Quantum Field Theory II
Exercise sheet 3

Exercise 3.1 Basics of QFT

Consider the Lagrangian for a real scalar field $\phi(x)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

a) The principle of least action states that the action $S = \int d^4x\mathcal{L}$ has to be an extremum, so that small variations vanish ($\delta S = 0$). Use this fact to derive the Euler-Lagrange equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Compute it explicitly for the real scalar field.

b) Use canonical quantisation to derive the Feynman propagator $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$.

Hint: first compute the Hamiltonian of the system, apply the usual commutation relations, and then rewrite the field ϕ in terms of creation and annihilation operators.

Exercise 3.2 Gaussian integrals

Using the known result for the Gaussian integral

$$\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$$

show that

$$\int_{-\infty}^{\infty} dx_1\ldots dx_N\ e^{-\frac{1}{2}x^TAx} = \sqrt{\frac{(2\pi)^N}{\det A}}$$

where $x^T = (x_1,\ldots,x_N)$ and A is a symmetric, positive definite $N\times N$ matrix.
Remark: taking the limit $N\to\infty$, one can see that this identity holds formally for operators, so that for the action

$$S_0 = -\frac{1}{2}\int d^4x\ \phi(\partial^2+m^2)\phi$$

one has

$$\int\mathcal{D}\phi\ e^{iS_0} = \text{const}\cdot\left[\det(\partial^2+m^2)\right]^{-1/2}$$

Exercise 3.3 Discretisation of the path integral

Consider the action for a free real scalar field of mass m

$$S_0 = \int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right]$$

We want to evaluate exactly the path integral

$$\int\mathcal{D}\phi e^{iS_0}$$

in a finite volume V.

First, we replace the field $\phi(x)$ defined on a continuum of points by variables $\phi(x_i)$ defined at the points x_i of a square lattice. The integration measure becomes

$$\mathcal{D}\phi = \prod_i d\phi(x_i)$$

up to an irrelevant constant.

a) The field values $\phi(x_i)$ can be represented by a discrete Fourier series:

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\phi(k_n)$$

where $k_n^\mu = 2\pi n^\mu/L$ with n^μ an integer, and $V = L^4$. Since $\phi(x)$ is real the Fourier coefficients have to obey $\phi^*(k) = \phi(-k)$.

Rewrite the action S_0 in terms of the Fourier coefficients $\phi(k_n)$.

b) Since the $\phi(k_n)$ are complex, one can integrate separately their real and imaginary part. Show that the integration measure can be written

$$\mathcal{D}\phi(x) = \prod_{k_n^0>0} d\,\mathrm{Re}\,\phi(k_n)d\,\mathrm{Im}\,\phi(k_n)$$

and write the action S_0 also in terms of $\mathrm{Re}\,\phi(k_n)$ and $\mathrm{Im}\,\phi(k_n)$.

c) Using the results of exercise 2, perform the path integral. You should obtain

$$\int \mathcal{D}\phi e^{iS_0} = \prod_{k_n}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}$$

Exercise 3.4 Four-point function

Consider the generating functional of the free Klein-Gordon theory

$$Z[J] = Z[0]\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\right]$$

The four-point function is then

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{Z[0]i^4}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}Z[J]\bigg|_{J=0}$$

Compute it and interpret the result pictorially.
Hint: use the basic axiom of functional derivation in 4 dimensions

$$\frac{\delta}{\delta J(x)}J(y) = \delta^{(4)}(x-y)$$

For more complicated functions of J, one simply uses the ordinary Leibniz rule for derivatives of composite function, e.g

$$\frac{\delta}{\delta J(x)}J(y)J(z) = \left(\frac{\delta}{\delta J(x)}J(y)\right)J(z) + J(y)\left(\frac{\delta}{\delta J(x)}J(z)\right) = \delta^{(4)}(x-y)J(z) + \delta^{(4)}(x-z)J(y)$$