# Renormalization

## Electrodynamics as example (1-loop)

The Lagrangian is (bare)

$$\mathcal{L} = -\frac{1}{4}F_0^2 + i\,\bar{\psi}_0(i\partial\!\!\!/_0 - m_0)\psi_0$$

$A_0, \psi_0, m_0$ are the "bare" parameters and are infinite.
One writes ($Z_A = Z_3$)

$$\psi = Z_2^{-1/2}\psi_0 \quad A = Z_A^{-1/2}A_0 \quad e = Z_3^{1/2}e \quad m = m_0 + \delta m$$

where the $\psi, A, e, m$ are the renormalized quantities.
The renormalization constants are infinite. Rewriting $\mathcal{L}$ in terms of the renormalized quantities gives

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

$$\mathcal{L}_0 + \mathcal{L}_1 = -\frac{1}{4}F^2 + i\,\bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\hookrightarrow \mathcal{L}_1 = e\bar{\psi}A\!\!\!/\psi$$

$$\mathcal{L}_2 = -\frac{1}{4}(Z_3-1)F^2 + (Z_2-1)i\,\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\partial\!\!\!/\psi$$
$$+ Z_2\,\delta m\,\bar{\psi}\psi + e(Z_2-1)\bar{\psi}A\!\!\!/\psi$$

Since we have $Z = 1 + O(\alpha), \ldots, Z-1$ is of order $\alpha$.
Equally $\delta m$ is of order $\alpha$. Thus $\mathcal{L}_2$ is of order $\alpha$. It is called counterterms and is adjusted to cancel the divergencies that come from loops with $\mathcal{L}_0, \mathcal{L}_1$.
The point of renormalizability is that $\mathcal{L}_2$ has the same terms as $\mathcal{L}_0, \mathcal{L}_1$ and no more.

Thus: $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{CT}$. If this has the form of $\mathcal{L}_R$ then the =

The Lagrangian must have no operators of dimension more than 4 to be "renormalizable". The Lagrangian density satisfies certain symmetry constraints, such that the $[O] \leq 4$ is sufficient.
However one must show that also the effective quantum action has only infinities that obey the same symmetry constraints. We have seen (Babis p. 84 p.p) that the effective action inherits the symmetries if they are linear. We now consider this in some detail (W, 17.2).

We define an action

$$\Gamma[\langle X\rangle, K] = W[J_i^K K] - \int d^4x \langle X\rangle J_K] \qquad (1)$$

$$e^{iW[J_i^K K]} = \int \mathcal{D}X \exp(iS + \int d^4x(\Delta K + XJ) \qquad (2)$$

where $\Delta$ is the BRST - variation of $X$,

$$\delta X = \theta\Delta^{*} \qquad (3)$$

and $K$ a "new" set of external fields. $J^K$ is the current required to give $X$ a fixed value $\langle X\rangle$ in the presence of $K$.**

We have shown that $\Gamma[X]$ satisfies

$$\int d^4x \; \langle \Delta \rangle_J \frac{\delta\Gamma}{\delta X} = 0 \qquad (S\text{-}T) \qquad (4)$$

We can also require for $\Gamma[\langle X\rangle K]$

$$\int d^4x \; \langle \Delta \rangle_{JK} \frac{\delta\Gamma[\langle X\rangle K]}{\delta\langle X\rangle} = 0 \qquad (5)$$

Since $\langle \Delta_{JK} \rangle = \delta\Gamma[\langle X\rangle, K]/\delta K \qquad (6)$

* In Babis, $\Delta$ is called $B$

** $\left.\dfrac{\delta W[J,K]}{\delta J}\right|_{J_K} = \langle X\rangle$

We can write ( $\delta_R$ because in $\Delta \cdot K$, $K$ is on the right, etc )

$$\int d^4x \, \frac{\delta_R \Gamma}{\delta K} \frac{\delta_L \Gamma}{\delta \langle X \rangle} = 0 \tag{7}$$

One defines

$$(F, G) = \int d^4x \left( \frac{\delta_R F}{\delta X} \frac{\delta_L G}{\delta K} - \frac{\delta_R F}{\delta K} \frac{\delta_L G}{\delta X} \right) \tag{8}$$

Since from (3) $\langle X \rangle$ and $\Delta$ have opposite spin statistics, so have $\langle X \rangle$ and $K$. Thus both terms in (8) are the same if $F = G$ *, and thus (Z-Justin)

$$(\Gamma, \Gamma) = 0 \tag{9}$$

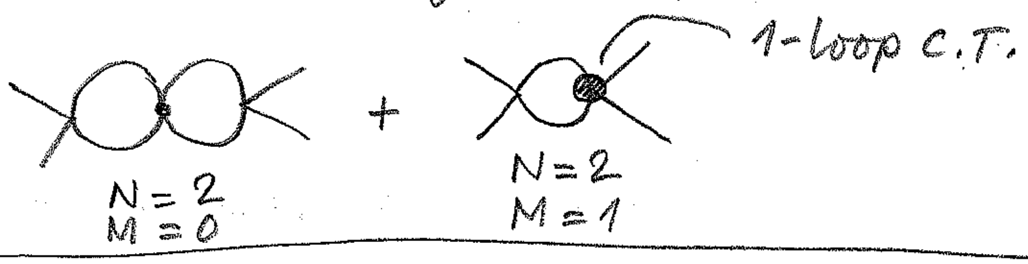
Next consider the action

$$\begin{aligned} S(X, K) &= S(X) + \int d^4x \, \Delta K \\ &= S_R(X, K] + S_\infty(X, K) \end{aligned} \tag{10}$$

where $S_R$ contains the renormalized values and $S_\infty$ the counterterms that cancel the divergences. We write them

$$\Gamma(X, K) = \sum_{N=0}^{\infty} \Gamma_N (X, K) \tag{11}$$

where $\Gamma_N$ contains $N$ loops of $N-M$ loops and C.T. which cancel graphs with $M$ loops

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* $\Delta K + XJ = -K\Delta + XJ$ if $X$ bosonic, $= K\Delta - JX$ if $X$ fermionic

We get from (9 )

$$\sum_{N'=0}^{N} (\Gamma_{N'}, \Gamma_{N-N'}) = 0 \qquad (12)$$

Since each loop brings a factor $\hbar$, therefore (9) holds at each fixed order in $\hbar$.

Let now all $\infty$ from M-loops be cancelled for $M \leq N-1$. Then infinities in (12) can only be for $N' = 0$ or $N' = N$. Since $\Gamma_0 = S_R$ (no loops), this gives

$$(S_R, \Gamma_\infty) = 0 \qquad (13)$$

where $\Gamma_\infty$ is the infinite part of $\Gamma_N$.

$\Gamma_\infty$ can only be composed of powers of the fields (Babis, p. 100), and has dimension 4 or less.

let now look at a BRST transformation

$$\delta X = \theta \Delta \tag{3}$$

If $X$ has dimension $d$, then ($\dim \theta = -1$), $\Delta$ has dimension $d+1$.*
Then $[K] = 3-d$ ($\mathcal{L}$ has always 4). If $X = A, \eta$ then $[K] = 2$, if $X = \psi$, $[K] = 3/2$.

The action $\Gamma_\infty[X,K]$ has dimension 4, thus can have at most 2 powers of $K$. For fermions, can have $K \cdot K \cdot \varphi$, $[\varphi] = 1$.

Since the "ghost" number is conserved, we use it to constrain $\Gamma_\infty$. The ghost numbers are $1$ for $\eta$, $-1$ for $\bar\eta$, $0$ for $A, \psi$ and $-1$ for $\theta$, and by definition $\gamma$ for $X$. Then $\Delta$ has $\gamma+1$ and $K$ $-(\gamma+1)$. Then we have the following ghost numbers

$$K_\eta = -2, \quad K_{\bar\eta} = 0, \quad K_A = -1, \quad K_\psi = -1 \tag{14}$$

Thus only a quadratic term in $K_{\bar\eta}$ is possible. Since however the extra term in $\Gamma$ is $\Delta K_{\bar\eta}$, and $\Delta$ is just $\omega$ (independent of $K$), everything is linear in $K$.

Now we have

$$\Gamma_\infty[X,K] = \Gamma_\infty[X,0] = \int d^4x\, \mathcal{D}[X,x] K(x) \tag{15}$$

and

$$S_R[X,K] = S_R[X] + \int d^4x\, \Delta[X,x] K(x) \tag{16}$$

From

$$(\Gamma_0, \Gamma_\infty) = 0 \quad \text{or} \quad (S_R, \Gamma_\infty) = 0 \tag{17}$$

$$\int d^4x \left( \frac{\delta S_R}{\delta X} \frac{\delta \Gamma_\infty}{\delta K} + \frac{\delta S_R}{\delta K} \frac{\delta \Gamma_\infty}{\delta X} \right) = 0$$

we have two equation for $K^0, K^1$:

$$\int d^4x \left( \frac{\delta S_R[X]}{\delta X} \mathcal{D} + \frac{\delta \Gamma[X,0]}{\delta X} \Delta \right) = 0 \qquad "K^0" \tag{18}$$

$$\int d^4x \left( \frac{\delta \mathcal{D}}{\delta X} \Delta + \frac{\delta \Delta}{\delta X} \mathcal{D} \right) = 0 \qquad "K^1" \tag{19}$$

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* In BRST, $\delta\psi = \theta\eta\psi$. Dimensions are: $[\eta] = 1$, $[\theta] = -1$, $[\psi] = 3/2$

These two equations essentially say that

$$\Gamma^{\varepsilon}[X] = S_R[X] + \varepsilon \Gamma_\infty[X, 0] \qquad (20)$$

is invariant under

$$X \to X + \theta(\overbrace{\Delta + \varepsilon \mathcal{D}}^{\Delta^{\varepsilon}}) = X + \underbrace{\delta X + \delta_\varepsilon X}_{\delta_t X} \qquad (21)$$

and that it is nilpotent:

$$* \quad \delta\Gamma^{\varepsilon} = \Big(\underbrace{\frac{\delta S_R}{\delta X}\Delta}_{} + \underbrace{\varepsilon\frac{\delta\Gamma_\infty}{\delta X}\Delta + \varepsilon\frac{\delta S_R}{\delta X}\mathcal{D}}_{0}\Big)\theta = 0 \qquad (22)$$

$$= 0 \; (S_R \text{ is BRST invariant})$$

$$* \quad \delta_t(\delta_t X) = \delta_t(\Delta + \varepsilon\mathcal{D})\theta = \delta_t(\delta X) + \delta_t(\varepsilon\mathcal{D}\theta)$$

$$= \underbrace{\delta(\delta X)}_{0} + \delta_\varepsilon \Delta\theta + \delta(\varepsilon\mathcal{D}\theta) + O(\varepsilon^2) \qquad (23)$$

$$\llcorner (BRST)^2 = 0 + \underbrace{\varepsilon\Big(\mathcal{D}\frac{\delta\Delta}{\delta X} + \frac{\delta\mathcal{D}}{\delta X}\Delta\Big)}_{0}\theta\theta = 0$$

The next step is to find the form of $\mathcal{D}$ (or $\Delta^{\varepsilon}$). From (17.1.4 W) and (17.2.5) and (17.2.5), $\mathcal{D}$ and $\Delta^{\varepsilon}$ can have at most the dimension of $\Delta$ (since $\Gamma_\infty$ has dimension 4 or less). Also the ghost number and Lorentz-properties are the same as for $\Delta$. Thus we set

$$\begin{aligned} \psi &\to \psi + i\theta T^a \eta_a \psi \\ A_a &\to \psi_a + \theta(B_{ab}\partial\eta_b + D_{abc}A_b\eta_c) \qquad (24) \\ \eta_a &\to \eta_a + \theta E_{abc}\eta_b\eta_c \end{aligned}$$

(see Babs, 281, 282, 277; B, D are constants that may be different from the original BRST), and E satisfies a cyclicity relation (from nilpotency)

$$E_{abc}E_{bde} + E_{abe}E_{bcd} + E_{abd}E_{bec} = 0 \qquad (25)$$

For $\varepsilon \to 0$, $E = f$( structure constants ), thus gives

$$E_{abc} = Z \cdot f_{abc} \tag{26}$$

(gauge invariance). Similarly we find

$$D_{abc} = Z \cdot f_{abc} \qquad B_{ab} = Z \cdot N \delta_{ab} \tag{27}$$

Calculation:

$$\delta A_a = B_{ab} \partial \eta_b + D_{abc} A_b \eta_c \tag{28}$$

$$\delta(\delta A_a) = B_{ab}\big(-\tfrac{1}{2} E_{bcd} \underbrace{\partial(\eta_c \eta_d)}_{\partial\eta_c \eta_d + \eta_c \partial \eta_d}\big) + D_{abc}\,(B_{bc'} \partial \eta_{c'} + D_{bcd} A_c \eta_d)\,\eta_c$$

$$+ D_{abc} A_b E_{cde} \eta_d \eta_e = 0$$

$$\to \text{"}BE + DB = 0\text{"} \quad \text{"}DD + DE = 0\text{"} \tag{29}$$

$$\delta(\delta\psi) = 0 \longrightarrow T^a = Z t^a$$

Thus, the modified transformation $\Delta^\varepsilon$ is just a BRST transformation with extra constants:

$$\begin{aligned} \delta\psi &= iZ\theta t^a \eta_a \psi \\ \delta A_a &= Z\theta(N\partial\eta_a + f_{abc} A_b \eta_c) \\ \delta \eta_a &= -\theta Z f_{abc} \eta_b \eta_c \end{aligned} \tag{30}$$

( and $\delta\bar{\eta} = \theta\omega$, $\delta\omega = 0$ )

Next we use these symmetries to constrain the modified action $\Gamma^\varepsilon[X]$. It contains $S_R$ and the counterterms in $\Gamma_\infty[X,0]$. We can set

$$\Gamma^\varepsilon[X] = \int d^4x\, \mathcal{L}^\varepsilon(X)$$

where $\mathcal{L}_0^\varepsilon$ is of dimension 4 or less. $\mathcal{L}_0$ is also invariant under linear symmetries of $\mathcal{L}$ (Babis, p. 86). The $\mathcal{L}$ we use is

$$\mathcal{L} = \bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F^2 - \partial\bar{\eta}D\eta + \omega g + \frac{\xi}{2}g^2 + \qquad (31)$$

(Babu, p. 64), where $g$ is the gauge condition, $\partial_\mu A^\mu$.

The linear transformations are Lorentz-invariance, (global) gauge invariance, anti-ghost shift $\bar{\eta} \to \bar{\eta}+c$ and ghost number $\delta\eta = \alpha\eta$, $\delta\bar{\eta} = -\alpha\bar{\eta}$.
The dimensions of $A, \eta, \bar{\eta}$ and $h$ are $1, 1, 1, 2$. We also need pairs of $\partial\bar{\eta}\cdot\eta$ (with dimension 3) to satisfy $\bar{\eta}\to\bar{\eta}+c$ and $\eta$-conservation. Thus

$$\mathcal{L}_0^\varepsilon \sim \bar{\psi}(i\not{D}-m)\psi - \frac{1}{2}F^2 + \frac{1}{2}\xi'\omega^2 + c\,\omega\,\partial_\mu A - \qquad (*)$$
$$e_{abc}\,\omega_a A_b A_c - Z_\eta(\partial\bar{\eta})(\partial\eta) - d_{abc}\,\partial\bar{\eta}\,\eta\,A \qquad (32)$$

BRST: (see eqs (30))

$$\delta(Z_\eta\,\partial\bar{\eta}\,\partial\eta) = \theta Z_\eta\,\partial\omega\,\partial\eta \ldots$$
$$\delta(c\,\omega\,\partial_\mu A) = \theta\, c\,\omega\, Z N\,\partial^2\eta + \ldots = -\theta c\,\partial\omega\,\partial\eta\, Z N + \ldots$$

$$\longrightarrow Z_\eta = c Z N \quad (\text{terms prop } \partial\omega\,\partial\eta) \qquad (33)$$

Similarly

$$d_{abc} = -\frac{Z_\eta}{N} f_{abc} \qquad (34)$$

We also find

$$e_{abc} = 0 \qquad (35)$$

For matter fields, (30) tells us that (30) is a gauge transformation with gauge parameter $\varepsilon = ZN\theta\eta$ and gauge coupling $\frac{1}{N}$ ($\delta\psi = g\lambda\psi$ $\delta A = \partial_\mu\lambda$).

Thus the most general form of $\mathcal{L}_0^\varepsilon$ is just the ordinary Lagrangian with renormalized couplings:

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(*) The gauge couplings might be modified, see below

$$\mathcal{L}_0^{\varepsilon} = \bar{\Psi}\left[\left(i\not{\partial} - \frac{t\not{A}}{N}\right) - m\right]\bar{\Psi} + \frac{1}{2}\xi' \omega^2 + \frac{Z_\eta}{NZ}\,\omega\,\partial_\mu A - Z_\eta(\partial\bar{\eta})(\partial\eta) + \frac{Z_\eta}{N} f\,\partial\bar{\eta}\,\eta A \qquad (36)$$

Thus, the modified Lagrangian has the form of $\mathcal{L}$ with new constants. They can be shifted, for instance, to make $\mathcal{L}_0^{\varepsilon} = \mathcal{L}_R$ (or $\Gamma^{\epsilon} = S_R$), thereby removing $\Gamma_\infty$. This shows that a few "counterterms at any order $N$ suffice to have a fully renormalized $\mathcal{L}$.