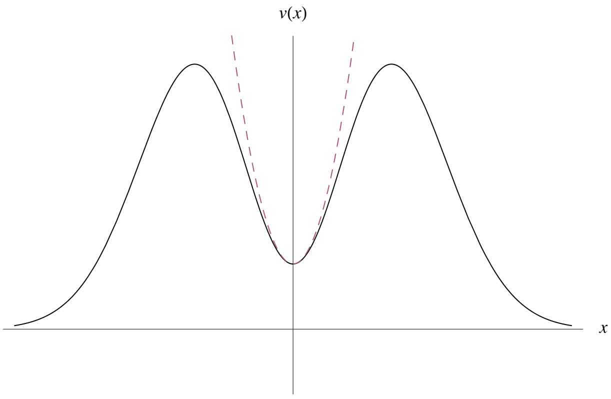
Quantenmechanik II
Serie 10

Aufgabe 10.1 Estimation of the lifetime of a shape resonance

Consider a quantum mechanical particle moving in one dimension under the influence of a potential v_θ, with

$$v_\theta(x) := \theta^2 v(x/\theta) \qquad 1 \leq \theta < \infty, \tag{1}$$

and $v(x)$ having the shape depicted in the figure below.



a) Analyze why it is $\sigma(H_\theta) = [0, +\infty)$.

b) In the vicinity of $x = 0$ it reasonable to express the Hamiltonian as

$$H_\theta = H_{1\theta} + \delta v_\theta, \tag{2}$$

where

$$H_{1\theta} = -\frac{1}{2}\frac{d^2}{dx^2} + \theta^2 v(0) + \frac{\omega^2 x^2}{2}. \tag{3}$$

Find $\sigma(H_{1\theta})$ and show that $\sigma(H_{1\theta}) \subset \sigma(H_\theta)$.

c) The system is initially prepared in ψ_i, which is an eigenstate of $H_{1\theta}$ that is localized inside the well of $v_\theta(x)$.

$$q_t := tr(e^{-iH_\theta t} P_i e^{+iH_\theta t} P_\theta) \tag{4}$$

is the probability of finding the particle in any state inside the well at time t, where P_i is the projector onto $|\psi_i\rangle$ and P_θ is the projector onto the subspace spanned by all the eigenstates of $H_{1\theta}$ that are contained inside the well of $v_\theta(x)$.

i) Estimate dq_t/dt using that $|tr(PA)| \leq (trP)||A||$

ii) Knowing that δv is small when x/θ is also small (more formally, it can be assumed that for $|x| < \epsilon\theta$ is $|\delta v| < e^{-\delta\theta^2}$), find an lower bound for τ, where τ is defined by $q_\tau = 1/2$.

iii) What is the physical meaning of τ?

HINT: You might need to think about how the height and the width of the well scale with θ. Also remember that the width of the n-th Hermite polynomial is $\sqrt{n}$.

Aufgabe 10.2 Bargmann coherent states

The phase space of a classical system with N degrees of freedom is given by $(q^1, \dots, q^N, p_1, \dots, p_N) \in \mathbb{R}^{2N}$. These canonical variables have the well known Poisson Brackets

$$\{q^i, q^j\}_{PB} = \{p_i, p_j\}_{PB} = 0 \qquad \{q^i, p_j\}_{PB} = \delta^i_j, \tag{5}$$

and upon applying Dirac quatization rule

$$\{\ ,\ \}_{PB} \to \frac{1}{i\hbar}[\ ,\ ] \tag{6}$$

the transition from classical to quantum mechanics is achieved.

a) Show that
$$z_j = \frac{1}{\sqrt{2}}(q^j + ip_j) \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(q^j - ip_j) \qquad j = 1, \dots, N \tag{7}$$
have the following Poisson brackets
$$\{z^i, z^j\}_{PB} = \{\bar{z}_i, \bar{z}_j\}_{PB} = 0 \qquad \{z^i, \bar{z}_j\}_{PB} = -i\delta^i_j, \tag{8}$$

b) Promote z_i and $\bar{z}_i$ to operators a_i and $a_i^\dagger$ respectively, and apply Dirac quantization rule to verify that these operators are the same creation and annihilation operators that arise in the quantum treatment of the simple harmonic oscillator.

c) Show that the coherent state
$$|z\rangle := e^{za^\dagger}|0\rangle \tag{9}$$
is an eigenstate of a with eigenvalue z.

d) To a given state $|\psi\rangle \in \mathcal{H} := L^2(q, dq)$ we associate a wave function
$$\psi(\bar{z}) := \langle z|\psi\rangle. \tag{10}$$
Evaluate $(a\psi)(\bar{z})$ and $(a^\dagger\psi)(\bar{z})$ to find a Schrödinger representation of a and $a^\dagger$.