Sheet I

Due: 15/03/11

Question 1 [*Path integral of the harmonic oscillator*]: For the harmonic oscillator the Lagrangian is given by

$$L(q,\dot{q}) = \frac{m}{2}(\dot{q}^2 - \omega^2 q^2) \ .$$

(i) Determine the propagator kernel,

$$K(t,q,q_0) = \langle q | e^{-itH/\hbar} | q_0 \rangle$$

using path integral methods.

Hints:

- Introduce the $(n-1)$ dimensional vectors

$$\xi = (q_{n-1}, q_{n-2}, \ldots, q_1) \ , \qquad \eta = (q, \underbrace{0, \ldots, 0}_{n-3}, q_0) \ ,$$

  and rewrite the action as

$$S(\xi,\eta) = \frac{m}{2}\left[\frac{1}{\epsilon}(\eta,\eta) + \frac{1}{\epsilon}(\xi, C\,\xi) - \frac{2}{\epsilon}(\xi,\eta) - \epsilon\,\omega^2\,q_0^2\right] \ ,$$

  where C is a $(n-1)\times(n-1)$ matrix.

- Expand around the extremum (*i.e.* the classical path) and integrate over ξ, using the generalized Gaussian integration formula.

- To compute the determinant $\det(C)$ one has to solve a recurrence relation of the form

$$a_n = A a_{n-1} + B a_{n-2} \ .$$

  Use the ansatz $a_n = r^n$ to get the characteristic equation of the recurrence relation, and solve for r to obtain the two roots λ_1, λ_2. In our case the roots are distinct, so we have the general solution

$$a_n = C\lambda_1^n + D\lambda_2^n \ .$$

- Expand $\det(C)$ for small ϵ, *i.e.* show that

$$\det(C) = \frac{\sin\omega t}{\epsilon\,\omega} + \mathcal{O}(1) \ , \qquad t/n = \epsilon \ .$$

(ii) At time $t = 0$ the particle is described by the wave-function

$$\psi(q_0) = (A + Bq_0)e^{-m\omega q^2/2\hbar} \ .$$

(Note that this is a certain linear combination of the ground state and the first excited state wave-function.) Using the propagator kernel, calculate the wave-function at time t. Compare with what you expect based on the usual solution of the harmonic oscillator.

Question 2 [*Four point function in free Klein-Gordon theory*]: By evaluating the path-integral formula

$$\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\Big)\,|\Omega\rangle$$
$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,\phi(x_3)\,\phi(x_4)\,\exp\Big[i\int_{-T}^{T} d^4x\,\mathcal{L}(\phi)\Big]}{\int \mathcal{D}\phi\,\exp\Big[i\int_{-T}^{T} d^4x\,\mathcal{L}(\phi)\Big]}$$

determine the 4-point function in the free Klein-Gordon theory.

Hint: The calculation can be done as for the case of the two-point function (see the lecture). However, you have to keep track carefully of the various terms that contribute (and their combinatorial factors). The final result is

$$\begin{aligned}\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\Big)|\Omega\rangle &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &+D_F(x_1-x_3)D_F(x_2-x_4)\\ &+D_F(x_1-x_4)D_F(x_2-x_3)\ ,\end{aligned}$$

where $D_F(x_1 - x_2)$ is the Feynman propagator.