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Advanced Field Theory
Solution 5

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Exercise 5.1 Goldberger-Treiman Relation

Starting from our parametrization in terms of three form factors

$$\langle N|J^{\mu 5a}(q)|N\rangle = \bar{u}(p')\left[\gamma^\mu\gamma^5 F_1^5(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\gamma^5 F_2^5(q^2) + q^\mu\gamma^5 F_3^5(q^2)\right]\tau^a u(p) \qquad (1)$$

we insert the current conservation in the form of $q_\mu J^{\mu 5a}(q)$ to have

$$0 = \bar{u}(p')\left[\not{q}\gamma^5 F_1^5(q^2) + \frac{i\sigma^{\mu\nu}q_\nu q_\mu}{2m}\gamma^5 F_2^5(q^2) + q^2\gamma^5 F_3^5(q^2)\right]\tau^a u(p)$$

the $\sigma^{\mu\nu}$ term vanishes because of its antisymmetry, we insert $q = p - p'$ and we use the equations of motion $\bar{u}(p')\not{p}' = m$ on both sides (we get $-2m_N$ due to the anticommutation of $\not{p}$ with γ^5)

$$0 = \bar{u}(p')\left[-2m_N\gamma^5 F_1^5(q^2) + q^2\gamma^5 F_3^5(q^2)\right]\tau^a u(p)$$

and therefore

$$F_1^5(0) = \lim_{q^2\to 0}\frac{q^2 F_3^5(q^2)}{2m_N}.$$

As we can see, $F_1^5(0)$ can only be nonzero if $F_3^5(q^2)$ has a pole in $q^2 = 0$, corresponding to the exchange of a massless pion. We have for the interaction:

$$\mathcal{M} = (iq^\mu f_\pi)\left(\frac{i}{q^2}\right)\left(-2g_{\pi NN}\bar{u}\gamma^5\tau^a u\right)$$

which we compare with (1) to conclude

$$F_3^5(q^2) \xrightarrow{q^2\to 0} \frac{2f_\pi g_{\pi NN}}{q^2}$$

and therefore

$$F_1^5(0) = \frac{f_\pi g_{\pi NN}}{m_N}.$$

Exercise 5.2 Gell-Mann–Okubo Mass Formula and Weinberg Ratio of Quark Masses

We expand up to second order in Φ:

$$U \approx 1 + i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2$$

which gives us the Lagrangian (inserting $D_\mu = \partial_\mu$ and $\chi = 2BM$ as well)

$$\frac{v^2}{4}\,\mathrm{Tr}\left(\frac{2}{v^2}\partial_\mu\Phi\partial^\mu\Phi + 2BM\left(1 - i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2\right) + 2BM\left(1 + i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2\right)\right)$$

we omit a constant term and have

$$\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}\left(\partial_\mu\Phi\partial_\mu\Phi\right) - \mathrm{Tr}\left(2BM\Phi^2\right).$$

We can write this Lagrangian as a sum of Lagrangians for scalar and complex fields plus a pion eta interaction

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^0)(\partial^\mu \pi^0) - \frac{1}{2}m_{\pi^0}^2(\pi^0)^2 + \frac{1}{2}(\partial_\mu \eta_8)(\partial^\mu \eta_8) - \frac{1}{2}m_{\eta_8}^2(\eta_8)^2 + (\partial_\mu \pi^+)(\partial^\mu \pi^-) - m_{\pi^+}^2 \pi^+ \pi^-$$
$$+ (\partial_\mu K^0)(\partial^\mu \bar{K}^0) - m_{K^0}^2 K^0 \bar{K}^0 + (\partial_\mu K^+)(\partial^\mu K^-) - m_{K^+}^2 K^+ K^- + \frac{2B}{\sqrt{3}}(m_d - m_u)(\pi^0 \eta_8)$$

with the mass parameters

$$m_{\pi^0}^2 = 2B(m_d + m_u), \quad m_{\eta_8}^2 = \frac{2B}{3}(m_u + m_d + 4m_s)$$
$$m_{\pi^+}^2 = 2B(m_u + m_d), \quad m_{K^+}^2 = 2B(m_u + m_s), \quad m_{K^0}^2 = 2B(m_d + m_s)$$

which obey the Gell-Mann–Okubo relation and the Weinberg ratio of quark masses.