Sheet 4

Due: 05/04/11

Question 1 [*Feynman rules for QCD*]:

Consider the parts of the QCD Lagrangian that involve gauge fields only

$$\mathcal{L}_{QCD}^{(YM+GF)} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 \tag{1}$$

and decompose it into an interacting and a free part by using the definition of the field strength tensor given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu. \tag{2}$$

(i) Show that

$$\mathcal{L}_{free} = \mathcal{L}|_{g=0} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}) - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2, \tag{3}$$

$$\mathcal{L}_{int} = -\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{\mu,b} A^{\nu,c} - \frac{g^2}{4} f^{abe} f^{cde} A^a_\mu A^b_\nu A^{\mu,c} A^{\nu,d}. \tag{4}$$

(ii) Use $\mathcal{L}_{free}$ to derive the gluon propagator as

$$\langle\Omega|T A^a_\mu(x_1) A^b_\nu(x_2)|\Omega\rangle = \left(\frac{1}{i}\frac{\delta}{\delta J^{\mu,a}(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^{\nu,b}(x_2)}\right) Z_0^G[J]|_{J=0}, \tag{5}$$

with

$$Z_0^G[J] = \int \mathcal{D}A\, exp\left\{ i \int d^4x \left[\mathcal{L}_{free} + J^{\mu,a} A^a_\mu\right]\right\}. \tag{6}$$

Hints

- Write $\mathcal{L}_{free}$ as $A^a_\mu \mathcal{O}^{\mu\nu} A^a_\nu$, where $\mathcal{O}$ is a quadratic operator
- Use the following shift in A^a_μ

$$A^a_\mu(x) \longrightarrow A^a_\mu(x) + \int d^4y D^{ab}_{\mu\nu}(x-y) J^{\nu,b}(y) \tag{7}$$

to complete the square and rewrite the generating functional $Z_0^G[J]$ in eq.(6) as

$$Z_0^G[J] = exp\left\{\frac{i}{2}\int d^4x\, d^4y\; J^{\mu,a}(x) D^{ab}_{\mu\nu}(x-y) J^{\nu,b}(y)\right\}, \tag{8}$$

with

$$D^{ab}_{\mu\nu}(x) = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 + i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{9}$$

(iii) Using the purely interacting part of the QCD Lagrangian obtained in part (i) and using $Z_0^G[J]$ as in eq.(8), show that the 3-gluon vertex with all three gluon momenta k_1,k_2,k_3 incoming (i.e, $k_3 = -k_1 - k_2$), given by

$$V^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2) = -igf^{a_1a_2a_3}\left[g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + g_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + g_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\right], \tag{10}$$

can be obtained from the three point correlation function $G_{3,\,\mu_1\mu_2\mu_3}^{\;\;a_1a_2a_3}(x_1,x_2,x_3)$ for gluons to order g, which reads

$$\begin{aligned} G_{3,\,\mu_1\mu_2\mu_3}^{\;\;a_1a_2a_3}(x_1,x_2,x_3) &\equiv \langle\Omega|TA^{a_1}_{\mu_1}(x_1)A^{a_2}_{\mu_2}(x_2)A^{a_3}_{\mu_3}(x_3)|\Omega\rangle \\ &\equiv -(i)^2\frac{\delta^3}{\delta J_1\delta J_2\delta J_3}\int d^4x\,\mathcal{L}^{3G}_{int}\left(\frac{\delta}{i\delta J^{\mu,a}}\right)Z_0^G[J]|_{J=0}, \end{aligned} \tag{11}$$

where $J_i = J^{\mu_i,a}(x_i)$ and

$$\mathcal{L}^{3G}_{int}\left(\frac{\delta}{i\delta J^{\mu,a}}\right) = -\frac{g}{2}f^{abc}\left(\partial_\mu\frac{\delta}{i\delta J^{\nu,a}} - \partial_\nu\frac{\delta}{i\delta J^{\mu,a}}\right)\frac{\delta}{i\delta J^b_\mu}\frac{\delta}{i\delta J^c_\nu}. \tag{12}$$

(iv) Show that the 4-gluon vertex given by

$$\begin{aligned} W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4} &= -g^2\left[(f^{13,24}-f^{14,32})g_{\mu_1\mu_2}g_{\mu_3\mu_4} + (f^{12,34}-f^{14,23})g_{\mu_1\mu_3}g_{\mu_2\mu_4}\right. \\ &\quad \left.+(f^{13,42}-f^{12,43})g_{\mu_1\mu_4}g_{\mu_3\mu_2}\right], \end{aligned} \tag{13}$$

where $f^{ij,kl}$ denotes the following combination

$$f^{ij,kl} = f^{a_ia_ja}f^{a_ka_la}, \tag{14}$$

can be obtained from the 4-point correlation function $G_{4,\,\mu_1\mu_2\mu_3\mu_4}^{\;\;a_1a_2a_3a_4}(x_1,x_2,x_3,x_4)$ for gluons to order g^2, which reads

$$\begin{aligned} G_{4,\,\mu_1\mu_2\mu_3\mu_4}^{\;\;a_1a_2a_3a_4}(x_1,x_2,x_3,x_4) &\equiv \langle\Omega|TA^{a_1}_{\mu_1}(x_1)A^{a_2}_{\mu_2}(x_2)A^{a_3}_{\mu_3}(x_3)A^{a_4}_{\mu_4}(x_4)|\Omega\rangle \\ &\equiv -i\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}\int d^4x\,\mathcal{L}^{4G}_{int}\left(\frac{\delta}{i\delta J^{\mu,a}}\right)Z_0^G[J]|_{J=0}, \end{aligned} \tag{15}$$

where $J_i = J^{\mu_i,a}(x_i)$ and

$$\mathcal{L}^{4G}_{int}\left(\frac{\delta}{i\delta J^{\mu,a}}\right) = -\frac{g^2}{4}f^{abe}f^{cde}\frac{\delta^4}{\delta J^{\mu,a}\delta J^{\nu,b}\delta J^c_\mu\delta J^d_\nu}. \tag{16}$$