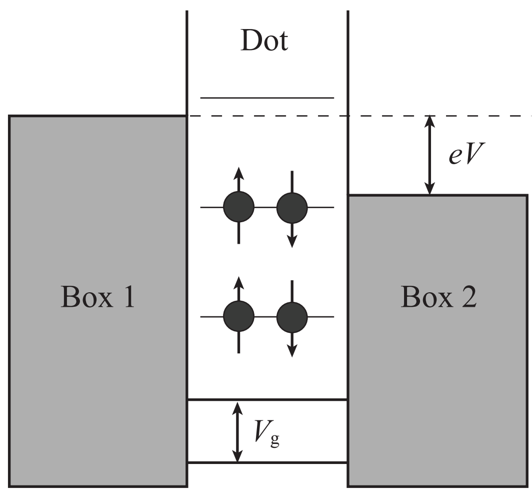
$$\int \rho(\varepsilon)(\dots)d\varepsilon, \tag{2}$$

where we have introduced the density of states $\rho(\varepsilon)$ which counts the number of states per volume in an energy shell $[\varepsilon, \varepsilon + d\varepsilon]$.
Find for non-interacting particles in a box with $L \to \infty$ and dimensions $d = 1, 2, 3$ the density of states and use your result to calculate the Fermi energy for a given density of electrons. To what temperature $T_F = \varepsilon_F/k_B$ does this correspond in 3 dimensions for a box of $L = 1\text{cm}$ and $N = 10^{23}$. What is the total energy of the system?
Hint: Write the ground state energy and the number of particles, which is again an expectation value of the many-particle wave-function, as a Riemann sum.

(iii) In a classical gas, the specific heat which is a measure for the number of particles that can be excited with an energy $k_B T$ is known to be $c_V \sim nk_B$. However, the specific heat of the electron gas when measured up to ambient temperatures turns out to be much smaller. Show how this problem can be solved by considering the particles that can be excited by $k_B T$ and using your result of (ii).

Exercise 9.3 Tunneling through a quantum dot

We model a one-dimensional quantum dot between two reservoirs as a harmonic oscillator (frequency ω_d) between two boxes with a tunable gate voltage V_g and applied voltage to the left reservoir V. Thus, the two reservoirs are filled with electrons up to an energy ε_F and $\varepsilon_F + eV$, respectively. Since the two reservoirs are coupled to the dot, the dot can be filled up to the edge of the left reservoir, i.e. $\varepsilon_F + eV$.



What condition has to be fulfilled that electrons can hop from the right to the left reservoir via the dot and when does the dot 'block'? Make a diagram with axes V and V_g showing regions where there is a current and regions where there's none.