

Exercise 4.1 Weak Decay of the Pion

1. For convenience, we define

$$\tau^+ = \frac{1}{\sqrt{2}} (\tau^1 + i\tau^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \bar{u}_L \gamma^\mu d_L &= \bar{u} \gamma^\mu P_L d \\ &= \bar{u} \gamma^\mu \frac{1}{2} (1 - \gamma^5) d \\ &= \frac{1}{2} (\bar{u} \bar{d}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (\gamma^\mu - \gamma^\mu \gamma^5) \begin{pmatrix} u \\ d \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (J^{\mu+} - J^{\mu 5+}). \end{aligned}$$

2. We recall

$$\begin{aligned} \langle 0 | J^{\mu 5-} | \pi^+(p) \rangle &= \left\langle 0 \left| \frac{1}{\sqrt{2}} (J^{\mu 5 1} - iJ^{\mu 5 2}) \right| \frac{1}{\sqrt{2}} (\pi^1 + i\pi^2) \right\rangle \\ &= -ip^\mu f_\pi e^{-ipx}. \end{aligned}$$

Due to $(\gamma^5)^\dagger = \gamma^5$, we have

$$(J^\mu)^\dagger = J^\mu \quad (J^{\mu 5})^\dagger = -J^{\mu 5}.$$

and therefore

$$\begin{aligned} (J^{\mu+})^\dagger &= (J^{\mu 1} + iJ^{\mu 2})^\dagger = J^{\mu-} \\ (J^{\mu 5+})^\dagger &= (J^{\mu 5 1} + iJ^{\mu 5 2})^\dagger = -J^{\mu 5-} \end{aligned}$$

We spell out \mathcal{L} :

$$\mathcal{L} = 2G_F ((\bar{\ell} \gamma^\mu P_L \nu)(J_\mu^+ - J_\mu^{5+}) + (\bar{\nu} \gamma^\mu P_L \ell)(J_\mu^- + J_\mu^{5-}))$$

and we write out $\langle \text{out} | L | \text{in} \rangle$ explicitly as

$$2G_F \left\langle \ell^+(k), \nu(q) \left| \int d^4x ((\bar{\ell} \gamma^\mu P_L \nu)(J_\mu^+ - J_\mu^{5+}) + (\bar{\nu} \gamma^\mu P_L \ell)(J_\mu^- + J_\mu^{5-})) \right| \pi^+(p) \right\rangle.$$

The first term vanishes because the fermions are on the wrong side, we execute $\int d^4x e^{i(k+q-p)x}$ to get the delta function:

$$\left\langle \ell^+(k), \nu(q) \left| \int d^4x \mathcal{L} \right| \pi^+(p) \right\rangle = 2G_F \bar{u}(q) \gamma^\mu P_L v(k) (-ip_\mu f_\pi) (2\pi)^4 \delta^{(4)}(p - q - k).$$

We split off the momentum conserving delta function, arriving at

$$i\mathcal{M} = 2G_F \bar{u}(q) \not{p} P_L v(k) \cdot f_\pi.$$

3. Now we want to calculate the decay width of $\pi^+ \rightarrow e^+\nu$. We sum over final-state spins to have

$$\sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2 f_\pi^2 \text{Tr} (\not{q} \not{p} (1 - \gamma^5) (\not{k} - m) (1 + \gamma^5) \not{p})$$

inserting $(1 - \gamma^5)m(1 + \gamma^5) = m(1 - \gamma^5\gamma^5) = 0$ and $(1 - \gamma^5)\not{k}(1 + \gamma^5) = (1 - \gamma^5)^2 \not{k} = 2(1 - \gamma^5)\not{k}$

$$= 2 G_F^2 f_\pi^2 \text{Tr} (\not{q} \not{p} (1 - \gamma^5) \not{k} \not{p})$$

since we have only three linearly independent momenta, the γ^5 trace has to vanish

$$= 8 G_F^2 f_\pi^2 (2(q \cdot p)(k \cdot p) - p^2(q \cdot k)).$$

Since we are dealing with a $1 \rightarrow 2$ process, all the invariants can be expressed in terms of masses. Starting from $p = q + k$, $p^2 = m_\pi^2$, $k^2 = m_\ell^2$, $q^2 = 0$ we have

$$(q \cdot p) = \frac{1}{2}(m_\pi^2 - m_\ell^2), \quad (k \cdot p) = \frac{1}{2}(m_\pi^2 + m_\ell^2), \quad (q \cdot k) = \frac{1}{2}(m_\pi^2 - m_\ell^2).$$

We insert this:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}|^2 &= 8 G_F^2 f_\pi^2 \left(\frac{1}{2}(m_\pi^2 - m_\ell^2)(m_\pi^2 + m_\ell^2) - \frac{1}{2}m_\pi^2(m_\pi^2 - m_\ell^2) \right) \\ &= 4 G_F^2 f_\pi^2 m_\pi^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2} \right). \end{aligned}$$

Since we have no angular dependence at all, we only need the integrated two-particle phase space

$$\int d\Pi_2 = \frac{1}{4\pi} \frac{|\mathbf{p}|}{E_{\text{cm}}} = \frac{1}{4\pi} \frac{1}{2} \left(1 - \frac{m_\ell^2}{m_\pi^2} \right)$$

which we insert into the expression for the total decay width

$$\begin{aligned} \Gamma &= \frac{1}{2m_\pi} |\mathcal{M}|^2 \Pi_2 \\ &= \frac{1}{4\pi} G_F^2 f_\pi^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2} \right)^2. \end{aligned}$$

From this we can see the suppression of the decay into electrons compared to the decay into muons and, using the values given on the exercise sheet, we can determine $f_\pi \approx 90$ MeV.