

Exercise 5.1 Specific Heat of a Semiconductor

Calculate the specific heat of a semiconductor under the assumption $k_B T \ll E_g$, where E_g is the band gap. Show that it is given by an ideal gas-like part $(3/2)n(T)k_B$ plus a correction, where $n(T)$ is the number of excitations. Is this correction small or large?

Hint: First, approximate the dispersion of both the conduction and the valence band parabolically, with the two effective masses m_v and m_c . Then, calculate the chemical potential μ from the condition, that the number of electrons in the conduction band ($n_e(T)$) must be equal to the number of holes in the valence band ($n_h(T)$).

Exercise 5.2 Linear Response Theory

We first want to reproduce the general form for the dielectric susceptibility given in the script. We consider a (external) scalar field $V(\vec{r}, t)$ that couples to the local density operator¹

$$\hat{n}(\vec{r}, t) = \hat{\psi}^\dagger(\vec{r}, t)\hat{\psi}(\vec{r}, t), \quad (1)$$

leading to a perturbation of the system of the form

$$\mathcal{H}' = \int d^3\vec{r} V(\vec{r}, t)\hat{n}(\vec{r}, t). \quad (2)$$

The linear response of the system is then given by

$$\langle \delta\hat{n}(\vec{r}, t) \rangle = \int dt' \int d^3\vec{r}' \chi(\vec{r} - \vec{r}', t - t') V(\vec{r}', t'), \quad (3)$$

where $\chi(\vec{r}, t)$ is the density-density correlation function

$$\chi(\vec{r} - \vec{r}', t - t') = \frac{i}{\hbar} \Theta(t - t') \langle [\hat{n}^\dagger(\vec{r}, t), \hat{n}(\vec{r}', t')] \rangle_{\mathcal{H}} \quad (4)$$

and $\langle \dots \rangle_{\mathcal{H}}$ denotes the thermal mean value with respect to the (unperturbed) Hamiltonian \mathcal{H} (see for example the script of Statistical Physics, HS08, Chapter 6).

In momentum and frequency space Eq. (3) simplifies to

$$\langle \delta\hat{n}(\vec{q}, \omega) \rangle = \chi(\vec{q}, \omega) V(\vec{q}, \omega). \quad (5)$$

Show that the dielectric susceptibility is given by

$$\chi(\vec{q}, \omega) = \sum_{n, n'} \frac{e^{-\beta\epsilon_n}}{Z} |\langle n | \hat{n}_{\vec{q}} | n' \rangle|^2 \left\{ \frac{1}{\hbar\omega - \epsilon_{n'} + \epsilon_n + i\hbar\eta} - \frac{1}{\hbar\omega - \epsilon_n + \epsilon_{n'} + i\hbar\eta} \right\}, \quad (6)$$

with Z the partition function,

$$\hat{n}_{\vec{q}} = \int d^3r \hat{n}(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} \quad (7)$$

the Fourier transform of the local density operator and the sum over n, n' runs over all many-particle states.

¹Since we are doing time-dependent perturbation theory, we have to use the interaction representation of the operators.

Exercise 5.3 Dielectric susceptibility of free electrons

For free electrons the field operators are given as

$$\hat{\psi}(\vec{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k},s} e^{i\vec{k}\cdot\vec{r}} \hat{c}_{\vec{k}s}, \quad \hat{\psi}^\dagger(\vec{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k},s} e^{-i\vec{k}\cdot\vec{r}} \hat{c}_{\vec{k}s}^\dagger, \quad (8)$$

with $\hat{c}_{\vec{k}s}$ ($\hat{c}_{\vec{k}s}^\dagger$) annihilating (creating) an electron with momentum \vec{k} and spin s . Derive the Lindhard function,

$$\chi(\vec{q}, \omega) = \frac{1}{\Omega} \sum_{\vec{k},s} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\omega - \epsilon_{\vec{k}} + \epsilon_{\vec{k}+\vec{q}} + i\hbar\eta}, \quad (9)$$

using linear response theory. As a “bonus” evaluate $\chi(\vec{q}, \omega = 0)$ for $T = 0$.