# Solid State Theory
# Exercise 6



## Exercise 6.1 Linear Response Theory

We first want to reproduce the general form for the dielectric susceptibility given in the script. We consider a (external) scalar field $V(\vec{r},t)$ that couples to the local density operator[1]

$$\hat{n}(\vec{r},t) = \hat{\psi}^\dagger(\vec{r},t)\hat{\psi}(\vec{r},t), \tag{1}$$

leading to a perturbation of the system of the form

$$\mathcal{H}' = \int d^3\vec{r}\, V(\vec{r},t)\hat{n}(\vec{r},t). \tag{2}$$

The linear response of the system is then given by

$$\langle \delta\hat{n}(\vec{r},t)\rangle = \int dt' \int d^3\vec{r}\,'\chi(\vec{r}-\vec{r}\,',t-t')V(\vec{r},t), \tag{3}$$

where $\chi(\vec{r},t)$ is the density-density correlation function

$$\chi(\vec{r}-\vec{r}\,',t-t') = \frac{i}{\hbar}\Theta(t-t')\langle[\hat{n}^\dagger(\vec{r},t),\hat{n}(\vec{r}\,',t')]\rangle_{\mathcal{H}} \tag{4}$$

and $\langle\ldots\rangle_{\mathcal{H}}$ denotes the thermal mean value with respect to the (unperturbed) Hamiltonian $\mathcal{H}$ (see for example the script of Statistical Physics, HS08, Chapter 6).
In momentum and frequency space Eq. (3) simplifies to

$$\langle \delta\hat{n}(\vec{q},\omega)\rangle = \chi(\vec{q},\omega)V(\vec{q},\omega). \tag{5}$$

Show that the dielectric susceptibility is given by

$$\chi(\vec{q},\omega) = \sum_{n,n'} \frac{e^{-\beta\epsilon_n}}{Z}|\langle n|\hat{n}_{\vec{q}}|n'\rangle|^2 \left\{ \frac{1}{\hbar\omega - \epsilon_{n'} + \epsilon_n + i\hbar\eta} - \frac{1}{\hbar\omega - \epsilon_n + \epsilon_{n'} + i\hbar\eta} \right\}, \tag{6}$$

with $Z$ the partition function,

$$\hat{n}_{\vec{q}} = \int d^3r\hat{n}(\vec{r})e^{-i\vec{q}\cdot\vec{r}} \tag{7}$$

the Fourier transform of the local density operator and the sum over $n, n'$ runs over all many-particle states.

## Exercise 6.2 Dielectric susceptibility of free electrons

For free electrons the field operators are given as

$$\hat{\psi}(\vec{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k},s} e^{i\vec{k}\cdot\vec{r}}\hat{c}_{\vec{k}s}, \qquad \hat{\psi}^\dagger(\vec{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k},s} e^{-i\vec{k}\cdot\vec{r}}\hat{c}^\dagger_{\vec{k}s}, \tag{8}$$

with $\hat{c}_{\vec{k}s}$ ($\hat{c}^\dagger_{\vec{k}s}$) annihilating (creating) an electron with momentum $\vec{k}$ and spin $s$.
Derive the Lindhard function,

$$\chi(\vec{q},\omega) = \frac{1}{\Omega}\sum_{\vec{k},s} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\omega - \epsilon_{\vec{k}} + \epsilon_{\vec{k}+\vec{q}} + i\hbar\eta}, \tag{9}$$

using linear response theory. As a "bonus" evaluate $\chi(\vec{q},\omega=0)$ for $T=0$.

[1] Since we are doing time-dependent perturbation theory, we have to use the interaction representation of the operators.