Quantum Field Theory II

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In case you still find mistakes, feel free to report them to Felix Haehl (haehlf@student.ethz.ch).

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Chapter 1

Path Integral Formalism

For the description of advanced topics in quantum field theory, in particular the quantization of non-abelian gauge theories, the formulation of quantum field theory in the path integral formulation is important. We begin by explaining the path integral formulation of quantum mechanics.

1.1 Path Integrals in Quantum Mechanics

In the Schrödinger picture the dynamics of quantum mechanics is described by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \ . \tag{1.1}$$

If the Hamilton operator is not explicitly time-dependent, the solution of this equation is simply

$$|\psi(t)\rangle = e^{-\frac{it}{\hbar}H}|\psi(0)\rangle \ . \tag{1.2}$$

Expanding the wave function in terms of position states, *i.e.* doing wave mechanics, we then have

$$\psi(t,q) \equiv \langle q|\psi(t)\rangle = \int dq_0 \, \langle q|e^{-itH/\hbar}|q_0\rangle \, \langle q_0|\psi(0)\rangle = \int dq_0 \, K(t,q,q_0)\, \psi(0,q_0) \ , \tag{1.3}$$

where we have used the completeness relation

$$\mathbf{1} = \int dq_0|q_0\rangle\,\langle q_0| \tag{1.4}$$

and introduced the propagator kernel

$$K(t,q,q_0) = \langle q|e^{-itH/\hbar}|q_0\rangle \ . \tag{1.5}$$

It describes the probability for a particle at q_0 at time $t=0$ to propagate to q at time t and will play an important role in the following.

By construction, the propagator satisfies the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}K(t,q,q_0) = HK(t,q,q_0) \ , \tag{1.6}$$

where H acts on q. It is furthermore characterised by the initial condition

$$\lim_{t\to 0} K(t,q,q_0) = \delta(q-q_0) \ . \tag{1.7}$$

For a free particle in one dimension with Hamilton operator

$$H_0 = \frac{1}{2m}p^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \tag{1.8}$$

the propagator, which is uniquely determined by (1.6) and (1.7), is

$$\boxed{K_0(t,q,q_0) = \langle q|e^{-itH_0/\hbar}|q_0\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} \exp\left(im\frac{(q-q_0)^2}{2\hbar t}\right) \ .} \tag{1.9}$$

To derive this formula, one can for example use a complete momentum basis; then

$$\begin{aligned}
\langle q|e^{-itH_0/\hbar}|q_0\rangle &= \frac{1}{2\pi\hbar}\int dp\, \langle q|p\rangle\, \langle p|e^{-itH_0/\hbar}|q_0\rangle \\
&= \frac{1}{2\pi\hbar}\int dp\, e^{iqp/\hbar}\, e^{-itp^2/2m\hbar}\, e^{-ipq_0/\hbar} \\
&= \frac{1}{2\pi\hbar}\exp\left(im\frac{(q-q_0)^2}{2\hbar t}\right)\int dp\, \exp\left[-\frac{it}{2m\hbar}\left(p-\frac{m(q-q_0)}{t}\right)^2\right] \ ,
\end{aligned}$$

which leads after Gaussian integration to (1.9). Here we have used that

$$\langle q|p\rangle = e^{\frac{i}{\hbar}qp} \ , \qquad \mathbf{1} = \frac{1}{2\pi\hbar}\int dp|p\rangle\langle p| \ . \tag{1.10}$$

The path integral is a method to calculate the propagator kernel for a general (non-free) quantum mechanical system. In order to derive it we need a small mathematical result.

1.1.1 Feynman-Kac Formula

The path integral formulation of quantum mechanics was first developed by Richard Feynman; the underlying mathematical technique had been previously developed by Marc Kac in the context of statistical physics.

The key formula underlying the whole formalism is the product formula of Trotter. In its simplest form (in which it was already proven by Lie) it states

$$e^{A+B} = \lim_{n\to\infty}\left(e^{A/n}\, e^{B/n}\right)^n \ , \tag{1.11}$$

where A and B are bounded operators on a Hilbert space. To prove it, we define

$$S_n = \exp\left[\frac{(A+B)}{n}\right] \ , \qquad T_n = \exp\left[\frac{A}{n}\right] \exp\left[\frac{B}{n}\right] \ . \tag{1.12}$$

Then we calculate

$$\begin{aligned} \|e^{A+B} - (e^{A/n}\, e^{B/n})^n\| &= \|S_n^n - T_n^n\| \\ &= \|S_n^{n-1}(S_n - T_n) + S_n^{n-2}(S_n - T_n)T_n + \cdots + (S_n - T_n)T_n^{n-1}\|. \end{aligned} \tag{1.13}$$

Since the norm of a product is always smaller or equal to the products of the norms, it follows (after using the triangle inequality $\|X + Y\| \leq \|X\| + \|Y\|$)

$$\|\exp(X)\| \leq \exp(\|X\|) \ . \tag{1.14}$$

Using the triangle inequality again it follows that

$$\|S_n\| \leq e^{(\|A\|+\|B\|)/n} \equiv a^{1/n} \ , \qquad \|T_n\| \leq e^{(\|A\|+\|B\|)/n} \equiv a^{1/n} \ . \tag{1.15}$$

Plugging into (1.13) leads, again after using the triangle inequality, to

$$\|S_n^n - T_n^n\| \leq n\, a^{(n-1)/n}\, \|S_n - T_n\| \ . \tag{1.16}$$

Finally, because of the Baker-Campbell-Hausdorff formula

$$S_n - T_n = -\frac{[A,B]}{2n^2} + \mathcal{O}(n^{-3}) \ , \tag{1.17}$$

and the product formula (1.11) follows.

If A and B are not bounded operators, the analysis is more difficult. If both A and B are self-adjoint (as is usually the case for the operators appearing in quantum mechanics), one can still prove that

$$e^{-it(A+B)} = \lim_{n\to\infty} \left(e^{-itA/n}\, e^{-itB/n}\right)^n \tag{1.18}$$

where the convergence is in the strong topology, *i.e.* the result holds when applied to any vector that lies in the domain of both A and B.

1.1.2 The Quantum Mechanical Path Integral

With these preparations we can now derive the path integral formulation of quantum mechanics. Let us assume that the Hamilton operator is of the form

$$H = H_0 + V(q) \qquad H_0 = \frac{p^2}{2m} \ , \tag{1.19}$$

where H_0 is the Hamilton operator of the free particle, and $V(q)$ is the potential. Applying the product formula (1.11) with $A = H_0/\hbar$ and $B = V/\hbar$ to (1.5) we obtain

$$\begin{aligned} K(t,q,q_0) &= \langle q|e^{-itH/\hbar}|q_0\rangle \\ &= \lim_{n\to\infty}\left\langle q\right|\left(e^{-itH_0/\hbar n}e^{-itV/\hbar n}\right)^n\left|q_0\right\rangle \\ &= \lim_{n\to\infty}\int dq_1\cdots dq_{n-1}\prod_{j=0}^{j=n-1}\left\langle q_{j+1}|e^{-itH_0/\hbar n}e^{-itV/\hbar n}|q_j\right\rangle \;, \end{aligned} \tag{1.20}$$

where $q \equiv q_n$, and we have, after each application of the exponential, introduced a partition of unity

$$\mathbf{1} = \int dq_j\; |q_j\rangle\langle q_j| \;. \tag{1.21}$$

Since the potential acts diagonally in the position representation, we now have

$$\left\langle q_{j+1}|e^{-itH_0/\hbar n}e^{-itV/\hbar n}|q_j\right\rangle = e^{-itV(q_j)/\hbar n}\left\langle q_{j+1}|e^{-itH_0/\hbar n}|q_j\right\rangle \;. \tag{1.22}$$

Thus we can use the propagator kernel of the free particle (1.9) to get, with $t/n = \epsilon$

$$\left\langle q_{j+1}|e^{-itH_0/\hbar n}e^{-itV/\hbar n}|q_j\right\rangle = \left(\frac{mn}{2\pi i\hbar t}\right)^{\frac{1}{2}}\exp\left[\frac{i\epsilon}{\hbar}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)\right] \;. \tag{1.23}$$

Hence we have for the complete propagator kernel the **Feynman-Kac formula**

$$K(t,q,q_0) = \lim_{n\to\infty}\int dq_1\cdots dq_{n-1}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{n}{2}}\exp\left[\frac{i\epsilon}{\hbar}\sum_{j=0}^{n-1}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)\right] \;. \tag{1.24}$$

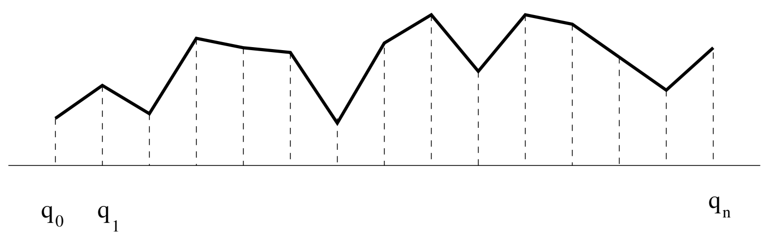


Figure 1.1: Interpretation as path integral.

1.1.3 The Interpretation as Path Integral

The interesting property of this formula is that it allows for an interpretation as a path integral. To understand this, we imagine that the points $q = q_0, q_1, \ldots, q_n$ are linked by straight lines, leading to piecewise linear functions (see fig. 1.1). We divide the time interval t into n subintervals of length $\epsilon = t/n$ each, and identify $q_k \equiv q(s = k\epsilon)$. The exponent of (1.24) can now be interpreted as the Riemann sum, which leads in the limit $\epsilon \to 0$ to the integral

$$\epsilon \sum_{j=0}^{n-1} \left(\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\epsilon} \right)^2 - V(q_j) \right) \sim \int_0^1 ds \left[\frac{m}{2} \left(\frac{dq}{ds} \right)^2 - V(q(s)) \right] . \tag{1.25}$$

This integral is now precisely the classical action of a particle (of mass m), moving along this path, since the integrand is just the Lagrange function.

$$L(q(s), \dot{q}(s)) = \frac{m}{2} \left(\frac{dq}{ds} \right)^2 - V(q(s)) , \tag{1.26}$$

whose action is

$$S[q(s)] = \int_{s_0}^{s_1} ds\, L(q(s), \dot{q}(s)) . \tag{1.27}$$

The multiple integrals $dq_1 \cdots dq_n$ imply that we are integrating over all possible (piecewise linear) paths, connecting q_0 and q. In the limit $n \to \infty$ the separate linear pieces become shorter and shorter, and we can approximate any continuous path from q_0 to q in this manner. The above formula thus sums over all possible paths beginning at time $t = 0$ at position q_0, and ending at time t at position q. The different paths are weighted by the phase factor

$$\exp \left[i \frac{S[q(s)]}{\hbar} \right] . \tag{1.28}$$

Formally, we may therefore write

$$\boxed{K(t, q, q_0) = C \int_{q(0)=q_0}^{q(t)=q} \mathcal{D}q \; e^{iS[q]/\hbar} ,} \tag{1.29}$$

where C is the formal expression

$$C = \lim_{n \to \infty} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{n}{2}} . \tag{1.30}$$

Here $C \cdot \mathcal{D}q$ corresponds to the limit of the integrals (1.24) for $n \to \infty$. As we will see, the divergent prefactor will cancel out of most calculations, and thus should not worry us too much. (However, mathematically, the definition of the path integral is somewhat subtle because of this.)

One of the nice features of the path integral formulation of quantum mechanics is that it gives a nice interpretation to the classical limit. The classical limit corresponds, at least formally, to $\hbar \to 0$. In this limit, the phase factor (1.28) of the integrand in the path integral formula (1.29) oscillates faster and faster. By the usual stationary phase method one therefore expects that only those paths contribute to the path integral whose exponents are stationary points. Since the exponent is just the classical action, the paths that contribute are hence characterised by the property to be critical points of the action. But because of the least action principle these are precisely the classical paths, *i.e.* the solutions of the Euler-Lagrange equations. In the classical limit, the path integral therefore localises on the classical solutions.

1.1.4 Amplitudes

The knowledge of the propagator kernel allows us to calculate other quantities of interest. In particular, in quantum mechanics we are usually interested in expectation values of operators, *i.e.* in quantities of the type

$$\langle\psi_f(t)|\,\mathcal{O}_1(\tau_1)\cdots\mathcal{O}_l(\tau_l)\,|\psi_i(0)\rangle\ , \tag{1.31}$$

where ψ_i and ψ_f are the initial and final state evaluated at $t=0$ and t, respectively, and $\mathcal{O}_i(\tau_i)$ is some operator that is evaluated at time $t=\tau_i$ with $0<\tau_l<\tau_{l-1}<\cdots<\tau_2<\tau_1<t$. Since we may expand any wavefunction in terms of position eigenstates, we can determine all such amplitudes (1.31), provided that we know the amplitudes

$$\langle q,t|\,\mathcal{O}_1(\tau_1)\cdots\mathcal{O}_l(\tau_l)\,|q_0,0\rangle\ . \tag{1.32}$$

Suppose now that $\mathcal{O}_i(\tau)$ can be expressed in terms of the position operator $\hat{q}(\tau)$, say $\mathcal{O}_i(\tau)=P_i(\hat{q}(\tau))$, where P_i is a polynomial. Then it follows immediately from the above derivation that (1.32) has the path-integral representation

$$\langle q,t|\,\mathcal{O}(\hat{q}(\tau_1))\cdots\mathcal{O}_l(\hat{q}(\tau_l))\,|q_0,0\rangle=\int_{q(0)=q_0}^{q(t)=q}\mathcal{D}q\,P_1(q(\tau_1))\cdots P_l(q(\tau_l))\,e^{iS[q]/\hbar}\ . \tag{1.33}$$

Indeed, we simply take l of the intermediate times to be equal to τ_i, $i=1,\ldots l$. At the corresponding intervals $P_i(q(\tau_i))$ acts as a multiplication operator, and we hence directly obtain (1.33).

A convenient compact way to describe these amplitudes is in terms of a suitable generating function. To this end, consider the modified path integral

$$I[J]=\int\mathcal{D}q\,\exp\Big[\frac{i}{\hbar}\int_0^t ds\Big(L(q,\dot{q},s)+J(s)q(s)\Big)\Big]\ , \tag{1.34}$$

where $J(s)$ is some arbitrary 'source' function. In order to obtain (1.33) from this we now only have to take functional derivatives with respect to $J(\tau_i)$, *i.e.*

$$\langle q,t|\,\mathcal{O}(\hat{q}(\tau_1))\cdots\mathcal{O}_l(\hat{q}(\tau_l))\,|q_0,0\rangle=P_1\left(\frac{\delta}{\delta J(\tau_1)}\right)\cdots P_l\left(\frac{\delta}{\delta J(\tau_l)}\right)I[J]\bigg|_{J=0}\ . \tag{1.35}$$

Here the functional derivative is defined by

$$\frac{\delta}{\delta J(\tau)}J(t)=\delta(\tau-t)\qquad\text{or}\qquad\frac{\delta}{\delta J(\tau)}\int dtJ(t)\phi(t)=\phi(\tau)\ , \tag{1.36}$$

which is the natural generalisation, to continuous functions, of the familiar

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij} \qquad \text{or} \qquad \frac{\partial}{\partial x_i} \sum_j x_j a_j = a_i \ . \tag{1.37}$$

Using these calculation rules it is then clear that (1.35) indeed reproduces (1.33). Often, introducing the generating function is not just a formal trick, but actually simplifies calculations since in many situations $I[J]$ is as difficult to compute as the original path integeral $I[0]$.

1.1.5 Generalisation to Arbitrary Hamiltonians

For the following we want to generalise the formula (1.29) to the case where the Hamiltonian is not necessarily of the form (1.19). We can still introduce a partition of unity, but now in each step we have to evaluate

$$\langle q_{j+1} | e^{-\frac{it}{\hbar n} H} | q_j \rangle \ , \tag{1.38}$$

where $H \equiv H(q, p)$ is a general function of q and p. We can always find a suitable ordering of the terms, the so-called *Weyl ordering* for which the q appears symmetrically on the left and right of p, so that

$$\langle q_{j+1} | H(q,p) | q_j \rangle = \int \frac{dp_j}{2\pi} H\left(\frac{q_{j+1} + q_j}{2}, p_j \right) e^{ip_j (q_{j+1} - q_j)/\hbar} \ . \tag{1.39}$$

Plugging this into (1.38) and using analogous arguments as above we find in the limit $n \to \infty$

$$\langle q_{j+1} | e^{-\frac{it}{\hbar n} H} | q_j \rangle = \int \frac{dp_j}{2\pi} e^{-\frac{it}{\hbar n} H\left(\frac{q_{j+1} + q_j}{2}, p_j \right)} e^{ip_j (q_{j+1} - q_j)/\hbar} \ . \tag{1.40}$$

Note that if H is of the form (1.19), $H = H_0 + V$, then we get

$$\int \frac{dp_j}{2\pi} e^{-\frac{it}{\hbar n} H_0(p_j)} e^{-\frac{it}{\hbar n} V\left(\frac{q_{j+1} + q_j}{2} \right)} e^{ip_j (q_{j+1} - q_j)/\hbar} = e^{-\frac{it}{\hbar n} V\left(\frac{q_{j+1} + q_j}{2} \right)} K_0(q_{j+1}, q_j) \ , \tag{1.41}$$

where K_0 is the free propagator kernel; this then agrees with (1.23). Using now (1.40) we obtain for the propagator kernel in the general case

$$K(t, q, q_0) = \lim_{n \to \infty} \int \prod_j \frac{dq_j dp_j}{2\pi} \exp\left[\frac{i}{\hbar} \sum_j p_j (q_{j+1} - q_j) - \frac{i\epsilon}{\hbar} H\left(\frac{q_{j+1} + q_j}{2}, p_j \right) \right] \ , \tag{1.42}$$

where $\epsilon = t/n$, as before. The exponent is now the Riemann sum of the integral

$$\int dt \left(p\dot{q} - H(q,p) \right) \ , \tag{1.43}$$

while the integration is over the full phase space. Formally, we can therefore write this as

$$K(t, q, q_0) = \int \mathcal{D}q \, \mathcal{D}p \, \exp\left[\frac{i}{\hbar} \int_0^t dt \, \left(p\dot{q} - H(q,p) \right) \right] \ . \tag{1.44}$$

1.2 Functional Quantization of Scalar Fields

Next we want to apply the functional integral formalism to the quantum theory of a scalar field. Our goal is to derive the Feynman rules for such a theory directly from functional integral expressions. From now we shall set $\hbar = 1$.

The general functional integral formula (1.44) holds for any quantum system, and thus we should also be able to apply it to a quantum field theory. To get a feeling for how this works, let us first consider the case of a scalar field theory. Here the analogue of the coordinates q_i are the field amplitudes $\phi(\mathbf{x})$, and the Hamiltonian is

$$H = \int d^3\mathbf{x}\,\left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right] . \tag{1.45}$$

Thus our formula becomes

$$\langle\phi_b(\mathbf{x})|e^{-iHt}|\phi_a(\mathbf{x})\rangle = \int \mathcal{D}\phi\,\mathcal{D}\pi\;\exp\Big[i\int_0^t d^4x\,\big(\pi\dot{\phi} - \tfrac{1}{2}\pi^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\big)\Big] , \tag{1.46}$$

where the functions over which we integrate are constrained to agree with $\phi_a(\mathbf{x})$ at $x^0 = 0$, and $\phi_b(\mathbf{x})$ at $x^0 = t$. Since the exponent is quadratic in π, we can complete the square and evaluate the $\mathcal{D}\pi$ integral, using

$$\int \mathcal{D}\pi\,\exp\Big[-\tfrac{i}{2}\int_0^t d^4x\,\big(\pi - \dot{\phi}\big)^2 + \tfrac{i}{2}\int_0^t d^4x\,\dot{\phi}^2\Big] = \exp\Big[i\int_0^t d^4x\,\tfrac{1}{2}\dot{\phi}^2\Big] . \tag{1.47}$$

(As always in the following, we shall ignore overall (field-independent) constants; as will become clear soon, they do not play any role in the calculation of physical quantities.) Then our formula becomes simply

$$\boxed{\langle\phi_b(\mathbf{x})|e^{-iHt}|\phi_a(\mathbf{x})\rangle = \int \mathcal{D}\phi\;\exp\Big[i\int_0^t d^4x\,\mathcal{L}(\phi,\pi)\Big] ,} \tag{1.48}$$

where $\mathcal{L}(\phi,\pi)$ is the Lagrange density

$$\mathcal{L}(\phi,\pi) = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) , \tag{1.49}$$

with $\partial_\mu\phi\,\partial^\mu\phi = \dot{\phi}^2 - (\nabla\phi)^2$

The time integral in the exponent goes from 0 to t, as determined by our choice of transition amplitude; in all other respects this formula is manifestly Lorentz invariant. Any other symmetries that the Lagrangian may have are also explicitly preserved by the functional integral.

1.2.1 Correlation Functions

Just as in the case of quantum mechanics, we can now also determine correlation functions which are also of primary importance in quantum field theory. Inspired by (1.33) let us consider the expression

$$\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, \exp\Big[i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)\Big]\ , \tag{1.50}$$

where the boundary conditions on the functional integral are $\phi(-T, \mathbf{x}) = \phi_a(\mathbf{x})$ and $\phi(T, \mathbf{x}) = \phi_b(\mathbf{x})$ for some given functions ϕ_a and ϕ_b. We would like to relate this quantity to the two-point correlation function of ϕ_1 and ϕ_2. Using essentially the same logic as before (but formulating it more formally now), we break up the functional integral as

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \int_{\substack{\phi(x_1^0,\mathbf{x}) = \phi_1(\mathbf{x}) \\ \phi(x_2^0,\mathbf{x}) = \phi_2(\mathbf{x})}} \mathcal{D}\phi\ . \tag{1.51}$$

The main functional integral $\int \mathcal{D}\phi$ is now constrained at times x_1^0 and x_2^0 (in addition to the endpoints $\pm T$), but we must integrate separately over the intermediate configurations $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$. After this decomposition, the extra factors $\phi(x_1)$ and $\phi(x_2)$ in (1.50) simply become $\phi_1(\mathbf{x}_1)$ and $\phi_2(\mathbf{x}_2)$, respectively, and can be taken outside the main integral. The main integral then factors into three propagating kernels, and we can write (1.50) as

$$\int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x})\, \phi_1(\mathbf{x}_1)\, \phi_2(\mathbf{x}_2) \\ \times \langle\phi_b|e^{-iH(T-x_2^0)}|\phi_2\rangle\, \langle\phi_2|e^{-iH(x_2^0-x_1^0)}|\phi_1\rangle\, \langle\phi_1|e^{-iH(x_1^0+T)}|\phi_a\rangle\ . \tag{1.52}$$

Using the completeness relation

$$\int \mathcal{D}\phi_1 |\phi_1\rangle\langle\phi_1| = \mathbf{1} \tag{1.53}$$

this can be simplified to

$$\langle\phi_b|e^{-iH(T-x_2^0)}\phi(\mathbf{x}_2)\, e^{-iH(x_2^0-x_1^0)}\, \phi(\mathbf{x}_1)e^{-iH(x_1^0+T)}|\phi_a\rangle\ , \tag{1.54}$$

where the operators $\phi(\mathbf{x}_1)$ and $\phi(\mathbf{x}_2)$ are time-independent, *i.e.* live in the Schrödinger picture, and we have assumed that $x_2^0 > x_1^0$ — otherwise the order of the operators $\phi(\mathbf{x}_1)$ and $\phi(\mathbf{x}_2)$ is reversed. The relation between Schrödinger and Heisenberg picture is

$$\phi_H(x) = e^{iHx^0}\phi(\mathbf{x})e^{-iHx^0}\ , \tag{1.55}$$

and thus (1.54) can be written as

$$\langle\phi_b|e^{-iHT}\mathcal{T}\Big(\phi_H(x_1)\, \phi_H(x_2)\Big)\, e^{-iHT}|\phi_a\rangle\ , \tag{1.56}$$

where $\mathcal{T}$ denotes the usual time ordering.

In quantum field theory one is usually interested in time-ordered vaccum correlation functions. In order to obtain this from the above, we want to take the limit $T \to \infty$, and replace ϕ_a and ϕ_b by the vacuum state Ω. Formally this can be done by taking the limit

$$T = s \cdot (1 - i\epsilon) \qquad s \to \infty \qquad - \quad \text{this will be abbreviated as } T \to \infty(1 - i\epsilon) \tag{1.57}$$

since a negative imaginary part of T implies that the exponential has the form

$$e^{-iHT} = e^{-iHs} e^{-sH} \,, \tag{1.58}$$

and hence projects in the limit $s \to \infty$ onto the state with smallest eigenvalue of H, namely the vacuum. In doing so we will obtain some awkward phases and overlap factors, but these cancel if we divide by the same quantitiy, but without the insertion of the two extra fields. Thus we obtain the simple formula

$$\boxed{\langle \Omega | \mathcal{T}\Big(\phi_H(x_1)\,\phi_H(x_2)\Big)\,|\Omega\rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, \exp\Big[i \int_{-T}^{T} d^4x\, \mathcal{L}(\phi)\Big]}{\int \mathcal{D}\phi\, \exp\Big[i \int_{-T}^{T} d^4x\, \mathcal{L}(\phi)\Big]}\,.} \tag{1.59}$$

This is our desired formula for the two-point correlation function in terms of functional integrals. Higher point functions are obtained similarly by inserting additional factors in the numerator. The other point worth stressing is that the final formula is indeed a ratio of path integrals, and hence does not depend on the precise overall normalisation of either of them. (This justifies why we can always be careless about overall normalisations.)

1.2.2 Feynman Rules

Our next aim is to show that the right-hand-side of (1.59) computes the same correlation functions as those that are obtained from the usual Feynman rules. We shall ignore in the following the infrared and ultraviolet divergences of the corresponding Feynman diagrams, but will only attempt to reproduce the same formal Feynman rules. In particular, this therefore shows that we do not introduce any new types of singularities in the functional integral formulation. First we discuss the free Klein-Gordon theory, before generalising our analysis to the ϕ^4 theory.

The action of the free Klein-Gordon theory is

$$S_0 = \int d^4x\, \mathcal{L}_0 = \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\right]. \tag{1.60}$$

Since $\mathcal{L}_0$ is quadratic in ϕ, the functional integrals take the form of generalised infinite-dimensional Gaussian integrals. We will therefore be able to do them exactly.

Since this is the first functional integral computation, we shall do it in a rather pedestrian manner — later on the relevant Gaussian integrals will be performed directly. In order to define the measure of the path integral we think of the theory as being defined on a (square) lattice with lattice spacing ϵ, taking $\epsilon \to 0$ in the end. We furthermore take the four-dimenisonal space-time to have volume L^4, where L is the size of each lattice direction. Up to an overall (irrelevant) factor, the path integral measure then equals

$$\mathcal{D}\phi = \prod_i d\phi(x_i) \ . \tag{1.61}$$

The field values $\phi(x_i)$ can be represented by a discrete Fourier series

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n x_i}\phi(k_n) \ , \tag{1.62}$$

where $k_n^\mu = 2\pi\frac{n^\mu}{L}$, with n^μ integer, $|k^\mu| < \pi/\epsilon$ and $V = L^4$. The separate Fourier coefficients are complex, but since $\phi(x)$ is real, we have the constraint $\phi^*(k) = \phi(-k)$. We will regard the real and imaginary parts of the $\phi(k_n)$ with $k_n^0 > 0$ as independent variables. The change of variables from the $\phi(x_i)$ to these new variables $\phi(k_n)$ is a unitary transformation, so we can rewrite the integrals as

$$\mathcal{D}\phi(x) = \prod_{k_n^0>0} d\,\mathrm{Re}\,\phi(k_n)\ d\,\mathrm{Im}\,\phi(k_n) \ . \tag{1.63}$$

Later we will take the limit $L \to \infty$, $\epsilon \to 0$. The effect of this limit is to convert discrete finite sums over k_n to continuous integrals over k

$$\frac{1}{V}\sum_n \to \int \frac{d^4k}{(2\pi)^4} \ . \tag{1.64}$$

With these preparations we can now compute the functional integral over ϕ. Rewriting the action (1.60) in terms of the Fourier modes we have

$$\begin{aligned} S_0 &= -\frac{1}{V}\sum_n \tfrac{1}{2}(m^2 - k_n^2)|\phi(k_n)|^2 \\ &= -\frac{1}{V}\sum_n \tfrac{1}{2}(m^2 - k_n^2)\left[(\mathrm{Re}\,\phi_n)^2 + (\mathrm{Im}\,\phi_n)^2\right] \ , \end{aligned} \tag{1.65}$$

where we have introduced the abbreviation $\phi_n \equiv \phi(k_n)$. The quantity $(m^2 - k_n^2) = (m^2 + |\mathbf{k}_n|^2 - (k_n^0)^2)$ is positive as long as k_n^0 is not too large. In the following we will only consider the case where $(m^2 - k_n^2) > 0$, *i.e.* k_n^0 is not too large; after doing the sum (or rather integral) we will then analytically continue our answer to arbitrary k_n^0.

Let us now do the path integral without any insertions of fields, *i.e.* the denominator of (1.59). This now takes the form of a product of Gaussian integrals since we can write

$$\begin{aligned}\int \mathcal{D}\phi\, e^{iS_0} &= \prod_{k_n^0>0}\int d\,\mathrm{Re}\,\phi_n\, d\,\mathrm{Im}\,\phi_n \;\; \exp\Big[-\frac{i}{V}\sum_{n|k_n^0>0}(m^2-k_n^2)|\phi_n|^2\Big] \\ &= \prod_{k_n^0>0}\int d\,\mathrm{Re}\,\phi_n\, \exp\Big[-\frac{i}{V}\sum_{n|k_n^0>0}(m^2-k_n^2)(\mathrm{Re}\,\phi_n)^2\Big] \\ &\qquad\times \prod_{k_n^0>0}\int d\,\mathrm{Im}\,\phi_n\, \exp\Big[-\frac{i}{V}\sum_{n|k_n^0>0}(m^2-k_n^2)(\mathrm{Im}\,\phi_n)^2\Big] \\ &= \prod_{k_n^0>0}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}\sqrt{\frac{-i\pi V}{m^2-k_n^2}} = \prod_{\mathrm{all}\,k_n}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}\;. \end{aligned} \tag{1.66}$$

The calculation of the Gaussian integrals in going to the last line is somewhat formal since the exponents are purely imaginary. However, in applying the formula to (1.59) we are interested in taking the time integral along a contour that is slightly rotated clockwise in the complex plane, $t \to t(1-i\epsilon)$. In terms of the Fourier modes this means that we should replace $k^0 \to k^0(1+i\epsilon)$ in all of these equations. Thus $(k^2-m^2) \to (k^2-m^2+i\epsilon)$, and the $i\epsilon$ term gives the necessary convergence factor for the Gaussian integrals.

To interpret the result of (1.66) let us consider as an analogy the general Gaussian integral

$$\prod_k \int d\xi_k \, \exp\big[-\xi_i B_{ij}\xi_j\big]\;, \tag{1.67}$$

where B_{ij} is a symmetric matrix with eigenvalues b_i. To evaluate this integral, we write $\xi_i = O_{ij}x_j$, where O_{ij} is the orthogonal matrix of eigenvectors that diagonalises B. Changing variables from ξ_i to the coefficients x_i we have

$$\begin{aligned}\prod_k \int d\xi_k\, \exp\big[-\xi_i B_{ij}\xi_j\big] &= \prod_k \int dx_k\, \exp\big[-\sum_i b_i x_i^2\big] \\ &= \prod_i \int dx_i\, \exp\big[-b_i x_i^2\big] = \prod_i \sqrt{\frac{\pi}{b_i}} = \mathrm{const}\times(\det B)^{-\frac{1}{2}}\;. \end{aligned} \tag{1.68}$$

We now want to argue that (1.66) is also of this form. To see this, we rewrite, using integration by parts

$$S_0 = \tfrac{1}{2}\int d^4x\, \phi(-\partial^2 - m^2)\phi \; + \text{surface terms}\;. \tag{1.69}$$

Thus our path integral in (1.66) is of the same form as (1.67) if we identify the operator B with

$$B = m^2 + \partial^2\;, \tag{1.70}$$

and thus formally write

$$\int \mathcal{D}\phi\, e^{iS_0} = \mathrm{const}\times\big[\det(m^2+\partial^2)\big]^{-\frac{1}{2}}\;. \tag{1.71}$$

This object is called a *functional determinant*. The actual result in (1.66) is quite ill-defined, but as we shall see, all the factors will cancel for the actual calculation in (1.59). There are, however, circumstances where also the functional determinant itself has a physical meaning.

Next we turn to the numerator of (1.59). The Fourier expansion of the two extra factors of ϕ equals

$$\phi(x_1)\,\phi(x_2) = \frac{1}{V}\sum_m e^{-ik_m\cdot x_1}\phi_m\,\frac{1}{V}\sum_l e^{-ik_l\cdot x_2}\phi_l\ . \tag{1.72}$$

Thus the numerator is

$$\frac{1}{V^2}\sum_{m,l} e^{-i(k_m\cdot x_1+k_l\cdot x_2)}\prod_{k_n^0>0}\int d\,\mathrm{Re}\,\phi_n\; d\,\mathrm{Im}\,\phi_n \tag{1.73}$$
$$\times\big(\mathrm{Re}\,\phi_m + i\mathrm{Im}\,\phi_m\big)\,\big(\mathrm{Re}\,\phi_l + i\mathrm{Im}\,\phi_l\big)\exp\Big[-\frac{i}{V}\sum_{n|k_n^0>0}(m^2-k_n^2)[(\mathrm{Re}\,\phi_n)^2+(\mathrm{Im}\,\phi_n)^2]\Big].$$

For most values of k_m and k_l this expression is zero since the extra factors of ϕ make the integrand odd; indeed it follows from the reality condition $\phi^*(-k)=\phi(k)$ that $\mathrm{Re}\,\phi_m$ is even, while $\mathrm{Im}\,\phi_m$ is odd. The situation is more complicated when $k_m=\pm k_l$. Suppose for example that $k_m^0>0$. Then if $k_l=+k_m$, the term involving $(\mathrm{Re}\,\phi_m)^2$ is non-zero, but is precisely cancelled by the term involving $(\mathrm{Im}\,\phi_m)^2$. If $k_l=-k_m$, however, we get an additional minus sign for the $(\mathrm{Im}\,\phi_m)^2$ term (since $\mathrm{Im}\,\phi_m$ is odd), and then the two terms add. The situation is identical for $k_m^0<0$, and thus we get altogether

$$(1.73) = \frac{1}{V^2}\sum_m e^{-ik_m\cdot(x_1-x_2)}\Big(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\Big)\frac{-iV}{m^2-k_m^2-i\epsilon}\ , \tag{1.74}$$

where we have used that

$$\begin{aligned}\int d\,\mathrm{Re}\,\phi_n\,(\mathrm{Re}\,\phi_n)^2\,\exp\big[-\frac{i}{V}(m^2-k_m^2)(\mathrm{Re}\,\phi_n)^2\big]&\\ = iV\frac{\partial}{\partial m^2}\int d\,\mathrm{Re}\,\phi_n\,\exp\big[-\frac{i}{V}(m^2-k_m^2)(\mathrm{Re}\,\phi_n)^2\big]&\\ = iV\frac{\partial}{\partial m^2}\sqrt{\frac{-i\pi V}{m^2-k_n^2-i\epsilon}} = \frac{1}{2}\sqrt{\frac{-i\pi V}{m^2-k_n^2-i\epsilon}}\,\frac{-iV}{m^2-k_n^2-i\epsilon}&\ .\end{aligned} \tag{1.75}$$

Now the factor in brackets in (1.74) is identical to the denominator, see (1.66), while the rest of the expression is the discretised form of the Feynman propagator. Indeed, taking the continuum limit (1.64) we get from (1.59)

$$\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\Big)|\Omega\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i\,e^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon} = D_F(x_1-x_2)\ . \tag{1.76}$$

This reproduces therefore exactly the correct Feynman propagator, including the $i\epsilon$ prescription.

In order to check that this reproduces the Feynman rules we now consider higher correlation functions. (We still consider just the free Klein-Gordon theory.) Inserting an extra factor of ϕ in the numerator of the path integral (1.73) we see that the three-point function vanishes since the integrand is now odd. All other odd correlation functions also vanish for the same reason.

The four point function, on the other hand, has four factors of ϕ in the numerator. Fourier-expanding the fields we obtain an expression similar to (1.73), but with a quadruple sum over indices that we will call m, l, p and q. The integrand contains the product

$$\big(\mathrm{Re}\,\phi_m + i\mathrm{Im}\,\phi_m\big)\,\big(\mathrm{Re}\,\phi_l + i\mathrm{Im}\,\phi_l\big)\,\big(\mathrm{Re}\,\phi_p + i\mathrm{Im}\,\phi_p\big)\,\big(\mathrm{Re}\,\phi_q + i\mathrm{Im}\,\phi_q\big)\ . \tag{1.77}$$

Again most terms vanish because the integrand is odd. One of the non-vanishing terms occurs when $k_l = -k_m$ and $k_q = -k_p$. After the Gaussian integrations this term of the numerator is then

$$\begin{aligned} &\frac{1}{V^4}\sum_{m,p} e^{-ik_m\cdot(x_1-x_2)}\,e^{-ik_p\cdot(x_1-x_2)}\,\Big(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\Big)\,\frac{-iV}{m^2-k_m^2-i\epsilon}\,\frac{-iV}{m^2-k_p^2-i\epsilon}\ ,\\ &\overset{V\to\infty}{\longrightarrow}\ \Big(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\Big)\,D_F(x_1-x_2)\,D_F(x_3-x_4)\ . \end{aligned} \tag{1.78}$$

Note that here we have here pretended that $m \neq p$ since otherwise we do not just get the square of (1.75) but rather

$$\int d\,\mathrm{Re}\,\phi_n\,(\mathrm{Re}\,\phi_n)^4\,\exp\big[-\frac{i}{V}(m^2-k_m^2)(\mathrm{Re}\,\phi_n)^2\big] = \frac{3}{4}\sqrt{\frac{-i\pi V}{m^2-k_n^2-i\epsilon}}\,\Big(\frac{-iV}{m^2-k_n^2-i\epsilon}\Big)^2\ . \tag{1.79}$$

The combinatorial factor of 3 by which this differs from the square of (1.75) is taken care of once we sum over the other ways of grouping the four momenta into pairs. Altogether we then get

$$\begin{aligned} \langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\Big)\,|\Omega\rangle \;=\;& D_F(x_1-x_2)\,D_F(x_3-x_4)\\ &+D_F(x_1-x_3)\,D_F(x_2-x_4)\\ &+D_F(x_1-x_4)\,D_F(x_2-x_3)\ . \end{aligned} \tag{1.80}$$

This agrees then exactly with the expression one obtains from applying Wick's theorem.

By the same methods we can also compute higher (even) correlation functions. In each case, the answer is just the sum of all possible contractions of the fields. The result is therefore identical to that obtained from Wick's theorem. This establishes that the correlation functions obtained from the path integral formulation agree (for the free Klein-Gordon theory) indeed with those obtained by applying the usual Feynman rules.

We are now ready to apply the same techniques to the ϕ^4 theory. For this we add to the Lagrangian of the free Klein-Gordon theory the ϕ^4 interaction

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4\ . \tag{1.81}$$

Assuming that λ is small, we can expand

$$\exp\Big[i\int d^4x\,\mathcal{L}\Big] = \exp\Big[i\int d^4x\,\mathcal{L}_0\Big]\Big(1 - i\int d^4x\,\frac{\lambda}{4!}\phi^4 + \cdots\Big)\ . \tag{1.82}$$

Making this substitution in both numerator and denominator of (1.59), we see that each term (aside from the constant factor (1.66) which again cancels between numerator and denominator) is expressed entirely in terms of free-field correlation functions. Furthermore, using that $i\int d^3x\,\mathcal{L}_{\rm int} = -iH_{\rm int}$, we can rewrite (1.59) as

$$\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\Big)\,|\Omega\rangle_{\phi^4} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,\exp\Big[i\int_{-T}^{T}dt\,H_{\rm int}(\phi)\Big]}{\int\mathcal{D}\phi\,\exp\Big[i\int_{-T}^{T}dt\,H_{\rm int}(\phi)\Big]}\ . \tag{1.83}$$

Since both numerator and denominator are just free field path integrals we can use the above results to replace them by the appropriate time-ordered correlation functions, *i.e.* we get

$$\boxed{\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\Big)\,|\Omega\rangle_{\phi^4} = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\exp\Big[i\int_{-T}^{T}dt\,H_{\rm int}(\phi)\Big]\Big)\,|\Omega\rangle_{\rm free}}{\langle\Omega|\mathcal{T}\Big(\exp\Big[i\int_{-T}^{T}dt\,H_{\rm int}(\phi)\Big]\Big)\,|\Omega\rangle_{\rm free}}\ .} \tag{1.84}$$

This then agrees precisely with the formula that was derived in QFT I.

1.2.3 Functional Derivatives and the Generating Functional

To conclude this section we shall now introduce a somewhat more elegant method to compute correlation functions in the path integral formulation. This will parallel our discussion for quantum mechanics from section 1.1.4.

First we generalise the functional derivative to functions of more variables, by defining

$$\frac{\delta}{\delta J(x)}J(y) = \delta^{(4)}(x-y) \qquad \text{or} \qquad \frac{\delta}{\delta J(x)}\int d^4y J(y)\phi(y) = \phi(x)\ . \tag{1.85}$$

Functional derivatives of more complicated functionals are defined by applying the usual product and chain rules of derivaties. So for example we have

$$\frac{\delta}{\delta J(x)}\,\exp\Big[i\int d^4y\,J(y)\phi(y)\Big] = i\phi(x)\,\exp\Big[i\int d^4y\,J(y)\phi(y)\Big]\ . \tag{1.86}$$

Furthermore, if the functional depends on the derivative of J, we integrate by parts before applying the functional derivative, *i.e.*

$$\frac{\delta}{\delta J(x)}\int d^4y\,\partial_\mu J(y)V^\mu(y) = -\partial_\mu V^\mu(x)\ . \tag{1.87}$$

As in section 1.1.4 we now introduce the *generating functional* of correlation functions $Z[J]$. (This is sometimes also called $W[J]$.) For the scalar field theories at hand, $Z[J]$ is defined as

$$Z[J] \equiv \int \mathcal{D}\phi \, \exp\Big[i \int d^4x \left(\mathcal{L} + J(x)\phi(x) \right) \Big] \ . \tag{1.88}$$

Note that this differs by the usual functional integral over $\mathcal{D}\phi$ by the source term $J(x)\phi(x)$ that has been added to the Lagrangian $\mathcal{L}$. Correlation functions can now be simply computed by taking functional derivatives of the generating functional. For example, the two point function is

$$\boxed{\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\Big)\,|\Omega\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\, Z[J]|_{J=0} \ ,} \tag{1.89}$$

where $Z_0 = Z[J=0]$. Here each functional derivative brings down a factor of ϕ in the numerator of $Z[J]$; setting $J = 0$ we then recover (1.59). To compute higher correlation functions, we simply take more functional derivatives.

The formula (1.89) is very useful because in a free field theoy $Z[J]$ can be rewritten in very explicit form. To see this, let us rewrite the exponent in the generating functional as

$$\int d^4x \left[\mathcal{L} + J\phi\right] = \int d^4x \left[\tfrac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi + J\phi\right] \ . \tag{1.90}$$

Here, the $i\epsilon$ term is the convergence factor for the functional integral we discussed above. We can complete the square by introducing a shifted field

$$\phi'(x) = \phi(x) - i\int d^4y D_F(x-y)J(y) \ . \tag{1.91}$$

Recall that D_F is a Green's function of the Klein Gordon operator, *i.e.*

$$(-\partial^2 - m^2 + i\epsilon)D_F(x-y) = i\,\delta^{(4)}(x-y) \ , \tag{1.92}$$

and hence that we can, more formally, write the change of variables as

$$\phi' = \phi + (-\partial^2 - m^2 + i\epsilon)^{-1}J \ . \tag{1.93}$$

Making this substitution we then get

$$\begin{aligned}
\int d^4x \left[\mathcal{L} + J\phi\right] &= \int d^4x \left[\tfrac{1}{2}\left(\phi' + i{\textstyle\int} D_F J\right)\left[-\partial^2 - m^2 + i\epsilon\right]\left(\phi' + i{\textstyle\int} D_F J\right) + J\phi\right] \\
&= \int d^4x \Big[\tfrac{1}{2}\phi'(-\partial^2 - m^2 + i\epsilon)\phi' - \phi' J \\
&\qquad - \tfrac{1}{2}{\textstyle\int} D_F J\,(-\partial^2 - m^2 + i\epsilon){\textstyle\int} D_F J + J\left(\phi' + i{\textstyle\int} D_F J\right)\Big] \\
&= \int d^4x \left[\tfrac{1}{2}\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] \\
&\qquad - \int d^4x\, d^4y\, \tfrac{1}{2}J(x)(-iD_F)(x-y)\,J(y) \ .
\end{aligned} \tag{1.94}$$

More formally, we can also write this as

$$\int d^4x \left[\mathcal{L} + J\phi\right] = \int d^4x \tfrac{1}{2}\Big[\phi'(-\partial^2 - m^2 + i\epsilon)\phi' - J(-\partial^2 - m^2 + i\epsilon)^{-1} J\Big] . \tag{1.95}$$

Now we change variables from ϕ to ϕ' in the functional integral (1.88) . This is just a shift, and hence the Jacobian of the transformation is 1. The result is therefore

$$Z[J] = \int \mathcal{D}\phi' \exp\Big[i \int d^4x\, \mathcal{L}_0(\phi')\Big] \exp\Big[-i \int d^4x\, d^4y\, \tfrac{1}{2} J(x)(-iD_F)(x-y)\, J(y)\Big] . \tag{1.96}$$

The second exponential factor is now independent of ϕ', while the remaining integral over ϕ' is precisely Z_0. Thus the generating function of the free Klein-Gordon theory is simply

$$\boxed{Z[J] = Z_0 \exp\Big[-\tfrac{1}{2} \int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)\Big] .} \tag{1.97}$$

Let us now use this to calculate the correlation functions, following (1.89). The two-point function is

$$\begin{aligned}
&\langle\Omega|\mathcal{T}\Big(\phi(x_1)\,\phi(x_2)\Big)\,|\Omega\rangle \\
&= -\frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)} \exp\Big[-\tfrac{1}{2}\int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)\Big]\bigg|_{J=0} \\
&= -\frac{\delta}{\delta J(x_1)}\Big[-\tfrac{1}{2}\int d^4y D_F(x_2-y)J(y) - \tfrac{1}{2}\int d^4x J(x) D_F(x-x_2)\Big]\frac{Z[J]}{Z_0}\bigg|_{J=0} \\
&= D_F(x_1-x_2) .
\end{aligned} \tag{1.98}$$

Note that in taking the second derivative only those terms survive where the functional derivative removes the J-factors outside the exponential since the other terms vanish upon setting $J = 0$. We therefore reproduce the correct formula.

It is instructive to work out the four-point function by this method as well. In order not to clutter the notation, let us introduce the abbreviations $\phi_1 \equiv \phi(x_1)$, $J_x = J(x)$, $D_{x4} = D_F(x - x_4)$, *etc.* Furthermore we shall use the convention that repeated subscripts will be integrated over. The four-point function is then

$$\begin{aligned}
&\langle\Omega|\mathcal{T}\Big(\phi_1\,\phi_2\,\phi_3\,\phi_4\Big)\,|\Omega\rangle \\
&= \frac{\delta}{\delta J_1}\,\frac{\delta}{\delta J_2}\,\frac{\delta}{\delta J_3}\,\big(-J_z D_{z4}\big)\exp[-\tfrac{1}{2}J_x D_{xy} J_y]\Big|_{J=0} \\
&= \frac{\delta}{\delta J_1}\,\frac{\delta}{\delta J_2}\,\big(-D_{34} + J_z D_{z4} J_u D_{u3}\big)\exp[-\tfrac{1}{2}J_x D_{xy} J_y]\Big|_{J=0} \\
&= \frac{\delta}{\delta J_1}\,\big(D_{34} J_z D_{z2} + D_{24} J_u D_{u3} + J_z D_{z4} D_{23}\big)\exp[-\tfrac{1}{2}J_x D_{xy} J_y]\Big|_{J=0} \\
&= \big(D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}\big) ,
\end{aligned} \tag{1.99}$$

in agreement with (1.80). The rules for differentiating the exponential give rise to the same familiar pattern: we get one term for each possible way of contracting the four points in pairs, with a factor of D_F for each contraction.

The generating functional method used above can also be used to represent the correlation functions of an interacting field theory. Indeed, formula (1.89) is equally true in an interacting theory. For an interacting theory, however, also the factor Z_0 is non-trivial. In fact, it is just given by the sum of vacuum diagrams. The combinatorical issues in the evaluation of the correlation functions is then exactly the same as in the Feynman diagrammatic approach.

1.3 Fermionic Path Integrals

For the application of the path integral methods to gauge theories we also need to be able to deal with fermionic fields. In order to do so we need to introduce a little bit of mathematical machinery, namely anti-commuting (or Grassmann) numbers. We will define them by giving algebraic rules for manipulating them. These rules are somewhat formal and may seem *ad hoc*; we will subsequently justify them by showing that they lead to the familiar quantum theory of the Dirac equation.

The basic property of anti-commuting numbers is — not surprisingly — that they anti-commute, *i.e.* if θ and η are anti-commuting numbers then

$$\theta\,\eta = -\eta\,\theta \ . \tag{1.100}$$

In particular, the square of a Grassmann number is zero

$$\theta\,\theta = 0 \ . \tag{1.101}$$

It is easy to see that a product of two Grassmann numbers $(\theta\,\eta)$ commutes with other Grassmann numbers. The Grassmann numbers form a complex vector space, *i.e.* we can add them and multiply them by complex numbers in the usual way; it is only among themselves that they anti-commute. It is convenient to define complex conjugation to reverse the order of the products, just like Hermitian conjugation

$$(\theta\,\eta)^* \equiv \eta^*\,\theta^* = -\theta^*\,\eta^* \ . \tag{1.102}$$

We will want to define some integral calculus for these anti-commuting numbers, *i.e.* we would like to define the expression

$$\int d\theta f(\theta) \ , \tag{1.103}$$

where $f(\theta)$ is a complex-valued function defined on the space of Grassmann numbers. We may expand the function $f(\theta)$ in a Taylor series as

$$f(\theta) = f(0) + \theta f'(0) \ , \tag{1.104}$$

and the series will terminate after the second term because of (1.101). Thus we have

$$\int d\theta\, f(\theta) = \int d\theta\big(f(0) + \theta f'(0)\big) \ . \tag{1.105}$$

The integral should be linear in f, *i.e.* it should be a linear combination of $f(0)$ and $f'(0)$. Furthermore, it should be invariant under shifting $\theta \mapsto \hat{\theta} = \theta + \eta$. Then we get

$$\int d\theta\, f(\theta) = \int d\theta\big(f(0) + \theta f'(0)\big) = \int d\hat{\theta}\, \Big(\big[f(0) - \eta f'(0)\big] + \hat{\theta} f'(0)\Big) \ . \tag{1.106}$$

Thus we conclude that

$$\int d\theta\, 1 = 0 \ , \qquad \int d\theta\, f(\theta) = \text{const} \times f'(0) \ . \tag{1.107}$$

We may fix the constant to be equal to one, *i.e.* we may normalise our integral so that

$$\int d\theta\, \theta = 1 \ . \tag{1.108}$$

Then $\int d\theta f(\theta) = f'(0)$, *i.e.* integration is effectively differentiation!

When we perform multiple integrals over more than one Grassmann variables a sign ambiguity arises; we shall adopt the convention that

$$\int d\theta \int d\eta\, \eta\, \theta = 1 \ , \tag{1.109}$$

i.e. the innermost integral is performed first, *etc.*

One of the key properties of these Grassmann integrals is their behaviour under changing variables. Suppose we want to integrate

$$\int d\theta_n \cdots d\theta_1 f(\theta_1, \ldots, \theta_n) \tag{1.110}$$

and we want to study the behaviour of the integral under a change of variables,

$$\theta_i = M_{ij} \eta_j \ , \tag{1.111}$$

where M_{ij} is a complex matrix. In order to determine the corresponding Jacobian, we write

$$\begin{aligned} 1 &= \int d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n \\ &= (\text{Jacobian}) \int d\eta_n \cdots d\eta_1\, M_{1j_1}\eta_{j_1} \cdots M_{nj_n}\eta_{j_n} \\ &= (\text{Jacobian})\ \det(M) \ , \end{aligned} \tag{1.112}$$

from which we conclude that

$$(\text{Jacobian}) = \det(M)^{-1} \ . \tag{1.113}$$

Note that this is precisely the inverse of what would have appeared for a usual commuting integral. This maybe surprising property is essentially a consequence of the fact that Grassmann integration is effectively differentiation, and hence behaves in the opposite way under coordinate transformations as normal integration.

For a complex Grassmann variable θ we can introduce real and imaginary part in the usual manner, *i.e.* we define

$$\theta_1 = \frac{1}{2}(\theta + \theta^*) \ , \qquad \theta_2 = \frac{1}{2i}(\theta - \theta^*) \ , \tag{1.114}$$

so that $\theta = \theta_1 + i\theta_2$. We can then treat θ_1 and θ_2 as independent variables, and hence define

$$\int d\theta_1 d\theta_2 \, \theta_2\theta_1 = 1 \ , \tag{1.115}$$

Written in terms of an integral over θ and θ^*, we then find

$$\int d\theta^* \, d\theta \, \theta \, \theta^* = 1 \ . \tag{1.116}$$

[In this case we have for the transformation matrix

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \ , \qquad \det(M) = \frac{i}{2} \ , \tag{1.117}$$

and hence the substitution formula becomes

$$1 = \frac{2}{i} \int d\theta \, d\theta^* \, \frac{1}{4i}(\theta - \theta^*)(\theta + \theta^*) = -\frac{1}{2} \int d\theta \, d\theta^* \left(\theta \, \theta^* - \theta^* \theta\right) \ , \tag{1.118}$$

which then leads to (1.116).]

In order to get a feeling for what these integrals are let us evaluate a Gaussian integral over a complex Gaussian variable

$$\int d\theta^* \, d\theta \, e^{-\theta^* b \theta} = \int d\theta^* \, d\theta (1 - \theta^* b \theta) = \int d\theta^* \, d\theta (1 + \theta \, \theta^* \, b) = b \ , \tag{1.119}$$

where b is a complex number. Note that unlike a usual (commuting) Gaussian integral the answer is proportional to b, rather than to $\frac{2\pi}{b}$. On the other hand, if we have an additional factor of $\theta\theta^*$ in the integrand, we get instead

$$\int d\theta^* \, d\theta \, \theta\theta^* \, e^{-\theta^* b \theta} = 1 = \frac{1}{b} \cdot b \ , \tag{1.120}$$

i.e. the extra $\theta\theta^*$ introduces a factor of $(1/b)$, just as in the bosonic case.

For the case of a general multidimensional Gaussian integral involving a Hermitian matrix B with eigenvalues b_i we then get

$$\int \prod_i d\theta_i^* \, d\theta_i \, e^{-\theta_i^* B_{ij} \theta_j} = \int \prod_i d\theta_i^* \, d\theta_i \, e^{-\theta_i^* b_i \theta_i} = \prod_i b_i = \det(B) \ , \tag{1.121}$$

where we have used that the Jacobian of the transformation putting the matrix into diagonal form, $(U^\dagger B U)_{ij} = \delta_{ij} b_i$ is $\det(U) \det(U)^* = 1$, since U is unitary. Similarly, one can show (**Exercise**) that

$$\int \prod_i d\theta_i^* d\theta_i \, \theta_k \theta_l^* e^{-\theta_i^* B_{ij} \theta_j} = (\det B) \, (B^{-1})_{kl} \ . \tag{1.122}$$

Inserting another pair $\theta_m \theta_n^*$ in the integrand would yield a second factor $(B^{-1})_{mn}$, as well as a second term in which the indices l and n are interchanged (the sum of all possible pairings). In general, except for the determinant appearing in the numerator rather than the denominator, Gaussian integrals over Grassmann variables behave exactly the same as in the usual commuting case.

1.3.1 The Dirac Propagator

A Grassmann *field* is a function of space-time whose values are anti-commuting numbers. More precisely, we can define a Grassmann field $\psi(x)$ in terms of a set of fixed Grassmann variables ψ_i,

$$\psi(x) = \sum_i \psi_i \phi_i(x) \ , \tag{1.123}$$

where the coefficient functions $\phi_i(x)$ are ordinary complex valued functions. For example, to describe the Dirac field, i will run from $i = 1, \ldots, 4$ and the ϕ_i can be identified with the four components of a spinor.

With these preparations we can now also formulate the correlation functions of fermions, in particular the Dirac fermion, in terms of a path integral. More specifically, we claim that the analogue of the generating functional (1.88) is

$$Z[\bar\eta, \eta] = \int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, \exp\Big[i \int d^4x \left(\bar\psi (i\partial\!\!\!/ - m)\psi + \bar\eta \, \psi + \bar\psi \eta \right) \Big] \ , \tag{1.124}$$

where η and $\bar\eta$ are Grassmann-valued source fields. As before we can complete the square by shifting

$$\psi \mapsto \hat\psi(x) = \psi(x) - i \int d^4y S_F(x-y) \, \eta(y) \ , \qquad \bar\psi \mapsto \hat{\bar\psi}(x) = \bar\psi(x) - i \int d^4y S_F(x-y) \, \bar\eta(y) \ , \tag{1.125}$$

where $S_F(x-y)$ is the Feynman propagator, *i.e.* the Green's function for

$$(i\partial\!\!\!/ - m) S_F(x-y) = i\delta^{(4)}(x-y) \ , \tag{1.126}$$

which is explicitly given by

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{\not{k} - m + i\epsilon} \; . \tag{1.127}$$

Since the Jacobian is trivial, we then get

$$\boxed{Z[\bar\eta,\eta] = Z_0 \cdot \exp\Big[i \int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\, \eta(y)\Big] \; ,} \tag{1.128}$$

where Z_0 is the value of the generating functional with vanishing sources, $\eta = \bar\eta = 0$. To obtain correlation functions, we will now differentiate $Z[\bar\eta,\eta]$ with respect to η and $\bar\eta$. First, however, we must adopt a sign convention for derivaties with respect to Grassmann numbers. If η and θ are anticommuting variables, we define

$$\frac{d}{d\eta}\theta\,\eta = -\frac{d}{d\eta}\eta\,\theta = -\theta \; . \tag{1.129}$$

Then the two-point function is given by

$$\begin{aligned} \langle\Omega|\mathcal{T}\Big(\psi(x_1)\,\bar\psi(x_2)\Big)\,|\Omega\rangle &= \frac{\int \mathcal{D}\bar\psi\mathcal{D}\psi\ \exp\big[i\int d^4x\, \bar\psi(i\not\partial - m)\psi\big]\ \psi(x_1)\,\bar\psi(x_2)}{\int \mathcal{D}\bar\psi\mathcal{D}\psi\ \exp\big[i\int d^4x\, \bar\psi(i\not\partial - m)\psi\big]} \\ &= Z_0^{-1}\left(-i\frac{\delta}{\delta\bar\eta(x_1)}\right)\left(+i\frac{\delta}{\delta\eta(x_2)}\right) Z[\bar\eta,\eta]\Big|_{\bar\eta=\eta=0} . \end{aligned} \tag{1.130}$$

Plugging in the explicit formula for (1.128) we then obtain

$$\langle\Omega|\mathcal{T}\Big(\psi(x_1)\,\bar\psi(x_2)\Big)\,|\Omega\rangle = S_F(x_1 - x_2) \; . \tag{1.131}$$

Alternatively, we can also do the ratio of path integrals directly. The denominator of the first line of (1.130) is formally equal to

$$\int \mathcal{D}\bar\psi\mathcal{D}\psi\ \exp\big[i\int d^4x\, \bar\psi(i\not\partial - m)\psi\big] = \det\big[-i(i\not\partial - m)\big] \; , \tag{1.132}$$

as follows from (1.121). On the other hand, according to (1.122), the numerator equals this same determinant times the inverse of the operator $-i(i\not\partial - m)$. Evaluating this inverse in Fourier space then leads directly to (1.127). Higher correlation functions of free Dirac fields can be evaluated in a similar manner. The answer is always just the sum of all possible full contractions of the operators, with a factor of S_F for each contraction. This then reproduces precisely the familiar result that one may obtain, for example, from Wick's theorem.

Chapter 2

Functional Quantization of Gauge Fields

The goal of this chapter is to apply the functional methods that we developed so far to gauge fields. We will derive the propagators of gauge fields, for example the photon field A^μ (*i.e.* QED) and the gauge fields $A^{\mu(a)}$ of non-Abelian gauge theories like QCD. This will lead to the Feynman rules for QED and QCD where we will see that the non-Abelian case is much more subtle. We begin this analysis by studying gauge invariance which we already know to be the main tool to make QED consistent (Ward identities etc.).

2.1 Non-Abelian Gauge Theories

The idea of gauge theories is to construct the Lagrangian of a theory by imposing symmetries that it should satisfy. In order to construct QED or QCD, we start from the free Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{2.1}$$

and require that certain gauge symmetries are fulfilled. The new idea is to take the gauge symmetry as the most fundamental ingredient of the theory and to take it as the starting point to determine the structure of the whole theory.

2.1.1 U(1) Gauge Invariance

Imposing a $U(1)$ gauge symmetry will lead to QED. The Lagrangian is said to be **invariant** under $U(1)$ if $\mathcal{L}$ does not change under the $U(1)$ gauge transformation

$$\Psi \to \Psi' = e^{i\alpha(x)}\Psi(x) \qquad \text{with } U(x) \equiv e^{i\alpha(x)} \in U(1) \tag{2.2}$$

where $U(1)$ denotes the 1×1 matrices U (i.e. complex numbers) that satisfy $UU^\dagger = 1$. This is a **local gauge transformation** because the gauge transformation parameter α of $U(x) \in U(1)$ is space-time dependent.

Note that the Lagrangian (2.1) is already invariant under a *global* $U(1)$-transformation. It is reasonable to impose that it should also be invariant under a *local* gauge transformation. These transformations correspond to multiplications with phase factors which have no observable effects. The term $m\bar{\Psi}\Psi$ is not a problem because it is obviously invariant also under local $U(1)$-transformations. A problem arises if we consider terms with derivatives:

$$\partial_\mu \Psi \longrightarrow \partial_\mu \Psi' = U(\partial_\mu \Psi) + (\partial_\mu U)\Psi. \tag{2.3}$$

We see that $\partial_\mu \Psi$ does not transform covariantly (i.e. like Ψ). The partial derivative $\partial_\mu \Psi$ is not even well defined. This is because the derivative of $\Psi(x)$ in the direction of a unit vector n^μ, given by

$$n^\mu \partial_\mu \Psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi(x + \varepsilon n) - \Psi(x) \right], \tag{2.4}$$

is not well defined itself. This expression contains the difference of two fields at different points in space-time which transform differently under the local gauge transformation. Thus the transformation behaviour of this object is not well defined.

We solve this problem by defining a scalar quantity $U(y, x)$ called **comparator** which compensates for the difference in phase transformations from one point to another. We impose that $U(y, x)$ transforms as

$$U(y, x) \longrightarrow e^{i\alpha(y)} U(y, x) e^{-i\alpha(x)} \qquad \text{with } U(y, y) = 1. \tag{2.5}$$

This implies that $U(y, x)\Psi(x)$ and $\Psi(y)$ have now the same transformation behaviour. We can now define a **covariant derivative** by

$$n^\mu D_\mu(\Psi) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi(x + \varepsilon n) - U(x + \varepsilon n, x)\Psi(x) \right] \tag{2.6}$$

which is well-defined because the two terms inside the brackets have the same transformation behaviour. Taking ε infinitesimal, we deduce that

$$\begin{aligned} U(x + \varepsilon n, x) &= \underbrace{U(x, x)}_{=1} + \varepsilon n^\mu \frac{\partial}{\partial y^\mu} U(y, x)\Big|_{y=x} + \mathcal{O}\left(\varepsilon^2\right) \\ &= 1 + ig\varepsilon n^\mu A_\mu(x) \end{aligned} \tag{2.7}$$

where g is a conventional constant and $A_\mu(x)$ a vector field. The covariant derivative of Ψ is

$$\boxed{D_\mu \Psi(x) = \partial_\mu \Psi(x) - igA_\mu \Psi(x)}. \tag{2.8}$$

Note the analogy to general relativity. We consider here a covariant derivative of a *field* Ψ. In general relativity, one considers covariant derivatives of vector fields:

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda. \tag{2.9}$$

So the gauge field A_μ takes the role which is taken by the Christoffel symbols in general relativity. We remember from differential geometry that the $\Gamma^\mu_{\lambda\nu}$ relate vector components at different points in the space-time manifold if the vectors are parallel transported. We can summarize this analogy as follows:

General Relativity		Gauge Theory
Coordinate transformations	$\leftrightarrow$	Gauge transformations
Connection $\Gamma^\mu_{\lambda\nu}$	$\leftrightarrow$	Gauge potential A^μ

If we replace the usual derivative ∂_μ by a covariant derivative D_μ in the Lagrangian (2.1), we want to get the following transformation rule for the derivative term:

$$\begin{aligned} &\bar{\Psi} \not{D} \Psi = \bar{\Psi}' \not{D}' \Psi' \\ \Leftrightarrow \quad & D'_\mu \Psi' = U(x) D_\mu \Psi. \end{aligned} \tag{2.10}$$

We thus have to require

$$D'_\mu \Psi' = (\partial_\mu - igA'_\mu) U(x)\Psi(x) \stackrel{!}{=} U(x) \left(\partial_\mu - igA_\mu\right) \Psi \tag{2.11}$$

This condition is satisfied if A_μ transforms as follows:

$$A'_\mu = U(x) A_\mu U(x)^{-1} - \frac{i}{g} U(x)^{-1} \left[\partial_\mu U(x)\right]. \tag{2.12}$$

Rewriting $U(x) = e^{i\alpha(x)}$ the $U(1)$-gauge transformation of the gauge field A^μ reads

$$\boxed{A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha(x)}. \tag{2.13}$$

We want to write down a consistent Lagrangian describing the interactions between the fermions (Ψ) and the gauge field (A^μ) for the photons. For this photon field to correspond to a physical field, we need a kinetic term for it. The building block that we have in order to construct such a kinetic term is essentially the covariant derivative. Because two covariant derivatives still transform covariantly, we can look at the commutator of covariant derivatives which is still a covariant object:

$$[D_\mu, D_\nu]\Psi(x) \longrightarrow U(x)[D_\mu, D_\nu]\Psi(x). \tag{2.14}$$

On the right-hand side $[D_\mu, D_\nu]$ appears as a multiplicative factor not acting on Ψ as a derivative. This is because the commutator of two covariant derivatives is in fact no longer a derivative:

$$\begin{aligned} [D_\mu, D_\nu]\Psi &= \left[\left(\partial_\mu - igA_\mu\right)\left(\partial_\nu - igA_\nu\right) - \left(\partial_\nu - igA_\nu\right)\left(\partial_\mu - igA_\mu\right)\right]\Psi(x) \\ &= -ig\left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right]\Psi(x) \\ &= -igF_{\mu\nu}\Psi(x) \end{aligned} \tag{2.15}$$

where $F_{\mu\nu}$ is the field strength tensor of the gauge field A^μ. It follows from these considerations that $F_{\mu\nu}$ is gauge invariant ($\rightarrow$ exercise). The kinetic term for A_μ reads $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ so that we finally have the following locally $U(1)$ gauge invariant QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.16}$$

We see immediately that the photon is massless because a mass term like $\frac{1}{2}m_\gamma A_\mu A^\mu$ would break gauge invariance. Note also that this Lagrangian is completely determined by the gauge invariance requirements: by demanding local $U(1)$ invariance of $\mathcal{L}$, we are forced to introduce a vector field A^μ (the photon field) which couples to the Dirac particle Ψ with charge $-g$. The transformation rule of this new vector field is then fixed by demanding the covariance of $D_\mu\Psi$. Local $U(1)$ invariance therefore completely dictates QED.

One could now easily derive the Feynman rules as in QFT I. One would obtain the usual factor of $-ig\gamma^\mu$ for the photon-fermion vertex. The gauge boson propagator would arise from $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. It reads $-\frac{ig^{\mu\nu}}{q^2}$. The fermion propagator is $\frac{i}{\not{p}-m}$ and it arises from $\bar{\Psi}(i\not{\partial} - m)\Psi$. We will soon be able to derive these results in the path integral formulation.

2.1.2 $SU(N)$ Gauge Invariance

$SU(N)$ Transformations

$SU(N)$ describes the non-Abelian group of all unitary transformations U in N dimensions which satisfy

$$UU^\dagger = 1 \qquad \text{and} \qquad \det U = 1. \tag{2.17}$$

Any $U \in SU(N)$ can be written as

$$U(x) = e^{i\alpha_a(x)T^a} \tag{2.18}$$

where $\alpha_a(x)$ are the group parameters (real functions) and T^a are called the generators of $SU(N)$.

These generators T^a can be written as $N \times N$ matrices which are hermitian ($(T^a)^\dagger = T^a$) and traceless ($\text{tr } T^a = 0$). They satisfy the Lie algebra commutation relations

$$[T^a, T^b] = if^{abc}T^c \tag{2.19}$$

with f^{abc} being the real, antisymmetric structure constants of $SU(N)$. We have

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0. \tag{2.20}$$

From this relation we also get

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0 \tag{2.21}$$

which is called the Jacobi identity.

The number of generators T^a, *i.e.* the dimension of the Lie algebra is

$$d = N^2 - 1 \tag{2.22}$$

as one can easily verify by considering the properties of hermitian, traceless $N \times N$ matrices: such a matrix has $2N^2$ real parameters of which N^2 are fixed by unitarity and one further parameter is fixed by the condition $\det U = 1$, such that $a = 1, ..., N^2 - 1$.

We have the following **representations of** $SU(N)$:

- **Fundamental Representation:**
  The fundamental representation is N-dimensional. It consists of $N \times N$ special unitary matrices T^a_{ij} acting on a space of complex vectors

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{pmatrix}. \tag{2.23}$$

- **Adjoint Representation:**
  The adjoint representation is $(N^2 - 1)$-dimensional. The generators are given by the structure constants:

$$T^b_{ac} = if^{abc} \tag{2.24}$$

  which are therefore $(N^2 - 1) \times (N^2 - 1)$ matrices. The Jacobi identity is trivially satisfied because

$$[T^b, T^c]_{ae} = if^{bcd} T^d_{ae}. \tag{2.25}$$

  Choice of basis: for the matrices T^a_{ij}, we consider (N^2-1) hermitian, traceless matrices which can be chosen such that

$$\mathrm{tr}(T^a T^b) = T_R \delta^{ab} \tag{2.26}$$

  where the constant T_R is a freely chosen normalization constant of the representation R. Here we consider $T_R = \frac{1}{2}$.

Local $SU(N)$ **Gauge Invariance of** $\mathcal{L}$

Consider an N-dimensional fermion multiplet

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{pmatrix}. \tag{2.27}$$

Note that in QCD, $N = 3$, so that Ψ is a triplet and $SU(3)$ describes the rotations in color space. As in the case of QED, we demand that $\mathcal{L}$ shall be invariant under a local $SU(N)$ transformation of Ψ characterized by

$$\Psi(x) \longrightarrow \Psi'(x) = V(x)\Psi(x) \equiv e^{i\alpha_a(x)T^a}\Psi(x). \tag{2.28}$$

The strategy is the same as in QED. We start by constructing a covariant derivative. We define a comparator $U(y, x)$ which is an $N \times N$ matrix transforming as

$$U(y, x) \longrightarrow V(y)U(y, x)V(x)^\dagger \qquad \text{with } U(y, y) = \mathbf{1} \tag{2.29}$$

such that $\Psi(y)$ and $U(y, x)\Psi(x)$ transform in the same way. The covariant derivative is again characterized by

$$n^\mu D_\mu \Psi = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left[\Psi(x + \varepsilon n) - U(x + \varepsilon n, x)\Psi(x)\right]. \tag{2.30}$$

We expand the comparator as a Taylor series in terms of the Hermitian operators near $U = \mathbf{1}$:

$$U(x + \varepsilon n, x) = 1 + ig\varepsilon n^\mu A^a_\mu T^a + \mathcal{O}(\varepsilon^2) \tag{2.31}$$

where $A^a_\mu T^a$ is a Lorentz vector field (the sum is over the generators of the gauge group, $a = 1, ..., N^2 - 1$). Therefore, we find

$$\boxed{D^{(SU(N))}_\mu = \partial_\mu - igA^a_\mu T^a.} \tag{2.32}$$

Requiring $SU(N)$ gauge invariance means that

$$\bar{\Psi}' \not{D}' \Psi' \stackrel{!}{=} \bar{\Psi} \not{D} \Psi. \tag{2.33}$$

So $D_\mu\Psi$ has to transform exactly like Ψ ("covariantly"),

$$D_\mu\Psi \longrightarrow D'_\mu\Psi' = e^{i\alpha^a(x)T^a} D_\mu\Psi. \tag{2.34}$$

Imposing the covariance of $D_\mu\Psi$ gives us a condition on how the gauge fields $A^{(a)}_\mu$ have to transform. We can either do this with the "full" transformation and see what the transformation $A^{(a)}_\mu \to A'^{(a)}_\mu$ has to look like. Or we derive the transformation of $A^{(a)}_\mu$ infinitesimally. We will follow the latter approach. In order to do so, we consider an infinitesimal transformation, given by

$$\begin{aligned} \Psi(x) &\longrightarrow \Psi'(x) = V(x)\Psi(x) \\ \text{with} \quad & V(x) = 1 + i\alpha^a(x)T^a. \end{aligned} \tag{2.35}$$

Under this transformation A_μ also transforms infinitesimally:

$$A_\mu \longrightarrow A'_\mu = A_\mu + \delta A_\mu. \tag{2.36}$$

The invariance condition reads

$$\begin{aligned} D'_\mu \Psi' &= \overbrace{\left(\partial_\mu - igT^c A'^c_\mu\right)}^{=D'_\mu} \overbrace{\left(1 + i\alpha^a T^a\right)\Psi}^{=V\Psi=\Psi'} \\ &\stackrel{!}{=} \underbrace{\left(1 + i\alpha^a T^a\right)}_{=V} \underbrace{\left(\partial_\mu - igT^a A^a_\mu\right)}_{=D_\mu} \Psi \end{aligned} \tag{2.37}$$

$$\begin{aligned} \Leftrightarrow \quad -igT^c \delta A^c_\mu + i(\partial_\mu \alpha^a)T^a - i^2 g T^c A'^c_\mu \alpha^a T^a &\stackrel{!}{=} -i^2 g \alpha^a T^a T^c A^c_\mu \\ \Leftrightarrow \quad T^c \delta A^c_\mu &= \frac{1}{g}(\partial_\mu \alpha^a) T^a + i[T^a, T^c]\alpha^a A^c_\mu \\ \Leftrightarrow \quad \delta A^a_\mu T^a &= \left[\frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu\right] T^a \end{aligned} \tag{2.38}$$

such that finally

$$\boxed{A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c} \tag{2.39}$$

the first two terms of which are analogous to QED and the last part corresponds to a term related to the non-Abelian nature of $SU(N)$.

Next, we need a kinetic term for A^a_μ (the analogue of $-F_{\mu\nu}F^{\mu\nu}$ in QED). To this end, observe that

$$[D_\mu, D_\nu]\Psi(x) \longrightarrow V(x)[D_\mu, D_\nu]\Psi(x). \tag{2.40}$$

Furthermore, we can write the commutator of two covariant derivatives as a field strength tensor (→ exercise):

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T^a \tag{2.41}$$

$$\text{with} \quad \boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}. \tag{2.42}$$

Note that (unlike in QED) $F^a_{\mu\nu}$ is not gauge invariant under the gauge transformation of A^a_μ. In fact, it transforms in the adjoint representation of $SU(N)$ (→ exercise):

$$F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} + \delta F^a_{\mu\nu} = F^a_{\mu\nu} - g f^{abc}\alpha^b F^c_{\mu\nu}. \tag{2.43}$$

Thus, in order to make $\mathcal{L}$ invariant, we do not use $F^a_{\mu\nu}$ directly but rather the trace, $\mathrm{tr}\left(F^{(a)}_{\mu\nu}F^{(a)\mu\nu}\right)$, which is indeed gauge invariant:

$$\begin{aligned} \delta\left(F^a_{\mu\nu}F^{\mu\nu a}\right) &= 2\left(\delta F^a_{\mu\nu}\right)F^{\mu\nu a} \\ &= -2g f^{abc}\alpha^b F^c_{\mu\nu}F^{\mu\nu a} \\ &= 0 \end{aligned} \tag{2.44}$$

where we used that f_{abc} is totally antisymmetric, whereas $F^a_{\mu\nu}F^{\mu\nu a}$ is symmetric under $a \leftrightarrow c$.

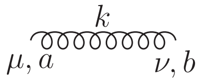
We now have all the ingredients to build a locally $SU(N)$ invariant Lagrangian. We just have to use covariant derivatives instead of usual derivatives and we have to make sure that $\mathcal{L}$ depends on gauge invariant terms like $F^a_{\mu\nu}F^{\mu\nu a}$. Of course, it has to be invariant also under global $SU(N)$ transformations.

For $N = 3$, the **classical QCD-Lagrangian** containing the **Yang-Mills part** reads

$$\boxed{\mathcal{L}^{\text{class.}}_{\text{QCD}} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}}_{\mathcal{L}_{\text{YM}}} + \underbrace{\bar{\Psi}(i\not{D} - m)\Psi}_{\mathcal{L}_F}} \tag{2.45}$$

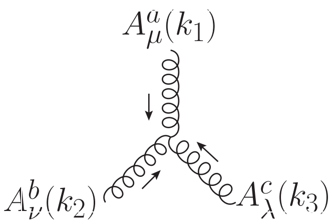
where $-\frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)}$ is a gauge-invariant kinetic term for $A^{(a)}_\mu$. It is called the *Yang-Mills term* and $\mathcal{L}^{\text{class.}}_{\text{QCD}}$ is called a *Yang-Mills* Lagrangian.

It follows a list of the propagators and vertices in $\mathcal{L}^{\text{class.}}_{\text{QCD}}$. The derivation of these using the functional approach will be sketched in section 2.3.1. Propagators for A^a_μ come from the term $(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})$:



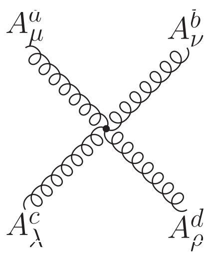
$$= -\frac{ig^{\mu\nu}}{k^2+i\varepsilon}\delta^{ab}$$

Three boson interaction terms arise from $(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(-gf^{abc}A^b_\mu A^c_\nu)$. The three gluon vertex reads ($\rightarrow$ exercise)



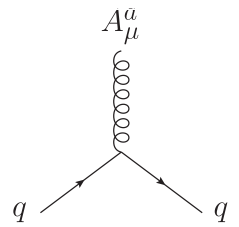
$$\begin{aligned} = gf^{abc}[&g^{\mu\nu}(k_1 - k_2)^\rho \\ &+g^{\nu\rho}(k_2 - k_3)^\mu \\ &+g^{\rho\mu}(k_3 - k_1)^\nu] \end{aligned}$$

The four gluon vertex reads



$$\begin{aligned} = -ig^2[&f^{abc}f^{cde}(g^{\mu\lambda}g^{\nu\rho} - g^{\mu\rho}g^{\nu\lambda}) \\ &+f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\rho}g^{\nu\lambda}) \\ &+f^{ade}f^{bce}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})] \end{aligned}$$

The second part of the Lagrangian, $\mathcal{L}_F$, gives rise to the well known propagator for fermions (coming from $\bar{\Psi}(i\partial\!\!\!/ - m)\Psi$). Furthermore we get a gauge boson-fermion interaction term which arises from $\bar{\Psi}_i g\gamma^\mu T^a_{ij}\Psi_j A^a_\mu$:



$= -ig\gamma^\mu T^a_{ij}$

Finally we make some remarks concerning the form of $\mathcal{L}^{\text{class.}}_{\text{QCD}}$. A priori the term $-\frac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\mu\nu}$ is not the only invariant term that one can add to the Lagrangian which can serve as kinetic term for $A^{(a)}_\mu$. We note that $\mathcal{L}^{\text{class.}}_{\text{QCD}}$ contains operators of mass dimension 4. Since

$$S = \int d^4x\, \mathcal{L} \tag{2.46}$$

has mass dimension 0, $\mathcal{L}$ has to have mass dimension 4 (d^4x has mass dimension -4). Other possible terms in $\mathcal{L}^{\text{class.}}_{\text{QCD}}$ could be

- of mass dimension 4: terms like $\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$. However, this term is not very useful because it violates P- and T-invariance and therefore also CPT.[1]
- of mass dimension higher than 4: Possible extra terms include $(F_{\mu\nu}F^{\mu\nu})^2$. However, to keep $\mathcal{L}$ of dimension 4, these terms have to be multiplied by couplings of negative mass dimensions. Such terms are forbidden by the requirement of renormalizability.

If we require CPT-invariance and renormalizability, then the kinetic term defined above ($-\frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)}$) is the only allowed term to be included in $\mathcal{L}^{\text{class.}}_{\text{QCD}}$.

2.1.3 Polarisation Vectors for the Gauge Fields

In this section we will outline a problem that arises due to the non-Abelian nature of the gauge fields of $SU(N)$-invariant theories. We start by deriving the equation of motion of A^μ, the free photon field of QED. In QFT I we derived the corresponding equation of motion starting from

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{2.47}$$

[1] The invariance of CPT is a well-established theorem. An experimental test is given by the measurement of K^0 and $\bar{K}^0$ masses. As a consequence of CPT the masses should be equal. Experiments show that $|m_{K^0} - m_{\bar{K}^0}| < 8 \times 10^{-9}$.

where the last term is the (Lorenz) gauge fixing term. The variation of $\mathcal{L}$ with respect to A^μ yields the equation of motion

$$\Box A^\mu = 0 \tag{2.48}$$

which has the usual plane wave solutions

$$A^\mu(k) = \varepsilon^\mu(k)e^{-ikx}. \tag{2.49}$$

We have seen that only two of the four components of ε^μ are physical. If we choose a particular representation for k^μ and ε^μ, these are given, for example, by

$$k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \quad \varepsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.50}$$

the *physical* components are $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$. Using n_μ given by $n_\mu = (k, 0, 0, k)$, such that $n \cdot k \neq 0$, $n \cdot \varepsilon^{(1,2)} = 0$, the "sum"[2] over all polarisation states is

$$\sum_{\lambda=0}^{3} \varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} \tag{2.51}$$

and the "sum" over non-physical (longitudinal and scalar) contributions ($\lambda = 0, 3$) is given by

$$\sum_{\lambda=0,3} \varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)} = \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k}. \tag{2.52}$$

In QED, an external photon couples always to a conserved current. Therefore in squared amplitudes like the one in fig. (2.1), one can use

$$M^\mu M^{*\nu} \sum_{\lambda=0,3} \varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)} = 0 \tag{2.53}$$

because after inserting (2.52), the $\lambda = 0, 3$ polarisations give a zero contribution due to the Ward identity (gauge invariance)

$$k_{1\mu} M^\mu = k_{2\nu} M^\nu = 0 \tag{2.54}$$

as seen in QFT I. In QED the unphysical polarisations therefore do not contribute to the process and in calculations one can use

$$\sum_{\substack{\text{phys.} \\ (\lambda=1,2)}} \varepsilon_\mu^{*(\lambda)}\varepsilon_\nu^{(\lambda)} = -g_{\mu\nu}. \tag{2.55}$$

[2]Note that (as seen in QFT I), the "sum" over polarisations λ is not really a sum. The time-like component is multiplied with a minus sign while the spatial components are multiplied with a plus sign implicitly.

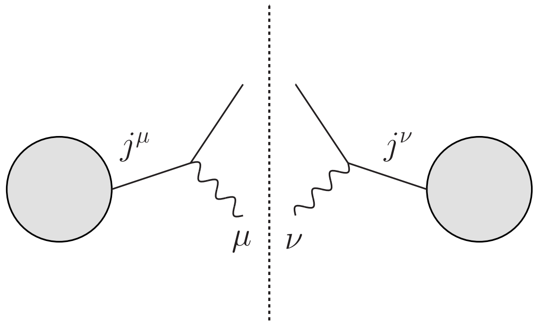


Figure 2.1: External photons couple to conserved currents in QED.

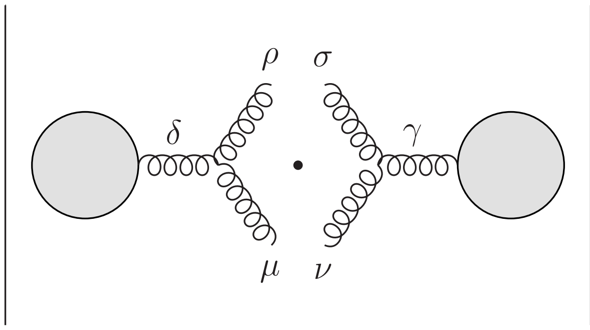


Figure 2.2: An amplitude which causes problems with QCD polarisation sums: the gluons couple to themselves in 3-gluon and 4-gluon vertices, so unphysical polarisations do not necessarily cancel a priori.

For non-Abelian Yang-Mills theories (like QCD) we also have $A_\mu^{(a)} = \varepsilon_\mu(k)e^{-ikx}$ with only two physical polarisations. However, due to the presence of 3-boson and 4-boson interactions we expect non-vanishing contributions from unphysical (scalar and longitudinal) polarisations ($\varepsilon^{(0)}$, $\varepsilon^{(3)}$) in processes with two external gluons.

As depicted in fig. 2.2, the bosons couple to themselves. We have

$$k_\mu V^{\mu\rho\delta}(k, ...) \neq 0 \tag{2.56}$$

so that using $\mathcal{L}_{\text{QCD}}^{\text{class.}}$, we have a priori a problem satisfying the Ward identities as given in Eq. (2.54) for the QED case.

2.2 Quantization of the QED Gauge Field A^μ

Before we turn to the quantization of non-Abelian gauge theories, we want to see how the procedure works in the simpler case of QED. We will first derive the photon propagator by means of finding the Green's function that is the inverse of the corresponding term in the Lagrangian. Afterwards we derive the propagator using functional techniques.

2.2.1 The Green's Function Approach

We start with the Lagrangian that contains a gauge fixing term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{2.57}$$

where ξ is a gauge parameter. The last (gauge fixing) term is added to $\mathcal{L}$ to remove physically equivalent field configurations. Gauge fields A_μ and $\widetilde{A}_\mu$ are equivalent if they differ by $\partial_\mu\alpha(x)$. In particular the fields $A_\mu = 0$ and $A'_\mu = \partial_\mu\alpha(x)$ are gauge equivalent and both lead to $\mathcal{L} = 0$. Fixing the gauge in this way is crucial because otherwise it would not be possible to find the desired propagator. We will make these statements more precise in the next section when we derive the propagator using functional integrals.

Using integration by parts and assuming that surface terms vanish, we have

$$\int d^4x\ \partial_\mu\phi\partial^\mu\phi = -\int d^4x\ \phi\Box\phi + \underbrace{\int d^4x\ \partial_\mu(\phi\partial_\mu\phi)}_{=0} \tag{2.58}$$

where $\Box = \partial_\mu\partial^\mu$. We can thus rewrite $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2}A^\mu\left[g_{\mu\nu}\Box + \left(\frac{1}{\xi} - 1\right)\partial_\mu\partial_\nu\right]A^\nu \tag{2.59}$$

corresponding to the Green's function in configuration space:

$$\left[g_{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]D_F^{\nu\lambda}(x-y) = \delta^{(4)}(x-y)\delta_\mu^\lambda. \tag{2.60}$$

In momentum space this relation reads

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right]D_F^{\nu\lambda}(k) = \delta_\mu^\lambda. \tag{2.61}$$

This equation can be inverted. Indeed, one easily verifies that

$$\boxed{iD_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).} \tag{2.62}$$

The choice $\xi = 1$ is called the **Feynman gauge**. The physics is unaffected by the choice of a gauge. In different contexts, a particular choice of gauge may be more convenient than an other.

2.2.2 Functional Method

The functional integral reads

$$\int \mathcal{D}A_\mu \; e^{iS[A_\mu]} = \int \mathcal{D}A_\mu \; e^{i\int d^4x\, \mathcal{L}} \tag{2.63}$$

and we have to determine $\mathcal{L}$ to render this integral finite. If we just use $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, we can draw inconsistent conclusions, because we have in this case

$$\int d^4x \, \mathcal{L} = \frac{1}{2}\int d^4x \; A_\mu(x) \left[g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right] A_\nu(x). \tag{2.64}$$

The associated Green's function should satisfy

$$[g_{\mu\nu}\Box - \partial_\mu\partial_\nu] iD^{\nu\lambda}(x-y) = \delta^{(4)}(x-y)\delta^\lambda_\mu. \tag{2.65}$$

Multiplication with ∂^μ yields

$$0 \cdot \partial_\nu iD_F^{\nu\lambda}(x-y) = \partial^\lambda \delta^{(4)}(x-y). \tag{2.66}$$

So we cannot find an inverse of $D_F^{\nu\lambda}(x-y)$, which is thus formally infinite. This is because $[g_{\mu\nu}\Box - \partial_\mu\partial_\nu]$ has no inverse:

$$[g_{\mu\nu}\Box - \partial_\mu\partial_\nu]\partial^\mu X = 0. \tag{2.67}$$

The underlying problem has to do with gauge invariance in the following sense. The integral $\int \mathcal{D}A_\mu$ integrates over all possible field configurations for A_μ including those which are equivalent (by gauge transformation). In order to perform the functional integral and to obtain a finite result, we need to isolate the physical (*i.e.* non-equivalent) field configurations and count them only once.

We will now introduce a gauge fixing method (**Faddeev-Popov**) which solves this problem. To this end, we consider a gauge fixing function $G(A^\mu)$ (for example $G(A^\mu) = \partial_\mu A^\mu$ for the Lorenz gauge). This function constrains the path integral to configurations which satisfy $G(A^\mu) = 0$. In order to include this constraint in the path integral, we need to introduce a δ-function that ensures the gauge condition. For the discretized path integral we would insert

$$1 = \int \left(\prod_i da_i\right) \; \delta^{(n)}(\mathbf{g}(\mathbf{a})) \det\left(\frac{\partial g_i}{\partial a_j}\right). \tag{2.68}$$

The continuum generalization reads

$$1 = \int \mathcal{D}\alpha(x) \; \delta\left(G(A^\alpha_\mu)\right) \det\left(\frac{\delta G(A^\alpha_\mu)}{\delta\alpha}\right) \tag{2.69}$$

where the determinant is a functional determinant and

$$A_\mu^\alpha(x) \equiv A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{2.70}$$

denotes a locally $U(1)$-gauge transformed field. In Lorenz gauge, the gauge fixing function reads

$$G(A_\mu^\alpha) = \partial_\mu A^\mu + \frac{1}{e}\partial^2 \alpha. \tag{2.71}$$

so that

$$\det\left(\frac{\delta G(A_\mu^\alpha)}{\delta\alpha}\right) = \frac{1}{e}\det\left(\partial^2\right) \tag{2.72}$$

which is independent of A^μ and independent of α and can thus be treated as a constant in the functional integral; it can be taken outside of this integral. The δ-function that we will introduce in the path integral ensures that only fields which satisfy $G(A_\mu^\alpha) = 0$ are integrated over and only the non-equivalent fields are considered. Inserting (2.69) in (2.63), we get

$$\begin{aligned}
\int \mathcal{D}A_\mu \, \exp\left[i\int d^4x \, \mathcal{L}\right] &= \det\left(\frac{\delta G(A_\mu^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha(x) \int \mathcal{D}A_\mu \, e^{iS[A_\mu]}\delta(G(A_\mu^\alpha)) \\
&= \det\left(\frac{\delta G(A_\mu^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha(x) \int \mathcal{D}A_\mu^\alpha \, e^{iS[A_\mu^\alpha]}\delta(G(A_\mu^\alpha)) \\
&= \det\left(\frac{\delta G(A_\mu^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha(x) \int \mathcal{D}A_\mu \, e^{iS[A_\mu]}\delta(G(A_\mu))
\end{aligned} \tag{2.73}$$

where we first shifted the field A_μ to A_μ^α ($S[A_\mu]$ is gauge invariant and thus $S[A_\mu] = S[A_\mu^\alpha]$) and dropped the dummy index α in the last step. We need to fix the function $G(A_\mu)$ and we want to do this by adding a scalar function to the Lorenz gauge condition in the following sense:

$$G(A_\mu) = \partial_\mu A^\mu(x) - w(x). \tag{2.74}$$

Since the determinant in (2.72) is independent of A_μ and α, we find for (2.73)

$$\int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \sim \det(\partial^2)\int \mathcal{D}\alpha(x) \int \mathcal{D}A_\mu \, e^{iS[A^\mu]}\delta\left(\partial_\mu A^\mu - w(x)\right) \tag{2.75}$$

which holds true for all functions $w(x)$ that generalize the Lorenz gauge condition. Therefore, the above relation also holds for linear combinations with different functions $w(x)$. We form an infinite linear combination by integrating the complete expression (2.75) over all $w(x)$ with a Gaussian damping factor which renders the w-integral finite:

$$w \longrightarrow \int \mathcal{D}w \, e^{-i\int d^4x \, \frac{w^2}{2\xi}} \tag{2.76}$$

where ξ is an arbitrary constant. Performing the w-integration using the δ-function $\delta(\partial_\mu A^\mu - w(x))$, we are left with the following integral:

$$\underbrace{N(\xi)\det\left(\frac{\delta G(A^\mu)}{\delta\alpha}\right)}_{=C(\xi)=\text{const.}}\int\mathcal{D}\alpha\int\mathcal{D}A_\mu\ e^{i\int d^4x\,\left[\mathcal{L}_0-\frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]}. \tag{2.77}$$

Ignoring all irrelevant (infinite) constants, as a net effect we have added a new term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to the Lagrangian that appears in the functional integral $\int\mathcal{D}A_\mu\ e^{iS[A_\mu]}$. The gauge fixed Lagrangian reads

$$\boxed{\mathcal{L}=\mathcal{L}_0-\frac{1}{2\xi}(\partial_\mu A^\mu)^2} \tag{2.78}$$

with $\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The Lagrangian (2.78) contains exactly the kind of gauge fixing term that we already know from the operator quantization method developed in section 2.2.1 and also seen in QFT I.

Using this gauge fixed Lagrangian, we can derive the Feynman rules using the generating functional

$$Z[J]=\int\mathcal{D}A_\mu\ \exp\left[i\int d^4x\ \mathcal{L}+J^\mu A_\mu\right] \tag{2.79}$$

where J^μ is the source term for the vector field A^μ. This enables us to calculate, for instance, correlation functions as derivatives of J^μ. For example, the two-point function reads

$$\langle\Omega|\mathcal{T}A_\mu(x_1)A_\mu(x_2)|\Omega\rangle=\frac{1}{Z_0}\left(\frac{\delta}{\delta J_\mu(x_1)}\frac{\delta}{\delta J_\mu(x_2)}\right)Z[J]\Big|_{J=0} \tag{2.80}$$

with $Z_0 = Z[J]|_{J=0}$. To evaluate (2.80), consider the following steps:

1. Write $\mathcal{L}$ as a quadratic operator in A^μ.

2. Write $Z[J]$ by completing the squares using a shift in A^μ:

$$A_\mu(x)\longrightarrow A_\mu(x)+\int d^4y\ D_{\mu\nu}(x-y)J^\nu(y). \tag{2.81}$$

This yields

$$Z[J]=\exp\left[\frac{i}{2}\int d^4xd^4y\ J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)\right]\cdot Z_0 \tag{2.82}$$

($\rightarrow$ exercise).

2.3 Quantization of the non-Abelian Gauge Field A^a_μ

We want to use similar methods as those in the previous section and apply them to non-Abelian gauge theories. In the pure gauge theory with only non-abelian gauge fields (A^a_μ), we have to make sense of the integral

$$\int \mathcal{D}A \, \exp\left[i \int d^4x \, \left(-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2\right)\right] \tag{2.83}$$

by restricting the generating functional path integral to regions of non-equivalent field configurations. We insert into this integral the identity

$$1 = \int \mathcal{D}\beta \, \delta\left(G(A^\beta_\mu)\right) \det\left(\frac{\delta G(A^\beta_\mu)}{\delta\beta}\right) \tag{2.84}$$

where A^β_μ is the gauge transformed field given for each a ($a = 1, ..., N^2 - 1$) by

$$\begin{aligned}(A^\beta_\mu)^a &= A^a_\mu + \frac{1}{g}\partial_\mu \beta^a + f^{abc} A^b_\mu \beta^c \\ &= A^a_\mu + \frac{1}{g}(D_\mu \beta)^a.\end{aligned} \tag{2.85}$$

The function $\beta^a(x)$ is analogous to the function $\alpha(x)$ in the QED-case but it carries an index a because we can choose a different function for every component A^a_μ of A_μ. The covariant derivative D_μ acts here on a field in the adjoint representation. We have to pay attention because we use two covariant derivatives in different representations:

$$D_\mu \Psi = (\partial_\mu - igA^a_\mu T^a)\Psi \qquad \text{(fundamental representation, c.f. Eq. (2.32))}, \tag{2.86}$$

$$(D_\mu \beta)^a = \partial_\mu \beta^a + g f^{abc} A^b_\mu \beta^c \qquad \text{(adjoint representation)}. \tag{2.87}$$

The expression for the adjoint covariant derivative can easily be found using $(T^a_{\text{Adj.}})^{bc} = i f^{bac}$.

We will use a gauge condition which is analogous to the Abelian case:

$$G(A^{(a)}_\mu) = \partial_\mu A^{\mu(a)} - w^a(x). \tag{2.88}$$

The essential difference to the Abelian case can be seen if we note that Eq. (2.85) implies

$$\frac{\delta G(A^\beta)}{\delta \beta} = \frac{1}{g}\partial_\mu D^\mu \tag{2.89}$$

where the right-hand side depends on A^μ. Therefore $\det\left(\frac{\delta G(A^\beta)}{\delta\beta}\right)$ cannot be pulled out of the functional integral and so it cannot be absorbed into the irrelevant constant. However, the determinant can be written as a functional integral over a set of anticommuting fields which live in the adjoint representation (cf. the discretized formula (1.112)):

$$\det\left(\frac{1}{g}\partial_\mu D^\mu\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\, \exp\left[i\int d^4x\, \bar{c}(-\partial_\mu D^\mu)c\right] \tag{2.90}$$

where we absorbed $\frac{1}{g}$ in the definition of c. The fields c and $\bar{c}$ are complex *fictitious* fields because on the one hand they obey the Grassmann algebra (a characteristic property of *fermions*), but on the other hand they are scalar fields (which is a property of *bosons*). This shows that they violate the spin statistics theorem and are thus unphysical. The fields c, $\bar{c}$ are called **Faddeev Popov ghost fields**.

In order to calculate the functional integral we perform the same trick as in the Abelian case: we insert a redundant 1 as defined in Eq. (2.84) and integrate over w^a with a Gaussian weighting function. This yields

$$\begin{aligned}\int \mathcal{D}A\, \exp\left[i\int d^4x\, \left(\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\right)\right] =& \\ = N(\xi)\int \mathcal{D}w^a\, \exp\left[-i\int d^4x\, \frac{(w^a)^2}{2\xi}\right]\int \mathcal{D}\beta\int \mathcal{D}A\, e^{i\int d^4x\,\mathcal{L}}\, \delta\left(\partial_\mu A^{\mu a} - w^a(x)\right)& \\ \times \underbrace{\det\left(\frac{1}{g}\partial_\mu D^\mu\right)}_{=\int \mathcal{D}c\mathcal{D}\bar{c}\ldots}&.\end{aligned} \tag{2.91}$$

Integrating over w^a using the δ-function (as in the case of QED), we find

$$(2.91) = N_\alpha \int \mathcal{D}A_\mu \mathcal{D}c\mathcal{D}\bar{c}\, \exp\left[i\int d^4x\left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \bar{c}(-\partial_\mu D^\mu)c\right)\right] \tag{2.92}$$

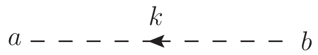
where the last two terms in the exponential are the gauge fixing and the ghost terms, respectively. Writing

$$-\partial_\mu D^\mu = -\partial^2 + g\partial_\mu f^{abc}A^{\mu b} \tag{2.93}$$

we find for the ghost term

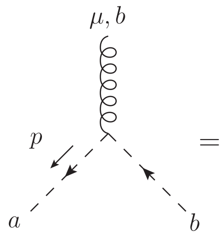
$$\begin{aligned}\int d^4x\, \mathcal{L}_{\text{ghost}} \equiv \int d^4x\, \bar{c}(-\partial_\mu D^\mu)c &= \int d^4x\, \bar{c}^a\left(-\partial^2\delta^{ac} + g\partial^\mu f^{abc}A^b_\mu\right)c^c \\ &= \int d^4x\, \bar{c}^a(-\partial^2\delta^{ac})c^c - g(\partial_\mu\bar{c}^a)f^{abc}A^{\mu b}c^c.\end{aligned} \tag{2.94}$$

We can now immediately deduce the Feynman rules for ghosts. The two-point interaction term $\bar{c}^a(-\partial^2\delta^{ac})c^c$ leads to the ghost propagator



$$= \frac{i\delta^{ab}}{k^2+i\varepsilon}$$

The ghost-ghost-gauge boson vertex comes from $-g(\partial_\mu \bar{c}^a) f^{abc} A^{\mu b} c^c$ and it reads



$$= -g f^{abc} p^\mu$$

The quantized Lagrangian (with only non-abelian gauge fields) is

$$\boxed{\mathcal{L}_{A_\mu^{(a)}} = \underbrace{-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}}_{\mathcal{L}_{\mathrm{YM}}} \underbrace{- \frac{1}{2\xi} (\partial_\mu A^\mu)^2}_{\mathcal{L}_{\text{gauge fix}}} + \underbrace{(\partial^\mu \bar{c}^a)\, D_\mu^{ac} c^c}_{\mathcal{L}_{\text{ghost}}} .} \tag{2.95}$$

We see that ghosts enable us to quantize the $A^{\mu(a)}$ fields such that the functional integral becomes finite.

For $N = 3$, the **quantized QCD Lagrangian** including also fermions reads

$$\boxed{\begin{aligned} \mathcal{L}_{\mathrm{QCD}} &= \mathcal{L}_{A_\mu^{(a)}} + \mathcal{L}_{\mathrm{F}} \\ \text{where} \quad \mathcal{L}_{\mathrm{F}} &= \bar{\Psi}^i \left(i\gamma^\mu D_\mu^{ij} - m\delta^{ij} \right) \Psi_j . \end{aligned}} \tag{2.96}$$

Note that, as we mentioned above, in $\mathcal{L}_{\mathrm{QCD}}$ there appear covariant derivatives living in different representations. The D_μ^{ij} in $\mathcal{L}_{\mathrm{F}}$ given in Eq. (2.45) lives in the fundamental representation of $SU(3)$ whereas the derivative in $\mathcal{L}_{\text{ghost}}$, lives in the adjoint representation.

We have now all ingredients to calculate Feynman diagrams in non-Abelian gauge theories, in particular in QCD. Ultimately the ghost field is just another field in the Lagrangian that can be accounted for in perturbation theory using the usual rules.

2.3.1 Feynman Rules for QCD

In order to derive the Feynman rules, we need to separate

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int.}} \equiv \mathcal{L}_{\mathrm{QCD}}\Big|_{g=0} + \mathcal{L}_{\text{int.}} . \tag{2.97}$$

For both $\mathcal{L}_{\text{free}}$ and $\mathcal{L}_{\mathrm{QCD}}$ we can define generating functionals $Z_0[J_{\phi_\alpha}]$ and $Z[J_{\phi_\alpha}]$ for all fields ϕ^α. From these one can then calculate n-point correlation functions as n-th derivatives of $Z[J_i]$ with respect to J_i. Propagators can then be obtained as second derivatives of $Z[J_{\phi_\alpha}]$ using $\mathcal{L}_{\text{free}}$. The vertices can be obtained using $\mathcal{L}_{\text{int.}}$ expanded in a perturbation series ($g \ll 1$).

Propagators

Formally, the full generating functional $Z[J_\alpha]$ with independent source terms J_α for all fields (fermions Ψ^i, anti-fermions $\bar{\Psi}^i$, ghosts η, anti-ghosts $\bar{\eta}$ and gauge fields $A_\mu^{(a)}$) is

$$Z[J^i_\Psi, J^i_{\bar{\Psi}}, J_\eta, J_{\bar{\eta}}, J_A] = \int \mathcal{D}\Psi^i \mathcal{D}\bar{\Psi}^i \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}A_\mu \; \exp\left[i \int d^4x \left(\mathcal{L}_{\text{QCD}} + J_A^{\mu a} A_\mu^a + J^i_{\bar{\Psi}} \Psi^i + \bar{\Psi}^i J^i_\Psi + J^a_{\bar{\eta}} \eta^a + \bar{\eta}^a J^a_\eta \right) \right] . \tag{2.98}$$

The free generating functional looks exactly the same but with $\mathcal{L}_{\text{free}}$ instead of $\mathcal{L}_{\text{QCD}}$ in the exponential. Note that in the exponential, $J_A^{\mu a}$ is the only commuting variable. All the source terms related to Ψ^i, $\bar{\Psi}^i$, η, $\bar{\eta}$ are Grassmann variables.

The free generating functional $Z_0[J^i_\Psi, J^i_{\bar{\Psi}}, J_\eta, J_{\bar{\eta}}, J_A]$ can be rewritten as a product of simpler generating functionals involving only one type of fields and only the corresponding part in $\mathcal{L}_{\text{free}}$:

$$Z_0[J^i_\Psi, J^i_{\bar{\Psi}}, J_\eta, J_{\bar{\eta}}, J_A] = Z_0[J_A] \cdot Z_0[J^i_\Psi, J^i_{\bar{\Psi}}] \cdot Z_0[J_\eta, J_{\bar{\eta}}] \tag{2.99}$$

where

$$Z_0[J_A] = \int \mathcal{D}A_\mu^{(a)} \; \exp\left[i \int d^4x \; \left(\mathcal{L}_0^{(G')} + J_A^{\mu a} A_\mu^a \right) \right] \tag{2.100}$$

$$\text{with } \mathcal{L}_0^{(G')} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gauge fix}},$$

$$Z_0[J^i_\Psi, J^i_{\bar{\Psi}}] = \int \mathcal{D}\Psi_i \mathcal{D}\bar{\Psi}_i \; \exp\left[i \int d^4x \; \left(\mathcal{L}_0^{(\Psi_i)} + J^i_{\bar{\Psi}} \Psi_i + \bar{\Psi}_i J^i_\Psi \right) \right] \tag{2.101}$$

$$\text{with } \mathcal{L}_0^{(\Psi_i)} = \mathcal{L}_F \text{ given in Eq. (2.96).}$$

$$Z_0[J_\eta, J_{\bar{\eta}}] = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} \; \exp\left[i \int d^4x \; \left(\mathcal{L}_0^{(\text{FP})} + J^a_{\bar{\eta}} \eta^a + \bar{\eta}^a J^a_\mu \right) \right] \tag{2.102}$$

$$\text{with } \mathcal{L}_0^{(\text{FP})} = \text{ free part of } \mathcal{L}^{(\text{FP})}. \tag{2.103}$$

[3] We need to get a suitable expression for $Z_0\left[J_{\phi_\alpha}\right]$ which allows for the above mentioned calculations. The complete calculations are rather lengthy and we will not do them here. Instead we outline the general procedure how one could derive propagators and vertices. Any propagator can be calculated by means of the following receipt:

1. For any field ϕ_α start by writing $\mathcal{L}_{\text{free}}(\phi_\alpha)$ in a form which is quadratic in the fields. For the gluon we obtain $\mathcal{L}_{\text{free}} \sim A_\mu^a \mathcal{O}^{\mu\nu} A_\nu^a$.

[3] $\mathcal{L}^{(\text{FP})}$ (FP: Faddeev Popov) is denoted by $\mathcal{L}_{\text{ghost}}$ in Eq. (2.95)

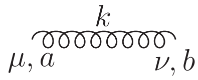
2. Then use a shift in the fields to complete the square and to obtain for each field a generating functional $Z_0\left[J_{\phi_\alpha}\right]$ of the form

$$Z_0\left[J_{\phi_\alpha}\right] = \exp\left[\frac{i}{2}\int d^4x d^4y\ \left(J_{\phi_\alpha}(x) D^{\phi_\alpha}(x-y) J_{\phi_\alpha}(y)\right)\right] \tag{2.104}$$

where $D^{\phi_\alpha}(x-y)$ satisfies the defining equation for Green's functions. One can show ($\rightarrow$ exercise) that for the gluon it assumes the form (in configuration space)

$$D^{ab}_{\mu\nu}(x) = \delta^{ab}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{2.105}$$

The gluon propagator thus reads (in momentum space)



$$= -\frac{i}{k^2+i\varepsilon}\delta^{ab}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{2.106}$$

Vertices

Having outlined the general procedure to find propagators, we now turn to the task of calculating vertex rules. The vertices can be obtained in a similar way using the generating functional for the full theory including interacting fields. It is given by the following formula which we will prove at the end of this section:

$$Z[J_{A_\mu}, J_\Psi, J_{\bar\Psi}, J_\eta, J_{\bar\eta}] = \\ = \exp\left[i\int d^4x\ \mathcal{L}_{\text{int.}}\left(\frac{\delta}{i\delta J_{A_\mu}}, \frac{\delta}{i\delta J_{\Psi_i}}, \frac{\delta}{i\delta J_{\bar\Psi_j}}, \frac{\delta}{i\delta J_{\eta_k}}, \frac{\delta}{i\delta J_{\bar\eta_l}}\right)\right]\cdot Z_0[J_{A_\mu}]\cdot Z_0[J_\Psi, J_{\bar\Psi}]\cdot Z_0[J_\eta, J_{\bar\eta}]. \tag{2.107}$$

To generate the perturbation series we expand the exponential. To obtain the vertices, we furthermore expand $\mathcal{L}_{\text{int.}}$ in a power series in the coupling ($g \ll 1$). For example, the first order term of $\mathcal{L}^{(A^{(a)}_\mu)}_{\text{int.}}$ reads

$$\mathcal{L}^{(3G)}_{\text{int.}}(A^a_\mu) = \frac{-g}{2} f^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{b\mu} A^{c\nu} \tag{2.108}$$

which can be obtained by expanding $F^a_{\mu\nu}F^{\mu\nu a}$. Using (2.108) and replacing the fields by the functional derivatives as in (2.107) gives rise to the three-gluon vertex which reads ($\rightarrow$ exercise)

$$\langle\Omega|\mathcal{T}\left(A^{a_1}_{\mu_1}(x_1)A^{a_2}_{\mu_2}(x_2)A^{a_3}_{\mu_3}(x_3)\right)|\Omega\rangle = (-i)^2\frac{\delta^3}{\delta J_1\delta J_2\delta J_3}\int d^4x\ \mathcal{L}^{(3G)}_{\text{int.}}\left(\frac{\delta}{i\delta J^a_{A_\mu}}\right) Z_0[J]\Big|_{J=0} \tag{2.109}$$

where $J_i = J^{a_i}_{\mu_i}(x_i)$. All Feynman rules for tree level diagrams can be derived by suitably using correlation functions taking the derivatives with respect to the source terms of the field involved as described above.

Additional rules for loops include:

- Sign factors for loops of quarks and ghosts (related to their anticommuting nature),
- Symmetry factors for identical field states.

Proof of the Formula (2.107) for the Generating Functional with Interacting Fields

First we note that S_0 is linear in the fields ϕ_α,

$$S_0[J_{\phi_\alpha}] = \int d^4x \left(\mathcal{L}_0 + \sum_\alpha \phi_\alpha J_{\phi_\alpha} \right), \tag{2.110}$$

and that the action for the interacting fields, $S_{\text{int.}}$ is a polynomial in the fields ϕ_α,

$$S_{\text{int.}} = \int d^4x \; g[\phi(x)]^\alpha. \tag{2.111}$$

For simplicity, we take only one field. The generalization to more fields is straightforward. So the formula that we want to prove for any field ϕ and its associated source term J_ϕ is

$$\boxed{Z[J_\phi] = e^{i\int d^4x\, \mathcal{L}_{\text{int.}}\left(\frac{\delta}{i\delta J_\phi}\right)} Z_0[J_\phi].} \tag{2.112}$$

To be precise, with one field ϕ we have

$$Z_0[J_\phi] = \int \mathcal{D}\phi \; \exp\left[i \int d^4x \, (\mathcal{L}_0 + J\phi) \right] = \int \mathcal{D}\phi \; e^{iS_0[\phi]} \tag{2.113}$$

$$Z[J_\phi] = \int \mathcal{D}\phi \; \exp\left[i \int d^4x \, (\mathcal{L}_0 + \mathcal{L}_{\text{int.}} + J\phi) \right] = \int \mathcal{D}\phi \; e^{i(S_0[\phi]+S_{\text{int.}}[\phi])}. \tag{2.114}$$

We get (denoting $Z_0[J_\alpha] = Z_0[J]$ and $S_0[\phi] = S_0$)

$$\frac{\delta}{i\delta J(x)} Z_0[J] = \frac{1}{i} \int \mathcal{D}\phi \left(\frac{\delta}{\delta J(x)} e^{iS_0} \right) = \int \mathcal{D}\phi \; \phi(x) e^{iS_0} = \phi(x) \int \mathcal{D}\phi \; e^{iS_0} \equiv \phi(x) Z_0[J] \tag{2.115}$$

remembering the property seen in Eq. (1.33). We can repeat this process forming infinite series as follows:

$$\sum_n \frac{1}{n!}\left[i\int d^4x\, g\left(\frac{\delta}{i\delta J(x)}\right)^{\alpha}\right]^n Z_0[J] = \int \mathcal{D}\phi \left(\sum_n \frac{1}{n!}\left[i\int d^4x\, g\phi^{\alpha}(x)\right]^n e^{iS_0}\right) \quad (2.116)$$

$$\Leftrightarrow \quad \sum_n \frac{1}{n!}\left[iS_{\text{int.}}\left(\frac{\delta}{i\delta J(x)}\right)\right]^n Z_0[J] = \int \mathcal{D}\phi\; e^{iS_{\text{int}}[\phi]}e^{iS_0} \quad (2.117)$$

$$\Leftrightarrow \quad e^{iS_{\text{int}}\left(\frac{\delta}{i\delta J(x)}\right)} Z_0[J] = \int \mathcal{D}\phi\; e^{i[S_{\text{int}}+S_0](\phi)} = Z[J] \quad (2.118)$$

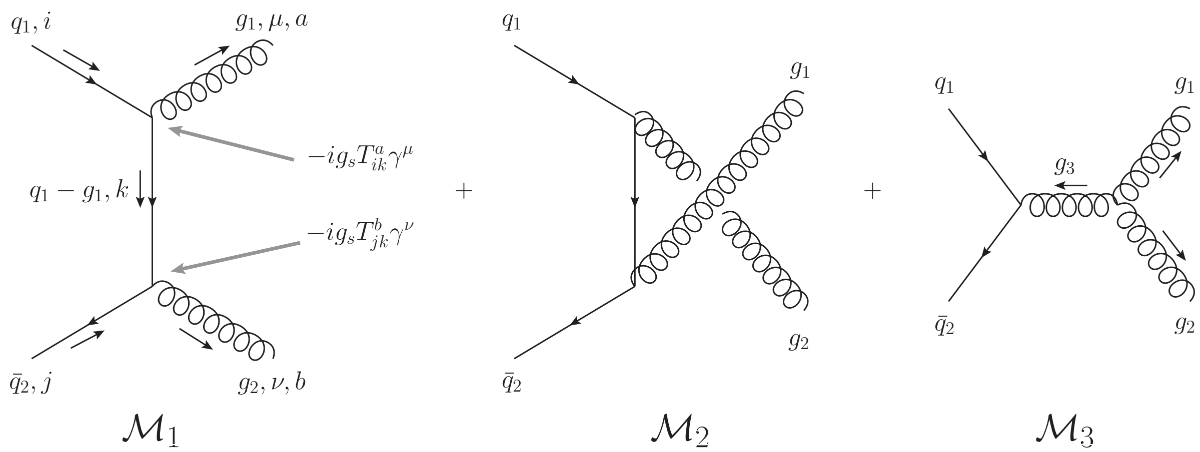
$$\Leftrightarrow \quad Z[J] = e^{i\int d^4x \mathcal{L}_{\text{int}}\left(\frac{\delta}{i\delta J(x)}\right)} Z_0[J] \quad (2.119)$$

which proves formula (2.112). From this, one can easily obtain the generating functional for the full theory (QCD) by generalizing (2.112) to more fields with a source term for each of them.

2.4 Ghosts and Gauge Invariance

2.4.1 QCD Ward Identity

Let us consider, for example, the process $q\bar{q} \to gg$ characterized by the following tree level graphs:



By analogy to the process $q\bar{q} \to \gamma\gamma$ (derived as exercise) the first two amplitudes add as follows:

$$i[\mathcal{M}_1+\mathcal{M}_2]\left(\varepsilon_1^*, \varepsilon_2^*\right) = -ig_s^2\bar{v}(\bar{q}_2)\left[T^b_{jk}T^a_{ki}\not{\varepsilon}_2^*\frac{1}{\not{q}_1 - \not{g}_1}\not{\varepsilon}_1^* + T^a_{jk}T^b_{ki}\not{\varepsilon}_1^*\frac{1}{\not{q}_1 - \not{g}_2}\not{\varepsilon}_2^*\right]u(q_1), \quad (2.120)$$

We can test gauge invariance (or the naïve Ward identity) by replacing $(\varepsilon_i)_\mu \to (g_i)_\mu$ and checking whether the amplitude vanishes. We will now do the case $(\varepsilon_2)_\mu \to (g_2)_\mu$.

Replacing the first denominator in Eq. (2.120) by $\not{g}_2 - \not{\bar{q}}_2$ and using the well known trick of adding momenta (which vanish by means of Dirac's equation) in order to cancel the denominators, one finds

$$\begin{aligned} i[\mathcal{M}_1 + \mathcal{M}_2]^{\mu\nu} \varepsilon^*_{1\mu} g_{2\nu} &= i g_s^2 [T^a, T^b] \bar{v}(\bar{q}_2) \not{\varepsilon}^*_1 u(q_1) \\ &= -g_s^2 f^{abc} T^c \bar{v}(\bar{q}_2) \not{\varepsilon}^*_1 u(q_1) \end{aligned} \tag{2.121}$$

which is not zero (in Abelian gauge theories it is zero because there are no generators T^a and the commutator is replaced by zero). As we will see, this contribution is only partially cancelled by a contribution from $\mathcal{M}_3$. For the amplitude corresponding to the third diagram, we find

$$i\mathcal{M}_3 = \bar{v}(\bar{q}_2) \left[i g_s T^c_{ij} \gamma_\sigma \right] u(q_1) \frac{-i}{g_3^2} \left[g_s f^{acb} V^{\mu\nu\sigma} \right] \varepsilon^*_{1\mu} \varepsilon^*_{2\nu} \tag{2.122}$$

where

$$V_{\mu\nu\sigma}(g_1, g_2, g_3) = \left[\eta_{\mu\nu}(-g_1 + g_2)_\sigma + \eta_{\nu\sigma}(-g_2 + g_3)_\mu + \eta_{\sigma\mu}(-g_3 + g_1)_\nu \right]. \tag{2.123}$$

This yields (replacing $\varepsilon^*_{2\nu}$ by $g_{2\nu}$)

$$i\mathcal{M}_3^{\mu\nu} \varepsilon^*_{1\mu} g_{2\nu} = g_s^2 f^{abc} T^c \bar{v}(\bar{q}_2) \left[\not{\varepsilon}^*_1 + \not{g}_2 \frac{g_1 \cdot \varepsilon^*_1}{2 g_1 \cdot g_2} \right] u(q_1). \tag{2.124}$$

For *physical* states we have $g_1 \cdot \varepsilon^*_1 = 0$ (note that g_1 is the gauge boson which is not the one whose Ward identity we check) such that the second term in the bracket vanishes and (2.124) and (2.121) exactly cancel each other. This means that gauge invariance or the "naïve" Ward identity is satisfied in this case. However, it is not particularly appealing just to assume that the gluon g_1 is physical (*i.e.* transverse polarized). As we shall see next, this deficiency can be avoided if we take ghosts into account.

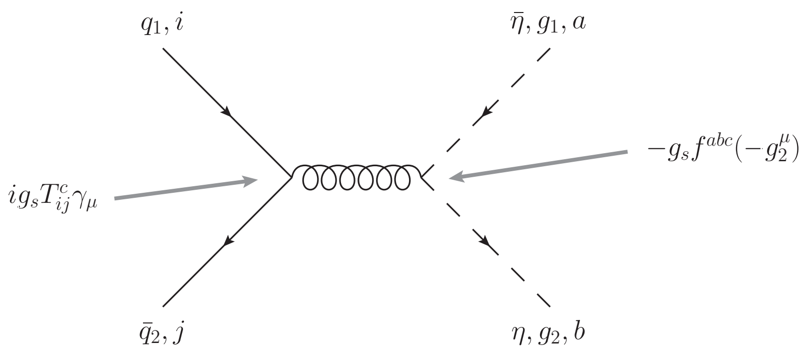
If the gluon states g_1, g_2 are not physical, then we cannot guarantee gauge invariance, i.e. the naïve Ward identity needs to be modified. (In QED this was different: gauge invariance was guaranteed because $g_{2\nu} M^{\mu\nu}$ and $g_{1\nu} M^{\mu\nu}$ vanish independently of whether g_1 and g_2 are physical or not.) If we write

$$\mathcal{M}_{q\bar{q}\to gg} = [\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3] (\varepsilon(g_1), \varepsilon(g_2)) = (\mathcal{M}_{q\bar{q}\to gg})^{\mu\nu} \varepsilon^*_\mu(g_1) \varepsilon^*_\nu(g_2) \tag{2.125}$$

we can easily derive that instead (not assuming that the gluons are physical)

$$\boxed{(\mathcal{M}_{q\bar{q}\to gg})^{\mu\nu} g_{2\nu} = g_1^\mu (\mathcal{M}_{\text{ghost}})} \tag{2.126}$$

which is the so-called **QCD Ward identity**. The amplitude $\mathcal{M}_{\text{ghost}}$ for $q\bar{q} \to \eta\bar{\eta}$ comes from the diagram



and it reads

$$i\mathcal{M}_{\text{ghost}} = g_s^2 \underbrace{f^{abc}T^c}_{=-i\left[T^a,T^b\right]} \bar{v}(\bar{q}_2)\frac{\not{g}_2}{(q_1+\bar{q}_2)^2}u(q_1) \tag{2.127}$$

where $(q_1+\bar{q}_2)^2 = (g_1+g_2)^2 = 2g_1 \cdot g_2$ which shows (2.126).

2.4.2 Physical States and Ghosts: Polarisation Sums Revisited

How do we treat the polarisation vector $\varepsilon^\mu(k)$ and its polarisation sum in QCD calculations?

As we saw before in section 2.1.3, the polarisation vectors $\varepsilon_\mu(k)$ of gluons with momenta k^μ satisfy the following polarisation sum:

$$\sum_{\substack{\text{phys.}\\(\lambda=1,2)}} \varepsilon^*_\mu(k)\varepsilon^*_\nu(k) = -g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{n\cdot k} - \frac{n^2 k_\mu k_\nu}{(n\cdot k)^2} \tag{2.128}$$

where n^μ is chosen such that $n \cdot k \neq 0$. In QED one can use that the sum over physical polarisations is equal to the sum over all polarisations which is equal to $-g_{\mu\nu}$ (since the longitudinal and scalar polarisation do not contribute in the calculations of S-matrix elements). In QCD processes all external gluons have to be physical. The sum over all polarisation states is given by (2.128) and one cannot ignore the contributions from the unphysical polarisation states ($\lambda = 0, 3$). If two or more external gluons are present in a process, using $-g_{\mu\nu}$ will lead to unwanted extra contributions coming from unphysical (longitudinal and scalar) contributions.

How can we quantify these unphysical contributions for our example process $q\bar{q} \to gg$? We use the difference established by using

$$\begin{aligned} &a) \quad -g_{\mu\nu} \\ &b) \quad -g_{\mu\nu} + f(k_\mu^{(1,2)}, n_\mu^{(1,2)}) + \mathcal{O}(n^2) \end{aligned}$$

with $k^{(1,2)}$ the momenta of the two gluons. The $\mathcal{O}(n^2)$ terms are given by

$$\left(n^{(1,2)}\right)^2 \frac{k_\mu^{(1,2)}k_\nu^{(1,2)}}{(n^{(1,2)}\cdot k^{(1,2)})^2}. \tag{2.129}$$

These terms can be neglected by a judicious choice of $n_\mu^{(1,2)}$ where $n_\mu^{(1,2)}$ is such that $n^{(1,2)} \cdot k^{(1,2)} \neq 0$. The amplitude squared for this process $q\bar{q} \to gg$ is then given by

$$\begin{aligned}
\mathcal{M}_{\sigma\tau}\mathcal{M}_{\sigma'\tau'} \cdot \left(\left[(-g^{\sigma\sigma'})(-g^{\tau\tau'}) \right] - \left[-g^{\sigma\sigma'} + \frac{k_1^\sigma n_1^{\sigma'} + n_1^\sigma k_1^{\sigma'}}{n_1 \cdot k_1} \right] \cdot \left[-g^{\tau\tau'} + \frac{k_2^\tau n_2^{\tau'} + n_2^\tau k_2^{\tau'}}{n_2 \cdot k_2} \right] \right) \\
= \left[i g_s^2 f^{abc} T_{ij}^c \frac{1}{2k_1 \cdot k_2} \bar{v}(q) \not{k}_2 u(q) \right]^2 \qquad (2.130) \\
= |\mathcal{M}_{\text{ghost}}|^2. \qquad (2.131)
\end{aligned}$$

The explicit derivation of Eq. (2.130) is straightforward but very lengthy.

Finally note that the expression (2.130) satisfies a symmetry under $g_1 \leftrightarrow g_2$. A priori it does not seem so. However, if we replace $g_1 \to q + \bar{q} - g_2$, using the massless Dirac equations we are left with the same expression except that $\not{g}_2 \to \not{g}_1$ and $f^{abc} \to f^{bac}$.

In practical QCD calculations (with more than two external gluons), the sum over only *physical* polarisations is quite cumbersome. Therefore one uses $-g_{\mu\nu}$ for the complete polarisation sum and accounts for the unphysical polarisation contributions by adding the ghost field contributions. From the example above we deduce that ghost contributions serve to cancel the unwanted (unphysical) scalar and longitudinal polarisation contributions of the gauge bosons.

Finally, we can summarize the role of ghosts as follows:

- They render $S_{A_\mu} = \int \mathcal{D}A_\mu$ finite.
- The QCD Ward identity (gauge invariance of QCD amplitudes involving gluons) is guaranteed to hold true as given in Eq. (2.126).
- They cancel the unphysical polarisation contributions in QCD calculations (shown here at tree level).

2.5 BRST Symmetry

2.5.1 The Definition of BRST-Symmetry

The aim of this section is to show how the ghosts play a crucial role in cancelling unphysical polarisations at all orders in perturbation theory, not only at tree level. The BRST symmetry [B=Becchi, R=Rouet, S=Stora, T=Tyutin] is in some sense a remnant of the gauge symmetry that persists even after the gauge has been fixed. Recall from eqs. (2.95) and (2.96) that the gauge-fixed Lagrangian (including the FP ghosts) has the form

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \bar{c}^a(-\partial^\mu D_\mu^{ab})c^b \; . \qquad (2.132)$$

In order to exhibit the BRST symmetry it is convenient to rewrite the Lagrangian in an equivalent form as

$$\boxed{\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}(F^a_{\mu\nu})^2 + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b} \tag{2.133}$$

Here we have introduced a commuting scalar field B^a. The field B^a is *auxiliary*, since it does not have a kinetic term, and hence has algebraic equations of motion. Indeed, the Euler-Lagrange equation coming from the variation of B^a is

$$\xi B^a + \partial^\mu A^a_\mu = 0 \quad \Rightarrow \quad B^a = -\frac{1}{\xi}(\partial^\mu A^a_\mu) \ . \tag{2.134}$$

Inserting this into (2.133), one finds the original form (2.132). Another way of arriving at the same conclusion is to integrate out B^a in the path integral. The relevant part of the path integral is of the form

$$\begin{aligned} &\int \mathcal{D}B^a \ \exp\left[i\int d^4x \ \left(\frac{\xi}{2}(B^a)^2 + B^a(\partial^\mu A^a_\mu)\right)\right] \\ &\quad = \int \mathcal{D}B^a \ \exp\left[\frac{i\xi}{2}\int d^4x \ \left(B^a + \frac{1}{\xi}\partial^\mu A^a_\mu\right)^2\right] \exp\left[i\int d^4x \ \left(-\frac{1}{2\xi}\right)(\partial^\mu A^a_\mu)^2\right] \ . \end{aligned} \tag{2.135}$$

The integral over B^a is now Gaussian and can be performed, leading to an irrelevant constant. The remaining term $-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$ then reproduces the third term of (2.132).

With these preparations we can now write down the **BRST symmetry transformations** for the fields that appear in the Lagrangian (2.133). We claim that the Lagrangian is invariant under the transformations

$$\delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \tag{2.136}$$
$$\delta\Psi = ig\varepsilon c^a T^a \Psi, \tag{2.137}$$
$$\delta c^a = -\tfrac{1}{2} g\varepsilon f^{abc} c^b c^c, \tag{2.138}$$
$$\delta\bar{c}^a = \varepsilon B^a, \tag{2.139}$$
$$\delta B^a = 0 \ . \tag{2.140}$$

Since the transformation of the gauge fields A^a_μ and of the fermions Ψ is formally just a gauge transformation with gauge parameter $\alpha^a = g\varepsilon c^a$, see eqs. (2.28) and (2.39),

$$\begin{aligned} \delta A^a_\mu &= D_\mu \alpha^a \\ \delta\Psi &= i\alpha_a T^a \Psi \end{aligned} \qquad \text{with } \alpha^a(x) = g\varepsilon c^a(x) \ , \tag{2.141}$$

we conclude that the first two terms in (2.133) are invariant under the BRST symmetry. The third term is trivially invariant due to the transformation law (2.140). It therefore remains to check the invariance of the last two terms. The fourth term transforms as

$$\delta(B^a\partial^\mu A^a_\mu) = \varepsilon B^a \partial^\mu(D^{ab}_\mu c^b) \ . \tag{2.142}$$

This is cancelled by the variation of $\bar{c}^a$ in the last term $\bar{c}^a(-\partial^\mu D_\mu^{ab})c^b$. The remaining variation of the last term is then proportional to

$$\begin{aligned}
\delta(D_\mu^{ac}c^c) &= D_\mu^{ac}\delta c^c + gf^{abc}\delta A_\mu^b c^c \\
&= -\tfrac{1}{2}g\varepsilon f^{abc}\partial_\mu(c^b c^c) - \tfrac{1}{2}g^2\varepsilon f^{abc}A_\mu^b f^{cde}c^d c^e + \varepsilon g f^{abc}(\partial_\mu c^b)c^c + \varepsilon g^2 f^{abc}f^{bde}A_\mu^d c^e c^c \\
&= -\tfrac{1}{2}g^2\varepsilon f^{abc}A_\mu^b f^{cde}c^d c^e + \varepsilon g^2 \underbrace{f^{abc}f^{bde}A_\mu^d c^e c^c}_{=f^{acb}f^{cde}A_\mu^d c^e c^b = -f^{abc}f^{cde}A_\mu^d\left(\frac{1}{2}(c^e c^b - c^b c^e)\right)} \\
&= -\tfrac{1}{2}g^2\varepsilon f^{abc}f^{cde}\left(A_\mu^b c^d c^e + A_\mu^d c^e c^b + A_\mu^e c^b c^d\right) \ , \qquad (2.143)
\end{aligned}$$

where we inserted the definition $D_\mu^{ac} = \partial_\mu\delta^{ac} + gf^{abc}A_\mu^b$ from (2.87) in the first step. We now show that this term vanishes by means of the Jacobi identity

$$\left[t^a,[t^b,t^c]\right] + \left[t^b,[t^c,t^a]\right] + \left[t^c,[t^a,t^b]\right] = 0 \qquad (2.144)$$

$$\Rightarrow \quad f^{bcd}f^{ade} + f^{cad}f^{bde} + f^{abd}f^{cde} = 0 \ . \qquad (2.145)$$

Reordering the indices and relabelling $d \leftrightarrow c$, this identity becomes

$$f^{bdc}f^{cae} + f^{dac}f^{cbe} + f^{abc}f^{cde} = f^{cbd}f^{aec} + f^{adc}f^{ceb} + f^{abc}f^{cde} = 0 \ , \qquad (2.146)$$

where we have reordered indices again in the last step. Thus we obtain for (2.143)

$$\delta(D_\mu^{ac}c^c) = -\tfrac{1}{2}g^2\varepsilon A_\mu^b c^d c^e\left(f^{abc}f^{cde} + f^{aec}f^{cbd} + f^{adc}f^{ceb}\right) = 0 \ . \qquad (2.147)$$

This completes the proof that the Lagrangian (2.133), which is equivalent to the gauge fixed Lagrangian including FP ghosts, is invariant under the BRST transformation defined by eqs. (2.136) – (2.140).

2.5.2 Implications of the BRST Symmetry

A very important property of the BRST symmetry is its *nilpotency*, *i.e.* the property that applying the transformation twice annihilates all fields. In order to formulate this, it is convenient to define an operator Q via

$$\delta\phi = \varepsilon Q\phi \ . \qquad (2.148)$$

(Thus Q is just the operator that gives the infinitesimal BRST transformation when multiplied with ε.) The nilpotency of the BRST tranformation then simply means that

$$\boxed{QQ\phi = 0} \qquad (2.149)$$

for all fields ϕ. Let us check this case by case. If $\phi = B$ or $\phi = \bar{c}$ this follows trivially from eqs. (2.139) and (2.140). From the calculation we did above it follows that

$$QQA_\mu^a = Q(D_\mu^{ac}c^c) = 0. \qquad (2.150)$$

Thus it only remains to check the claim for $\phi = c^a$ and $\phi = \Psi$. In the former case we find

$$\begin{aligned} QQc^a &= -\tfrac{1}{2} g f^{abc} Q(c^b c^c) \\ &= -\tfrac{1}{2} g f^{abc} \left((Qc^b)c^c - c^b Q c^c \right) \\ &= \tfrac{1}{4}\, g^2 f^{abc} \left(f^{bde} c^d c^e c^c - f^{cde} c^b c^d c^e \right) \\ &= -2 \cdot \tfrac{1}{4}\, g^2 f^{abc} f^{cde} \left(c^b c^d c^e \right) \\ &\overset{\text{Jacobi}}{=} 0 \ , \end{aligned} \tag{2.151}$$

where, in the penultimate line, we have relabelled $b \leftrightarrow c$ in the first term. We have also used that the action of Q on products of fields equals

$$Q(\phi_1 \phi_2) = (Q\phi_1)\phi_2 + (-1)^{f_1} \phi_1 (Q\phi_2) \ , \tag{2.152}$$

where f_1 is the fermion number of ϕ_1 (*i.e.* $f_1 = 0$ if ϕ_1 is bosonic and $f_1 = 1$ if ϕ_1 is fermionic). [This is a direct consequence of the fact that ε in (2.148) is a Grassmann variable.] Finally, for the fermion field $\phi = \Psi$, the calculation is similar

$$\begin{aligned} QQ\Psi &= Q(igc^a T^a)\Psi \\ &= -\tfrac{i}{2} g^2 f^{abc} c^b c^c T^a \Psi + g^2 c^a t^a c^b T^b \Psi \\ &= -\tfrac{i}{2} g^2 f^{abc} c^b c^c T^a \Psi + \frac{1}{2} g^2 \big(c^a c^b \underbrace{(T^a T^b - T^b T^a)}_{= i f^{abc} T^c} \big) \Psi \\ &= 0 \ . \end{aligned} \tag{2.153}$$

In order to understand the physical significance of the BRST symmetry let us analyse the action of Q on the gauge-fixing term in the Langrangian

$$\mathcal{L} = \mathcal{L}_{\text{phys.}} + \mathcal{L}_{\text{gauge-fix}} \ . \tag{2.154}$$

Recall that the physical part of the Lagrangian $\mathcal{L}_{\text{phys.}}$ is trivially BRST invariant (since Q acts as a gauge transformation on the physical fields). In order to see that the whole $\mathcal{L}$ is Q-invariant, we showed above, by a brute force calculation, that $\mathcal{L}_{\text{gauge-fix}}$ is Q-invariant. However, there is a much more elegant way to see this. To this end we introduce the following terminology: a field ϕ is called

- BRST-closed if $Q\phi = 0$,
- BRST-exact if $\phi = Q\chi$ for some field χ .

We note that every BRST-exact field is BRST-closed since

$$\phi = Q\chi \quad \Rightarrow \quad Q\phi = QQ\chi = 0 \ . \tag{2.155}$$

On the other hand, not every closed field is necessarily exact. It is therefore natural to define the **cohomology**[4] of Q which is just the quotient space

$$\text{cohomology}(Q) = \frac{Q\text{-closed fields}}{Q\text{-exact fields}} \; . \tag{2.156}$$

We shall denote the cohomology class that contains ϕ by $[\phi]$. As we shall see, the cohomology of Q corresponds to the physical degrees of freedom.

Next we want to show that the gauge-fixing term in the Lagrangian is not just BRST-invariant (*i.e.* BRST-closed), but actually BRST-exact. This means that there exists a field χ such that

$$Q\chi = \mathcal{L}_{\text{gauge-fix}} \; . \tag{2.157}$$

In order to find χ we note that

$$\begin{aligned} Q\left(G^a(A^b_\mu)(x)\right) &= \int d^4y \, \frac{\partial G^a(x)}{\partial A^b_\mu(y)} Q A^b_\mu(y) \\ &= \int d^4y \; \delta^{ab} \partial^\mu \delta^{(4)}(x-y) Q A^b_\mu(y) \\ &= -\partial^\nu D^{ac}_\nu c^c \; . \end{aligned} \tag{2.158}$$

With this in mind we now take $\chi = \bar{c}^a \left(G^a + \frac{\xi}{2} B^a\right)$ and find

$$Q\chi \equiv Q\left(\bar{c}^a\left(G^a + \tfrac{\xi}{2}B^a\right)\right) = B^a G^a + \frac{\xi}{2} B^a B^a - \bar{c}^a \partial^\mu D^{ac}_\mu c^c = \mathcal{L}_{\text{gauge-fix}} \; . \tag{2.159}$$

Thus we conclude that the gauge-fixing term in the Lagrangian is BRST-exact. In particular it is therefore trivial in the BRST-cohomology.

The next step is to show that BRST-exact terms do not contribute to any amplitudes of BRST-closed states. (The fact that the gauge-fixing term is BRST-exact then means that it does not contribute to the amplitudes!) Amplitudes can be calculated from the inner product of the underlying vector space. Let us assume – this follows essentially from the reality of the BRST transformation — that the BRST operator Q is hermitian with respect to this inner product. Then it follows that the cohomology of Q inherits an inner product from the underlying vector space

$$\langle [\phi_1] | [\phi_2] \rangle := \langle \phi_1, \phi_2 \rangle \; . \tag{2.160}$$

To see that this inner product is well-defined we only need to show that the definition is independent of the chosen representative, *i.e.* that

$$\begin{aligned} \langle \phi_1 + Q\chi_1 | \phi_2 + Q\chi_2 \rangle &= \langle \phi_1 | \phi_2 \rangle + \langle Q\chi_1 | \phi_2 + Q\chi_2 \rangle + \langle \phi_1 | Q\chi_2 \rangle \\ &= \langle \phi_1 | \phi_2 \rangle + \langle \chi_1 | \underbrace{Q(\phi_2 + Q\chi_2)}_{=0} \rangle + \langle \underbrace{Q\phi_1}_{=0} | \chi_2 \rangle \; . \end{aligned} \tag{2.161}$$

[4] Our notion of the cohomology of Q is a special case of the general notion of cohomology that is frequently used in mathematics. For example, the de Rham cohomology is defined analogously for closed and exact p-forms on differentiable manifolds.

In particular, it follows from this calculation that BRST-trivial terms do not contribute in amplitudes with BRST-closed terms.

With these preparations we can now identify the physical states with the BRST-cohomology. As we have seen above we can think of Q as inducing gauge transformations since Q acts as a gauge transformation on A and Ψ. If we restrict to Q-closed states we are thus considering the gauge invariant fields. Furthermore, by going to the BRST-cohomology, we are removing the redundancy due to gauge transformations since the equivalence class $[\phi]$ of ϕ contains all gauge equivalent fields which can be reached by a gauge transformation acting on ϕ. Thus it is natural to identify the 'physical' states with the BRST-cohomology,

$$\boxed{\{\text{Physical states }\} = \text{ cohomology}(Q)} \ . \tag{2.162}$$

The above argument then shows that the 2-point functions of the physical states are independent of the particular gauge that is chosen.

It is now fairly easy to see how this argument can be generalised to arbitrary n-point functions. Suppose that ϕ_1 is a Q-closed field while $\phi_2 = Q\chi_2$ is Q-exact. Then the product $\phi_1\phi_2$ is again Q-exact

$$Q(\phi_1\chi_2) = \underbrace{(Q\phi_1)}_{=0}\chi_2 \pm \phi_1 Q\chi_2 = \pm\phi_1\phi_2 \ , \tag{2.163}$$

where the sign depends on whether ϕ_1 is bosonic or fermionic. Thus the operator product of the underlying vector space defines a product on the BRST-cohomology. Iterating this procedure it follows that in an n-point product, changing the gauge for any of the individual terms leads to an overall BRST-exact term that does not contribute in amplitudes with physical (and hence BRST-closed) states. Since loop diagrams can be obtained by integrating higher point functions this argument then shows quite generally that all physical amplitudes will be independent of any gauge choices.

Chapter 3

Renormalisation Group

In QFT I we have seen how renormalisation works in some simple cases. The basic mechanism is that UV divergences can be absorbed into redefining a few bare parameters, such as the masses or coupling constants. A priori it is somewhat surprising that high-momenta virtual quanta have so little effect on a theory, *i.e.* that they only affect these few parameters.

In this section we want to understand this phenomenon more conceptually, by explaining the physical picture of renormalisation that has been advocated by Ken Wilson. It suggests that all parameters of a renormalisable field theory can usefully be thought of as scale-dependent quantities. The scale dependence is described by differential equations, the renormalisation group (RG) equations. Solving these equations we will obtain new physical predications, in particular, we will be able to show that (at least some) non-abelain gauge theories are asymptotically free, *i.e.* become more weakly coupled as we go to higher energies or smaller distances.

3.1 Wilson's Approach to Renormalisation

Wilson's method is based on the functional integral approach to quantum field theory (which is the reason why it was not already explained in QFT I). In this approach we can study the origin of the UV divergencies by isolating the dependence of the functional integral on the short-distance degrees of freedom. In the following we want to illustrate this for a simple example, the ϕ^4-theory, that is not plagued by various technical difficulties; the same ideas can, however, also be applied to more complicated theories such as gauge theories.

Let us work with a sharp momentum cut-off regularisation scheme. (This is one of the places where simplifications arise for ϕ^4; for gauge theories a sharp momentum cut-off is problematic since it does not respect the Ward identities.) Recall that the Green's functions can be obtained from the generating functional

$$Z[J] = \int \mathcal{D}\phi \; e^{i\int \mathcal{L}+J\phi} = \int \prod_k d\phi(k) \; e^{i\int \mathcal{L}+J\phi} \; , \tag{3.1}$$

where the last formulation is to remind us of the definition of the path integral as the limit of a product of integrals over discrete momentum modes.

Now we impose a sharp momentum cut-off Λ, *i.e.* we restrict the integration variables to those $\phi(k)$ with $|k| < \Lambda$ and set $\phi(k) \equiv 0$ for $|k| > \Lambda$. Note that in order for this to actually remove the large momenta, we should consider the Euclidean theory with signature $(+,+,+,+)$.[1] If we are given a theory in Minkowski space, we can always go to the Euclidean theory by a Wick rotation $x^0 \to x^0_E = -ix^0$. The action defined by the (Minkowski-)Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.2}$$

gets Wick-rotated into

$$i\int d^4x\, \mathcal{L} = i\int dx^0_E d^3x\; i\mathcal{L} = -\int d^4x_E\, \left(\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right). \tag{3.3}$$

We will drop the subscript E in the Euclidean coordinates x_E from now on. Restricting to $J = 0$ for simplicity we then obtain

$$Z = \int [\mathcal{D}\phi]_\Lambda\; \exp\left[-\int d^4x \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right]\;, \tag{3.4}$$

where $[\mathcal{D}\phi]_\Lambda$ is meant to remind us of the fact that the discrete momenta in the last expression of eq. (3.1) are restricted to be smaller than Λ. We want to understand how this path integral depends on Λ. In order to do so we now split the path integral into the 'high momentum modes' on the one hand, and the rest on the other. The high momentum modes are those which satisfy $b\Lambda < |k| < \Lambda$ with $b \lesssim 1$. Let us introduce the notation

$$\hat{\phi}(k) = \begin{cases} \phi(k) & (b\Lambda < |k| < \Lambda) \\ 0 & (\text{otherwise})\;. \end{cases} \tag{3.5}$$

Generalising the action to a d-dimensional integral, we can then rewrite Z as

$$\begin{aligned} Z &= \int [\mathcal{D}\phi]_{b\Lambda}\mathcal{D}\hat{\phi}\; \exp\left[-\int d^dx \left(\frac{1}{2}(\partial_\mu\phi + \partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2(\phi+\hat{\phi})^2 + \frac{\lambda}{4!}(\phi+\hat{\phi})^4\right)\right] \\ &= \int [\mathcal{D}\phi]_{b\Lambda}\; e^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi}\; \exp\Bigg[-\int d^dx \Bigg(\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + \\ &\qquad + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\Bigg)\Bigg]\;, \end{aligned} \tag{3.6}$$

where we have collected all terms which depend only on ϕ outside the $\hat{\phi}$-integral — they then reproduce simply $\mathcal{L}(\phi)$. Note that we have dropped the quadratic terms of the form

[1] In Minkowski space a cut-off criterion like $|k| < \Lambda$ would not be particularly sensible because there are light-like momentum vectors whose Lorentzian norm stays small while the individual entries diverge.

$\phi\hat{\phi}$ because their integral vanishes (since the Fourier modes of $\hat{\phi}$ and ϕ are orthogonal). Even without performing the explicit $\hat{\phi}$-integrations, we can already see that the final result will be of the form

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} \; \exp\left[- \int d^d x \; \mathcal{L}_{\text{eff.}} \right] \;, \tag{3.7}$$

where $\mathcal{L}_{\text{eff.}} = \mathcal{L}+$(correction terms). Thus we will be able to rewrite the full theory with cut-off Λ in terms of an effective theory with cut-off $b\Lambda$. The correction terms in $\mathcal{L}_{\text{eff.}}$ compensate for the removal of the large momentum components $\hat{\phi}$ by supplying interactions among the remaining $\phi(k)$ that were previously mediated by the fluctuations of $\hat{\phi}$.

In order to get a feeling for how $\mathcal{L}_{\text{eff.}}$ looks explicitly, let us work out the $\hat{\phi}$-integrals in perturbation theory. We want to think of the kinetic part $\frac{1}{2}(\partial_\mu\hat{\phi})^2$ in the exponential of (3.6) as being the 'free' part, and treat the rest as perturbations — this is justified since the $\hat{\phi}$ integral only involves large momenta for which $m^2 \ll b^2\Lambda^2$. The propagator is then simply

$$\langle \hat{\phi}(k)\hat{\phi}(p) \rangle = \frac{\int \mathcal{D}\hat{\phi} \; e^{-\int \mathcal{L}_0} \hat{\phi}(k)\hat{\phi}(p)}{\int \mathcal{D}\hat{\phi} \; e^{-\int \mathcal{L}_0}} = \frac{1}{k^2}(2\pi)^d \delta^{(d)}(p+k)\Theta(k) \;, \tag{3.8}$$

where $\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\hat{\phi})^2$ and

$$\Theta(k) = \begin{cases} 1 & (b\Lambda \le |k| < \Lambda) \\ 0 & (\text{otherwise}) \;. \end{cases} \tag{3.9}$$

Let us first look at the simplest perturbative correction term $\frac{\lambda}{4}\phi^2\hat{\phi}^2$. Thinking of ϕ as an external field we can calculate the corresponding term in (3.6) as

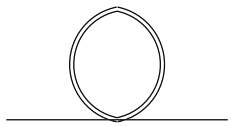
$$- \int d^d x \; \frac{\lambda}{4} \phi^2 \hat{\phi}^2 = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \, \mu \, \phi(k)\phi(-k) \;, \tag{3.10}$$

where we have gone to momentum space (for all four fields), and contracted the $\hat{\phi}$ fields according to (3.8). Here the parameter μ arises from integrating the propagator (3.8) over one of the momentum space integrals

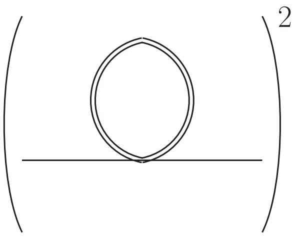
$$\begin{aligned} \mu &= \frac{\lambda}{2} \int_{b\Lambda<|k|<\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \\ &= \frac{\lambda}{2} \int \frac{d\Omega_d}{(2\pi)^d} \int_{b\Lambda}^{\Lambda} dk \; k^{d-3} \\ &= \frac{\lambda}{(4\pi)^{d/2}\Gamma(d/2)} \frac{1-b^{d-2}}{d-2} \Lambda^{d-2} \;. \end{aligned} \tag{3.11}$$

The term (3.10) looks like a mass term for the ϕ-modes. It therefore provides a correction term (in $\mathcal{L}_{\text{eff.}}$) to the mass m that appeared in $\mathcal{L}$.

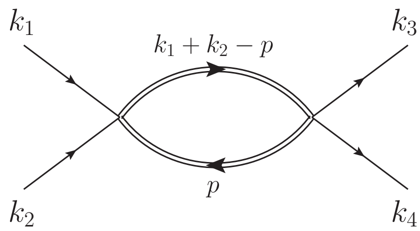
The other terms can be dealt with similarly. In order to do these calculations systematically we can introduce a diagrammatic notation (just as for the case of the usual Feynman rules). We denote external fields by lines, while doubles lines describe $\hat{\phi}$-fields and lead to the propagator $\frac{1}{k^2}$. For example, the diagram we have just computed and which corrects the mass m at order λ is



At $\mathcal{O}(\lambda^2)$, the perturbative corrections which come from the contraction of two $\frac{\lambda}{4}\phi^2\hat{\phi}^2$-terms have the diagrammatic representation



and



The first diagram gives rise to a term $\frac{\mu^2}{2}$ which is just the square of the $\mathcal{O}(\lambda)$-term that we just computed; this is precisely the term that is required for the above μ term to modify the mass of $\mathcal{L}$ in the exponential. The second diagram, on the other hand, leads to

$$\frac{2}{2!}\left(\frac{\lambda}{4}\right)^2 \int_{b\Lambda<|p|<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(k_1+k_2-p)^2}\, \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta^{(d)}(k_1+k_2+k_3+k_4). \tag{3.12}$$

In the limit where we ignore the external momenta of the ϕ (assuming that they are small compared to $b\Lambda$) the propagator becomes $(k^2)^{-2}$, and we get

$$(3.12) \longrightarrow -\frac{1}{4!}\zeta \int d^dx\, \phi^4(x) \tag{3.13}$$

where

$$\zeta = \frac{-3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)} \frac{(1-b^{d-4})}{d-4}\Lambda^{d-4} \quad \overset{d\to 4}{\longrightarrow} \quad -\frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}. \tag{3.14}$$

Recall from QFT I that in the usual perturbative treatment, we would encounter a similar diagram integrated over a range of momenta from zero to Λ, producing an UV divergence.

The diagrammatic perturbation theory will also produce higher order terms in ϕ that can be calculated similarly. There are also derivative interactions which arise when we no longer neglect external momenta in the calculation, *i.e.* for the above ϕ^4 term we will also get terms of the form

$$-\frac{1}{4}\int d^dx\,\eta\,\phi^2(\partial_\mu\phi)^2 + \text{terms with higher derivatives}\ . \tag{3.15}$$

In general we will thus produce all possible interactions of fields and their derivatives. These diagrammatic corrections can be simplified by resumming them as an exponential; in particular, the diagrammatic expansion generates the appropriate disconnected terms which just produce the exponential series. Altogether we therefore get for $\mathcal{L}_{\text{eff.}}$

$$\mathcal{L}_{\text{eff.}} = -\int d^4x\,\left(\mathcal{L}(\phi) + \frac{\mu}{2}\phi^2(x) + \zeta\frac{1}{4!}\phi^4(x) + \cdots\ + (\phi^2\partial^2\phi^2) + \cdots +\ \phi^6(x) + \cdots\right)\ . \tag{3.16}$$

For example, the ϕ^6 term can come from a the contraction of two $\frac{\lambda}{6}\phi^3\hat{\phi}$ terms, *etc.*

We can now use the new effective Lagrangian $\mathcal{L}_{\text{eff.}}$ in order to compute correlation functions of $\phi(k)$ or S-matrix elements. Since $\phi(k)$ only include momenta up to $b\Lambda$, loop integrals only run up to $|k| < b\Lambda$. The correction terms in $\mathcal{L}_{\text{eff.}}$ precisely account for this change. The effective Lagrangian point of view is therefore ultimately equivalent to the standard approach (where we work with $\mathcal{L}$ and integrate loop momenta up to $|k| < \Lambda$). However, if we are interested in correlation functions of fields whose momenta $|p_i| \ll \Lambda$ the effective Lagrangian point of view is more useful: loop effects appear already at tree level (since the coefficients of $\mathcal{L}_{\text{eff.}}$ depend on them) rather than as large 1-loop corrections.

One may, however, be puzzled by the higher-dimension operators that appear in $\mathcal{L}_{\text{eff.}}$; in particular, all possible interactions are generated when we integrate out $\hat{\phi}$, and most of the resulting terms are non-renormalisable. As will become clear momentarily, our procedure actually keeps the contributions of these non-renormalisable interactions under control, and we will see that they have a negligible effect on the physics at scales much less than Λ. This will explain, in particular, why high-momenta virtual quanta only affect a few parameters of the theory.

3.2 Renormalisation Group Flows

In order to understand better how the non-renormalisable terms can be controlled, let us make a more careful comparison of the functional integrals

$$Z = \int [D\phi]_{b\Lambda}\, e^{-\mathcal{L}_{\text{eff.}}} \qquad \leftrightarrow \qquad Z = \int [D\phi]_{\Lambda}\, e^{-\mathcal{L}}\ . \tag{3.17}$$

To relate them to one another let us rescale the distances and momenta as

$$k' = \frac{k}{b} \quad \text{and} \quad x' = bx\ , \tag{3.18}$$

so that the variable k' is integrated up to $|k'| < \Lambda$ if k is integrated up to $|k| < b\Lambda$. Next we rewrite the effective action (written in terms of the variables x and k) in terms of the x' variables as

$$\begin{aligned}\int d^dx\, \mathcal{L}_{\text{eff.}} &= \int d^dx \left[\frac{1}{2}(1+\Delta Z)(\partial_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \right.\\ &\qquad\qquad \left. + \Delta C(\partial_\mu\phi)^4 + \Delta D\phi^6 + \cdots\right]\\ &= \int d^dx'\, b^{-d}\left[\frac{1}{2}(1+\Delta Z)b^2(\partial'_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \right.\\ &\qquad\qquad \left. + \Delta C b^4(\partial'_\mu\phi)^4 + \Delta D\phi^6 + \cdots\right]\\ &= \int d^dx' \left[\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{\lambda'}{4!}\phi'^4 + C'(\partial'_\mu\phi')^4 + D'\phi'^6 + \cdots\right], \end{aligned} \tag{3.19}$$

where we have used that

$$d^dx = b^{-d}d^dx' , \qquad \partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = b^{-1}\partial_\mu . \tag{3.20}$$

In the third line we have furthermore rescaled the field ϕ, so that the kinetic term takes again the canonical form

$$\phi' = \left(b^{2-d}(1+\Delta Z)\right)^{1/2}\phi \qquad \Leftrightarrow \qquad \phi = \left(b^{d-2}(1+\Delta Z)^{-1}\right)^{1/2}\phi' , \tag{3.21}$$

and have defined the coefficients

$$m'^2 = (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2} \tag{3.22}$$
$$\lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4} \tag{3.23}$$
$$C' = (C+\Delta C)(1+\Delta Z)^{-2}b^{d} \tag{3.24}$$
$$D' = (D+\Delta D)(1+\Delta Z)^{-3}b^{2d-6} \quad \textit{etc.} \tag{3.25}$$

All corrections Δm^2, $\Delta\lambda$, *etc.* arise from diagrams, and are thus small compared to the leading terms (if perturbation theory is justified). Note that the variables k' are again integrated up to Λ, so that we have the identity

$$\int [\mathcal{D}\phi]_{b\Lambda}\, e^{-\int d^dx\, \mathcal{L}_{\text{eff.}}} = \int [\mathcal{D}\phi]_{\Lambda}\, e^{-\int d^dx'\, \mathcal{L}_{\text{eff.}}(m^2\to m'^2,\cdots)} . \tag{3.26}$$

We can now iterate this procedure, going from $\Lambda \to b\Lambda \to b^2\Lambda$, *etc.* If we take b close to 1, the shells in momentum space are infinitesimally thin, and we get a continuous transformation. This is the renormalisation group. (Technically, this is not quite a group since the operation of integrating out degrees of freedom is not invertible; it has therefore more the structure of a semigroup rather than that of a group.)

The parameters of the effective Lagrangain may change quite significantly as we iterate the transformation $\Lambda \to b\Lambda$ many times. To understand how the effective Lagrangian varies under RG transformations, consider the Lagrangian in the vicinity of the free point where

$$m^2 = \lambda = C = D = \cdots = 0 \ . \tag{3.27}$$

Then $(m')^2 = \lambda' = C' = D' = \cdots = 0$, *i.e.* the free theory is a *fixed point* under the RG transformations. In order to study the nature of the fixed point suppose that all of these parameters are small. Working to linear order we then have

$$(m')^2 = m^2\, b^{-2} \tag{3.28}$$
$$\lambda' = \lambda\, b^{d-4} \tag{3.29}$$
$$C' = C\, b^{d} \tag{3.30}$$
$$D' = D\, b^{2d-6} \qquad \textit{etc.} \tag{3.31}$$

Since $b < 1$, the parameters that are multiplied by negative powers of b grow, while those that are multiplied by positive powers of b decay. If the Lagrangian contains growing coefficients, then the fixed point is unstable and the Lagrangian will eventually move away from $\mathcal{L}$.

Let us classify the relevant terms in the effective Lagrangian according to their behaviour under the RG flow. We call the corresponding operator

- **relevant** — if the coefficient grows (*e.g.* $\frac{1}{2}m^2\phi^2$)
- **irrelevant** — if the coefficient decays (*e.g.* $\lambda\phi^4$ for $d > 4$)
- **marginal** — if the coefficient goes as b^0 (*e.g.* $\lambda\phi^4$ for $d = 4$)

In the last case, the leading order analysis is not sufficient in oder to determine the actual behaviour of the fixed point. In fact, most of what we shall do later will concern this case.

In general, the b-behaviour of the coefficient depends on the power N of ϕ, and on the number M of derivatives in the corresponding term in the effective Lagrangian. The relevant coefficient then goes as

$$\text{coefficient} \sim b^{M-d} b^{\left(\frac{d}{2}-1\right)N} \ . \tag{3.32}$$

If we denote by d_i the mass dimension of the operator, then

$$d_i - d = \left(N\left(\frac{d}{2} - 1\right) + M \right) - d \tag{3.33}$$

is precisely the power of b in eq. (3.32). (This relation also follows directly from dimensional reasoning since the action must be dimensionless.) We conclude that the mass dimension of an operator determines whether it is relevant, irrelevant or marginal

- **relevant** if the mass dimension is smaller than d: $d_i - d < 0$,

- **marginal** if the mass dimension is equal to d: $d_i - d = 0$,
- **irrelevant** if the mass dimension is larger than d: $d_i - d > 0$.

We recall from QFT I that these definitions correspond precisely to the notions of super-renormalisable, renormalisable and non-renormalisable, respectively. However, now our perspective is in some sense opposite to that taken in QFT I. From the Wilsonian point of view, any quantum field theory is defined fundamentally with a cut-off. (For example, we may think of the cut-off in the context of high energy physics as the Planck scale where gravity effects need to be taken into account; in the context of statistical physics, the cut-off is something like the inverse atomic spacing of some condensed matter system.) This cut-off scale is very high relative to present-day experiments. For the purpose of understanding these experiments it is therefore more appropriate to work with an effective Lagrangian where we have integrated out all the higher momentum modes. But in this effective Lagrangian *only* the relevant and marginal terms appear since the coefficients of the other terms have gone to zero after integrating out many momentum shells. Given the above correspondence this then *explains* why the effective Lagrangian (that is a good description of the physics at the low momenta of present-day experiments) only contains renormalisable and super-renormalisable terms!

We should mention though that these simple mechanisms can change if there are sufficiently strong field theory interactions: away from the free-field fixed point the RG transformations have also higher order corrections (in powers of coupling constants), and these can modify the above simple scaling behaviour.

3.3 Callan-Symanzik Equation

Next we want to study the scale dependence of the marginal parameters of our effective Lagrangian. Our first aim is to derive the Callan-Symanzik equation which describes the dependence of the n-point Green's functions on the energy scale.

3.3.1 The ϕ^4-Theory

For concreteness let us consider again the ϕ^4 theory with $m = 0$ in $d = 4$, for which the ϕ^4 operator is marginal (and hence the scale dependence of λ is not determined by the leading order analysis of the previous section). Suppose the theory is fundamentally defined at some very high scale Λ_0, where the coupling constant takes the value λ_0, and we are interested in the theory at scale $\Lambda \ll \Lambda_0$. Following the discussion above, we should then analyse the theory using the effective Lagrangian at scale Λ, *i.e.* we consider the effective Lagrangian obtained from the fundamental Lagrangian upon integrating out all modes with momenta $\Lambda < |k| < \Lambda_0$. The resulting effective Lagrangian $\mathcal{L}_{\text{eff.}}$ then only depends on λ, where λ is the coupling constant that appears in the effective Lagrangian at scale Λ.

Using this effective description we can calculate renormalised Green's functions, and they will only depend on the arbitrary renormalisation scale $M = \Lambda$, as well as on λ, but not directly on the fundamental scale Λ_0 (nor on λ_0). [Indeed, in our effective Lagrangian, the coupling constant is λ, and $M = \Lambda$ enters since it determines the cut-off scale of the loop momenta.] We want to understand the behaviour of these Green's functions as a function of the renormalisation scale $M = \Lambda$. To this end we consider changing M as

$$M \longrightarrow M + \delta M \ . \tag{3.34}$$

Performing this change of scale amounts to introducing the shell $b\Lambda < |k| < \Lambda$ and integrating out the corresponding momentum modes. As we have seen above, this will then change the coupling constant λ as well as the normalisation of the fields, *i.e.* it will lead to

$$\lambda \longrightarrow \lambda + \delta\lambda \tag{3.35}$$

$$\phi \longrightarrow \phi(1 + \delta\eta) \ . \tag{3.36}$$

The rescaling of the fields will lead to a change in the n-point Green's function $G^{(n)}$ as

$$G^{(n)} \longrightarrow (1 + n\delta\eta)G^{(n)} \ . \tag{3.37}$$

As we have explained above, the renormalised Green's functions only depend on M and λ, $G^{(n)} \equiv G^{(n)}(M, \lambda)$, since these are the only parameters that enter the calculation when we use the effective Lagrangian at scale Λ. Thus we have

$$\delta G^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda \stackrel{!}{=} n\delta\eta G^{(n)} \ . \tag{3.38}$$

Observing that λ is dimensionless, it is convenient to use, instead of $\delta\lambda$ and $\delta\eta$, the dimensionless parameters

$$\beta = \frac{M}{\delta M}\delta\lambda \qquad \text{and} \qquad \gamma = -\frac{M}{\delta M}\delta\eta \ . \tag{3.39}$$

Eq. (3.38) then becomes

$$M\frac{\partial G^{(n)}}{\partial M} + \beta\frac{\partial G^{(n)}}{\partial \lambda} + \gamma n G^{(n)} = 0 \ . \tag{3.40}$$

Obviously the parameters β and γ are the same for every n, and thus must be independent of the insertion points x_i at which the n fields are evaluated. Furthermore, β and γ are by construction dimensionless, and they can therefore only depend on dimensionless quantities. The only mass-scale parameters are M and Λ_0, and hence β and γ can only depend on the ratio M/Λ_0, not on M alone. On the other hand, since the fundamental cut-off scale Λ_0 does not appear in our effective Lagrangian, β and γ (which come from integrating out momentum modes in the shell $M < |k| < M + \delta M$ in the theory described by $\mathcal{L}_{\text{eff.}}$) do not depend on Λ_0, and hence cannot depend on M! Thus we conclude that β and γ are *only* functions of λ

$$\beta \equiv \beta\left(\frac{M}{\Lambda_0}, \lambda\right) \equiv \beta(\lambda) \ , \qquad \gamma \equiv \gamma\left(\frac{M}{\Lambda_0}, \lambda\right) \equiv \gamma(\lambda) \ , \tag{3.41}$$

and eq. (3.40) is actually of the form

$$\boxed{\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma(\lambda) n\right] G^{(n)} = 0} \tag{3.42}$$

This is the celebrated **Callan-Symanzik equation**. The β-function

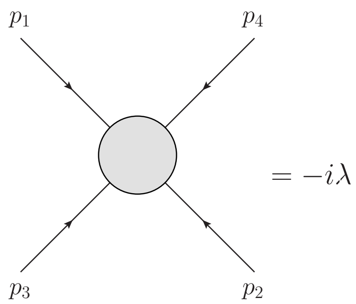
$$\beta = \frac{M}{\delta M}\delta\lambda \tag{3.43}$$

tells us how the λ-parameter in the effective action changes as we change the scale. In particular, the *sign* of the β-function determines whether the marginal term grows or decays as we integrate out momentum modes. Thus the β-function is of major interest since it captures how the dimensionless coupling 'constants' depend on the energy scale. (For the dimensionful coupling constants, the leading order analysis of the last section is sufficient: relevant parameters grow while irrelevant parameters decay.)

The procedure we have outlined above is conceptually very clean, but computationally rather unwieldy. We shall therefore make the basic assumption that the above Wilsonian scheme defined in the Wick rotated (Euclidean) theory at scale $\Lambda = M^2$ is equivalent to imposing standard renormalisation conditions of the Minkowski theory for typical particle momenta at $p^2 = -M^2$. (The basic intuition behind this identification is that the effective Lagrangian describes correctly the physics at energy scale M, *i.e.* for typical particle momenta with $p^2 = -M^2$.) We should note that for the case at hand this is a slightly unusual scheme since we are working with the massless ϕ^4-theory, but impose the renormalisation condition on the propagator,

$$\langle\Omega|\phi(p)\phi(-p)|\Omega\rangle = \frac{i}{p^2} \qquad \text{at} \ \ p^2 = -M^2 \ , \tag{3.44}$$

and not for $p^2 = 0$, as we would have imposed usually. Similarly, the value of the coupling constant

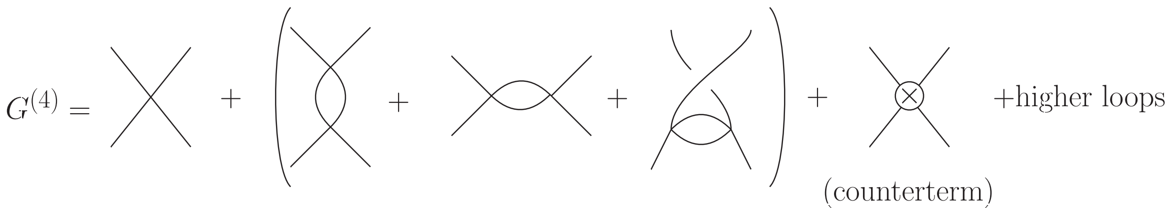


is imposed not for the 'on-shell' Mandelstam variables $s = t = u = 0$, but rather for $s = t = u = -M^2$.

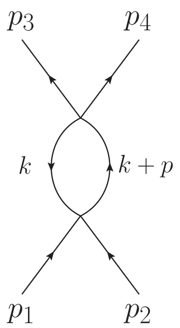
In order to get a feeling for how this actually works, let us apply this scheme to the ϕ^4-theory and calculate the β-function. This can be done by calculating some Green's functions and demanding that they satisfy the Callan-Symanzik equation. The simplest interesting Green's function to consider is the 4-point function. Then the Callan-Symanzik equation reads (with $n = 4$)

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 4\gamma(\lambda)\right] G^{(4)} = 0 \ . \tag{3.45}$$

The leading contributions to $G^{(4)}$ in perturbation theory are



The first diagram is the four-point vertex at tree level and it is given by $-i\lambda$. The first 1-loop contribution in the bracket reads



$$= \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(k+p)^2} = (-i\lambda)^2 iV(p^2)$$

where $p = p_1 + p_2$ and

$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx\ \left[\left(\frac{2}{\varepsilon} - \gamma\right) + \log(4\pi) - \log(0 - x(1-x)p^2)\right] \tag{3.46}$$

with $\varepsilon = 4 - d$ and $\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$. (Here $\gamma \sim 0.5772$ is the Euler-Mascheroni constant.) The other two terms in the bracket can be obtained by replacing $s \to t$ and $s \to u$. Evaluating for $s = t = u = -M^2$ (which is the scale where the renormalisation conditions are defined and the counterterm has to cancel the loop divergences), we obtain

$$\begin{aligned}\Big(\cdots\Big)\Big|_{s=t=u=-M^2} &= (-i\lambda)^2 i\left(V(s)+V(t)+V(u)\right)\big|_{s=t=u=-M^2}\\ &= i(-i\lambda)^2 3V(-M^2)\\ &= -i(-i\lambda)^2\frac{3}{2(4\pi)^2}\left[\frac{2}{4-d}-\log M^2+\text{finite}\right] ,\end{aligned} \tag{3.47}$$

where the finite part does not depend on M. The counterterm δ_λ must cancel the logarithmic M^2 dependence, and since it contributes to the 4-point amplitude as

$$G^{(4)} = \left[-i\lambda + (-i\lambda)^2 i\left(V(s)+V(t)+V(u)\right) - i\delta_\lambda\right]\cdot\prod_{i=1}^{4}\frac{i}{p_i^2} \tag{3.48}$$

we conclude that

$$\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left[\frac{2}{4-d}-\log M^2+\text{finite}\right] \tag{3.49}$$

such that the renormalization condition is preserved (*i.e.* the counterterm cancels the M-dependence of the other four diagrams at the scale $s=t=u=-M^2$ at $\mathcal{O}(\lambda^2)$). The only M-dependence in this counterterm is via $\log M^2$.

Next we observe that the 2-point function is not corrected at order λ. From the Callan-Symanzik equation for $G^{(2)}$ we then conclude that γ must be of the form

$$\gamma = 0 + \mathcal{O}(\lambda^2) , \tag{3.50}$$

and hence the Callan-Symanzik equation for $G^{(4)}$ has the structure

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 4\mathcal{O}(\lambda^2)\right] G^{(4)} = 0 . \tag{3.51}$$

Because the 4-point function

$$G^{(4)} = -i\lambda\prod_{i=1}^{4}\frac{i}{p_i^2} + \mathcal{O}(\lambda^2) \tag{3.52}$$

contains terms which are at least of $\mathcal{O}(\lambda)$, the third term in (3.51) produces an $\mathcal{O}(\lambda^3)$ term which we can neglect at leading order — the leading order here is $\mathcal{O}(\lambda^2)$. In order to evaluate (3.51), we need to take the derivative of $G^{(4)}$ with respect to M. The only M-dependent term is in the counterterm (3.49). But as we have seen, $M\cdot\frac{\partial\log M^2}{\partial M}=2$, and the M-dependence of β cancels indeed (as our general argument above predicted). In any case, eq. (3.51) then becomes

$$(-i)\left[-\frac{3\lambda^2}{(4\pi)^2} + \beta(\lambda) + \mathcal{O}(\lambda^3)\right]\prod_{i=1}^{4}\left(\frac{i}{p^2}\right) = 0 , \tag{3.53}$$

from which we read off that

$$\boxed{\beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)} \ . \tag{3.54}$$

It is worth noticing that the finite parts of the counterterms are independent of M and therefore do not contribute to β or γ. If we are only interested in the β-function, we therefore do not have to care about these finite contributions, but only need to extract the regulated divergencies of the relevant diagrams.

3.3.2 The General Structure

Before we apply these methods to non-abelian gauge theories, let us understand the structure of the terms that contribute at lowest order in perturbation theory. Suppose we are interested in the β-function of a generic dimensionless coupling constant g that is associated to an n-point vertex. (There will be a β-function for each dimensionless coupling constant, and a scaling function γ for each type of field.) Consider the corresponding (connected) n-point Green's function at 1-loop. Its contributions will be of the schematic form

$$\begin{aligned} G^{(n)} &= \begin{pmatrix}\text{tree level}\\ \text{diagram}\end{pmatrix} + \begin{pmatrix}\text{1PI 1-loop}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{vertex}\\ \text{counterterm}\end{pmatrix} + \begin{pmatrix}\text{external line}\\ \text{corrections}\end{pmatrix} \\ &= \left(\prod_j \frac{i}{p_j^2}\right)\left[-ig - iB\log\frac{\Lambda}{(-p^2)} - i\delta_g + (-ig)\sum_j \left(A_j \log\frac{\Lambda^2}{(-p_j^2)} - \delta_{Z_j}\right)\right] \ . \end{aligned} \tag{3.55}$$

Here we have rescaled the external fields as

$$\phi \to Z_\phi^{-1/2}\phi = \left(1 - \frac{1}{2}\delta_{Z_\phi}\right)\phi \ , \tag{3.56}$$

and p^2 is a typical invariant built from the external momenta p_i. Our renormalisation conditions are defined for $p^2 = -M^2$, and thus the counterterm δ_g will be of the form

$$\delta_g = -B\log\frac{\Lambda^2}{M^2} + \text{finite} \tag{3.57}$$

so that it cancels the Λ-dependence of the B-term at the renormalisation point $p^2 = -M^2$. Similarly, the Z_j-counterterm must be of the form

$$\delta_{Z_j} = A_j \log\frac{\Lambda^2}{M^2} + \text{finite} \tag{3.58}$$

so that the Λ-dependences of the A_j-terms are also cancelled. At leading order, the Callan-Symanzik equation then takes the form

$$(-i)\left[M\frac{\partial}{\partial M}\left(\delta_g - g\sum_j \delta_{Z_j}\right) + \beta(g) + g\sum_j \frac{1}{2}M\frac{\partial}{\partial M}\delta_{Z_j}\right] = 0 \ , \tag{3.59}$$

where we replaced the term $n\gamma$ by $\sum_j \gamma_j$ because in general each field comes with a different γ. The third term comes from the observation that the change of the ϕ-field,

$$\phi \longrightarrow \phi - \frac{1}{2}\delta_{Z_\phi}\phi, \tag{3.60}$$

implies $\delta\eta_\phi = -\frac{1}{2}\delta_{Z_\phi}$ and then

$$\gamma_\phi = -\frac{M}{\delta M}\delta\eta_\phi = \frac{1}{2}M\frac{\partial}{\partial M}\delta_{Z_\phi}. \tag{3.61}$$

Solving eq. (3.59) for $\beta(g)$ we then find

$$\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_j \delta_{Z_j}\right). \tag{3.62}$$

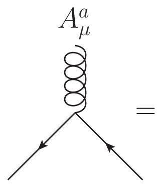
We therefore conclude that β depends only on the coefficients of the divergent logarithms, see eqs. (3.57) and (3.58)

$$\boxed{\beta(g) = -2B - g\sum_j A_j}. \tag{3.63}$$

Thus the determination of the β-function is reduced to obtaining B and A_j from an explicit 1-loop calculation.

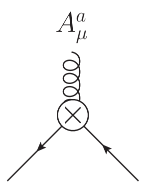
3.4 Asymptotic Freedom

With these preparations we now want to calculate the β-function $\beta(g)$ that describes the dependence of the coupling g on the renormalisation scale for $SU(N)$ Yang-Mills theory (in particular for QCD). The simplest diagram from which we can extract the lowest non-trivial order of the β-function is the 3-point function



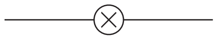
$= igt^a\gamma_\mu$

In order to do the calculation at 1-loop we will need vertex and propagator counterterms (that will absorb the corresponding divergencies). We denote the counterterm for the vertex by



$= igt^a\gamma^\mu\delta_1$

while those for the propagators are


$= i\not{p}\delta_2$

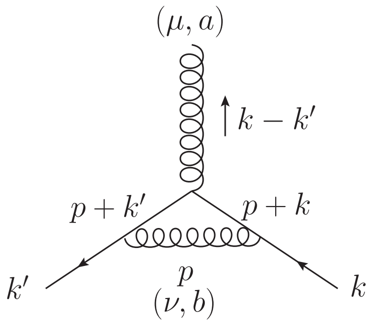

$= -i(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}\delta_3$

We will work in the **Feynman-'t Hooft gauge** which means that we set $\xi = 1$; the gluon propagator then has the form $-i\delta^{ab} g_{\mu\nu}\frac{1}{k^2}$ (*c.f.* the rules in section 2.1.2). Given our general analysis from above, it is clear that the β-function will be of the form

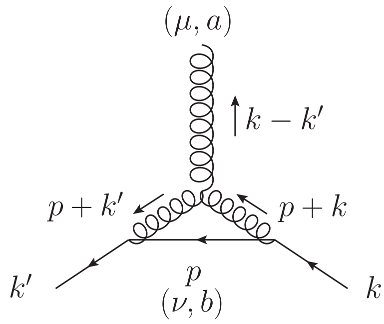
$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right) . \tag{3.64}$$

The counterterms δ_i are defined by their property that they absorb the divergencies of the loop corrections to the vertex and the propagators. The task is therefore to compute the relevant 1-loop diagrams which correct the vertex and the propagators and to extract their divergent pieces.

First we determine the corrections to the vertex at 1-loop. They arise from the diagrams



$[A]$



$[B]$

The first diagram gives

$$[A] = \int \frac{d^d p}{(2\pi)^d}(ig)^3 T^b T^a T^b \frac{(-i)}{p^2}\gamma_\nu \frac{i(\not{p} + \not{k}')}{(p+k')^2}\gamma^\mu \frac{i(\not{p} + \not{k})}{(p+k)^2}\gamma^\nu. \tag{3.65}$$

We observe that

$$\begin{aligned} T^b T^a T^b &= T^b T^b T^a + T^b[T^a, T^b] \\ &= C_R T^a + T^b i f^{abc} T^c \\ &= C_R T^a + \frac{i}{2} f^{abc}[T^b, T^c] \\ &= C_R T^a + \frac{i}{2} i f^{abc} f^{bcd} T^d \\ &= \left(C_R - \frac{1}{2}C_{\text{Adj.}}\right) T^a , \end{aligned} \tag{3.66}$$

where $C_R = T^b T^b$ is the quadratic Casimir operator of $\mathfrak{su}(N)$ evaluated in the representation R in which the fermions transform. Similarly, $C_{\text{Adj.}}$ is the quadratic Casimir evaluated in the adjoint representation, which by definition equals

$$C_{\text{Adj.}} \delta^{ad} = f^{abc} f^{dbc} = f^{abc} f^{bcd} \, . \tag{3.67}$$

For the case of $\mathfrak{su}(N)$, $C_{\text{Adj.}} = N$. The quadratic Casimir in the adjoint representation is always equal to the so-called dual Coxeter number.

In order to extract the logarithmic divergence of (3.65), we ignore k and k' relative to p — since we are only interested in the coefficient of the logarithmic divergence of the integral this is sufficient. Furthermore, we make the usual substitution

$$p^\rho p^\sigma \longrightarrow g^{\rho\sigma} \frac{p^2}{d} \tag{3.68}$$

inside the momentum integral. These two tricks simplify the calculation but do not change the leading logarithmic divergence. Up to corrections which are finite and not M-dependent, we find for (3.65):

$$\begin{aligned}
[A] &\sim \int \frac{d^d p}{(2\pi)^d} i(ig)^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \frac{1}{(p^2)^3} \gamma_\nu p^\rho \gamma_\rho \gamma^\mu p^\sigma \gamma_\sigma \gamma^\nu \\
&= \int \frac{d^d p}{(2\pi)^d} i(ig)^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \frac{1}{d} \frac{1}{(p^2)^2} \left[\gamma_\nu \gamma_\rho \gamma^\mu \gamma^\rho \gamma^\nu \right] \\
&= \int \frac{d^d p}{(2\pi)^d} i(ig)^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \frac{1}{d} \frac{1}{(p^2)^2} \left[(2-d) \gamma_\nu \gamma^\mu \gamma^\nu \right] \\
&= \int \frac{d^d p}{(2\pi)^d} i(ig)^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \frac{1}{d} \frac{1}{(p^2)^2} (2-d)^2 \gamma^\mu \\
&= g^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \gamma^\mu \underbrace{\frac{(2-d)^2}{d}}_{\to 1} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \\
&= i g^3 \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) T^a \gamma^\mu \frac{1}{(2\pi)^d} \int d\Omega_d \int_1^{\sqrt{\Lambda^2/M^2}} dp \; p^{d-5} \\
&= \frac{i g^3}{(4\pi)^2} \left(C_R - \frac{1}{2} C_{\text{Adj.}} \right) \log \left(\frac{\Lambda^2}{M^2} \right) T^a \gamma^\mu \, ,
\end{aligned} \tag{3.69}$$

where we used $\gamma_\rho \gamma^\mu \gamma^\rho = (2-d)\gamma^\mu$ twice and performed a Wick rotation to get the second last line. This completes the computation of the interesting part of the first diagram [A].

The other 1-loop correction to the vertex comes from diagram [B] for which we find

$$\begin{aligned}
[B] &= \int \frac{d^dp}{(2\pi)^d} \left(igT^b\gamma_\nu\right) \frac{i\not{p}}{p^2} \left(igT^c\gamma_\rho\right) \frac{(-i)}{(k-p)^2} \frac{(-i)}{(k'-p)^2} \times \\
&\qquad \times g f^{abc} \left[g^{\mu\nu}(2k'-k-p)^\rho + g^{\nu\rho}(-k'-k+2p)^\mu + g^{\rho\mu}(2k-p-k')^\nu\right] \\
&= \frac{g^3}{2} C_{\text{Adj.}} T^a \int \frac{d^dp}{(2\pi)^d}\, \gamma_\nu \not{p} \gamma_\rho \left(g^{\mu\nu}p^\rho - 2g^{\nu\rho}p^\mu + g^{\rho\mu}p^\nu\right) \frac{1}{(p^2)^3} \\
&= \frac{g^3}{2} C_{\text{Adj.}} T^a \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2)^2} \frac{1}{d} \Big(\gamma^\mu \underbrace{\gamma^\rho\gamma_\rho}_{=d} - 2 \underbrace{\gamma^\rho\gamma^\mu\gamma_\rho}_{=(2-d)\gamma^\mu} + \underbrace{\gamma_\nu\gamma^\nu}_{=d} \gamma^\mu \Big) \\
&= \frac{g^3}{2d} C_{\text{Adj.}} T^a (2d - 4 + 2d)\gamma^\mu \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2)^2} \\
&= \frac{ig^3}{(4\pi)^2} \frac{3}{2} C_{\text{Adj.}} T^a \gamma^\mu \log\left(\frac{\Lambda^2}{M^2}\right) ,
\end{aligned} \tag{3.70}$$

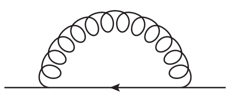
where we have used that the group theoretical part of the first line works out as

$$T^bT^cf^{abc} = \frac{1}{2}[T^b, T^c]f^{abc} = \frac{i}{2}f^{bcd}f^{abc}T^d = \frac{i}{2}C_{\text{Adj.}}T^a \ . \tag{3.71}$$

By the same arguments as above we have also ignored k and k' relative to p, and made the replacement (3.68). Adding the two contributions together we conclude that the vertex counterterm at 1-loop order is

$$\delta_1 = -\frac{g^2}{(4\pi)^2} \left(C_R + C_{\text{Adj.}}\right) \log \frac{\Lambda^2}{M^2}. \tag{3.72}$$

The calculation of the propagator counterterms is slightly more involved. There are two propagators we have to consider, namely the fermion and the gluon propagator. In the following we shall not perform all of these calculations in detail, but only sketch the relevant ideas, exhibiting the gauge group parts of the various contributions. (The details of the full calculation can be found in Section 16.5 of [PS].)
For the fermion propagator the relevant 1-loop correction comes from the term

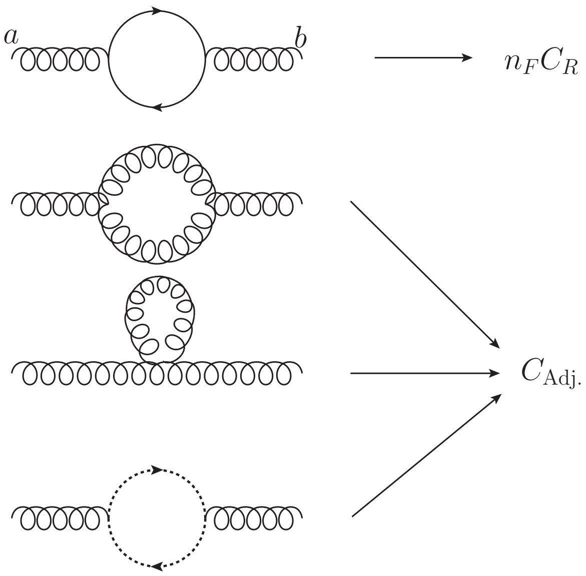


$$= \int \frac{d^4p}{(2\pi)^4} (ig)^2 \gamma^\mu T^a \frac{i(\not{p}+\not{k})}{(p+k)^2} \gamma_\mu T^a \frac{(-i)}{p^2}$$

Since $T^aT^a = C_R$, we find

$$\delta_2 = -\frac{g^2}{(4\pi)^2} C_R \log\left(\frac{\Lambda^2}{M^2}\right) . \tag{3.73}$$

The 1-loop correction to the gluon propagator is much more complicated. Now there are four 1-loop diagrams we have to consider, and their gauge group indices lead to the Casimir structures



After a lengthy calculation one finds that in order for the divergencies of these loop corrections to be cancelled by the counterterms, we need to have

$$\delta_3 = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3}C_{\text{Adj.}} - \frac{4}{3}n_F C_R\right]\log\left(\frac{\Lambda^2}{M^2}\right) . \tag{3.74}$$

Putting everything together, we then get for the complete β-function (3.64)

$$\begin{aligned}\beta(g) &= gM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right)\\ &= gM\frac{\partial}{\partial M}\left(\frac{g^2}{(4\pi)^2}\log\left(\frac{\Lambda^2}{M^2}\right)\right)\left[\frac{11}{6}C_{\text{Adj.}} - \frac{2}{3}n_F C_R\right]\\ \Rightarrow \quad & \boxed{\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_{\text{Adj.}} - \frac{4}{3}n_F C_R\right)}\end{aligned} \tag{3.75}$$

where we obtained the second line by summing eqs. (3.72)-(3.74) according to

$$-\delta_1 + \delta_2 + \frac{1}{2}\delta_3 = \frac{g^2}{(4\pi)^2}\log\left(\frac{\Lambda^2}{M^2}\right)\left[\frac{5}{6}C_{\text{Adj.}} - \frac{2}{3}n_F C_R - C_R + C_{\text{Adj.}} + C_R\right] . \tag{3.76}$$

For general $SU(N)$ Yang-Mills theory the values are

$$C_{\text{Adj.}} = N, \qquad C_{\text{R=fundamental}} = \frac{1}{2} \tag{3.77}$$

$$\Rightarrow \quad \boxed{\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_F\right)} . \tag{3.78}$$

If n_F (the number of flavours) is not too large, the β-function is negative due to the negative contributions from the Casimir of the adjoint representation $C_{\mathrm{Adj.}}$. This is for example the case for QCD, where $N = 3$ and $n_F = 3$. Note that the negative contribution to the β-function is proportional to $C_{\mathrm{Adj.}}$, which is only non-zero for non-abelian theories. (For abelian theories, such as QED, the first term vanishes, and the β-function has only a contribution from the second term, which is always positive.)

Theories with negative β-function are called **asymptotically free**, since the coupling gets weaker at large energies. In particular, such theories can be treated perturbatively at large energies. On the other hand, asymptotically free theories have the property that the coupling becomes strong at low energies or large distances; in this regime the theory must then be treated non-perturbatively.

For the demonstration that QCD is asymptotically free, Gross, Politzer and Wilzcek were awarded the Nobel Prize in Physics in 2004.

Chapter 4

Spontaneous Symmetry Breaking and the Weinberg-Salam Model of the Electroweak Interactions

The aim of this chapter is to derive the Lagrangian of the Standard Model of particle physics describing the known field content of matter particles (leptons and quarks) and the electroweak interactions (weak bosons and photons) from a classical field approach. We start by reviewing the electroweak interactions.

4.1 Electroweak Interactions

4.1.1 Characteristics of Weak Interactions

A classic example for the relevance of the weak interaction is the β-decay, i.e. $n \to p^+ e^- \bar{\nu}_e$. Weak interactions have the following characteristics:

- violate parity conservation since the weak gauge bosons couple only to left-handed fermions.
- The left-handed fermions are arranged in doublets of $SU(2)_{\mathrm{L}}$, whereas right-handed fermions transform as singlets under $SU(2)_{\mathrm{L}}$.
- The gauge group of the weak interaction is $SU(2)_{\mathrm{L}}$ and we refer to it as the **weak isospin group**.

The free fermion Lagrangian,

$$\mathcal{L}_{\text{free}}^{(\text{fermion})} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{4.1}$$

is invariant under $SU(2)_{\mathrm{L}}$ transformations

$$\Psi_L \;\longrightarrow\; \Psi'_L = e^{i\theta_a T^a}\Psi_L, \tag{4.2}$$

where $T^a = \frac{\sigma^a}{2}$ are the generators of $SU(2)$ (σ^a are the Pauli matrices) and θ_a is the gauge parameter, provided the derivatives ∂_μ is replaced by the covariant derivatives D_μ in as in the case of $SU(N)$ gauge theory seen in chapter 2. The fermions Ψ_L are defined by the decomposition of Ψ in left- and right-handed components:

$$\Psi_{\substack{L\\R}} = \frac{1 \pm \gamma_5}{2}\Psi. \tag{4.3}$$

The lepton doublet of $SU(2)_{\mathrm{L}}$ is defined as

$$\Psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}. \tag{4.4}$$

The **weak isospin** of this doublet is $T = 1/2$ and the third component T_3 of T can take values $\pm 1/2$. Considering only one quark-flavour family, the corresponding quark doublet reads analogously

$$\Psi_L = \begin{pmatrix} \Psi_u \\ \Psi_d \end{pmatrix}_L. \tag{4.5}$$

The left-handed spinor Ψ_L transforms under $SU(2)_{\mathrm{L}}$ as described in Eq. (4.2) whereas the right-handed component is invariant under $SU(2)_{\mathrm{L}}$:

$$\Psi_R = e_R \longrightarrow e'_R = e_R. \tag{4.6}$$

The gauge bosons of the $SU(2)_{\mathrm{L}}$ symmetry are denoted by W^i_μ ($i = 1, ..., 3$). $\boldsymbol{W}_\mu$ is therefore a triplet of weak isospin vectors.

$$\boldsymbol{W}_\mu = \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \end{pmatrix}. \tag{4.7}$$

The interaction Lagrangian between fermions and weak gauge bosons obtained by explicitly writing the covariant derivative D_μ in terms of the derivative ∂_μ reads

$$\mathcal{L}_{\text{int.}} \equiv \mathcal{L}^{\text{(fermions-gauge bosons)}}_{\text{int.}} = -ig\sum_{i=1}^{3} J^i_\mu W^\mu_i \qquad \text{with } J^i_\mu = \bar{L}\gamma_\mu \frac{\sigma_i}{2} L \tag{4.8}$$

where J^i_μ is the triplet of left-handed $SU(2)_{\mathrm{L}}$ currents and we used the shorthand $\Psi_L \equiv L$. Using the *charged vector bosons*

$$W^\pm = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right), \tag{4.9}$$

we can also write $\mathcal{L}_{\text{int.}}$ in terms of charged and neutral fields. The Lagrangian describing fermions, weak bosons and their interactions is therefore given by

$$\mathcal{L}_{\text{weak+fermions}} = \mathcal{L}^{\text{(fermions)}}_{\text{free}} + \mathcal{L}^{\text{(W)}}_{\text{kin.}} + \mathcal{L}_{\text{int.}} \tag{4.10}$$

$$\text{where } \mathcal{L}^{\text{(W)}}_{\text{kin.}} = -\frac{1}{4}\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu} \tag{4.11}$$

$$\text{and } \boldsymbol{W}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu - g\boldsymbol{W}_\mu \wedge \boldsymbol{W}_\nu. \tag{4.12}$$

The last term in (4.12) has the interpretation that the W-bosons couple to each other due to the non-Abelian nature of $SU(2)$. The Lagrangian (4.10) is $SU(2)_{\mathrm{L}}$-invariant.

4.1.2 Electroweak Interactions

The electroweak interaction are described by the gauge group $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$. In order to unify electromagnetic and weak interactions we cannot just add $\mathcal{L}_{\mathrm{QED}}$ to $\mathcal{L}_{\mathrm{weak}}$. The reason is that $\mathcal{L}_{\mathrm{QED}}^{\mathrm{int.}}$ is parity conserving and treats left- and right-handed fermions equally. A fact which is not compatible with the nature of the weak interactions. Indeed, $\mathcal{L}_{\mathrm{QED}}^{\mathrm{int.}}$, describing the inteaction between a photon (A_μ) and an electron (Ψ_e) of charge Q_e contains terms of the form

$$\left(Q_e \bar{\Psi}_e \gamma^\mu \Psi_e\right) A_\mu = \underbrace{Q_e \left(\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L\right)}_{=J^\mu_{\mathrm{QED}}} A_\mu, \tag{4.13}$$

which contains e_L instead of the doublet (ν_L, e_L). Therefore, this term is $U(1)_{\mathrm{e.m.}}$-invariant but it violates $SU(2)_{\mathrm{L}}$-invariance. We can solve this problem by introducing a new current J_Y^μ associated to $U(1)_{\mathrm{Y}}$ which preserve the $SU(2)_{\mathrm{L}}$ symmetry. The corresponding conserved quantity Y is called **hypercharge**. The current J_Y^μ couples to a vector gauge boson B^μ. Restricting ourselves to one family of leptons (ν, e), we write it as

$$J_Y^\mu = 2\left(J^\mu_{\mathrm{QED}} - J_3^\mu\right) = \bar{e}_R \gamma^\mu Y_{R,e} e_R + \bar{\nu}_R \gamma^\mu Y_{R,\nu} \nu_R + \underbrace{\bar{e}_L \gamma^\mu Y_{L,\chi} e_L + \bar{\nu}_L \gamma^\mu Y_{L,\chi} \nu_L}_{=\bar{\chi}_L \gamma^\mu Y_{L,\chi} \chi_L}, \tag{4.14}$$

where $Y_{R,e}$ ($Y_{R,\nu}$) is the hypercharge of e_R (ν_R) and $Y_{L,\chi}$ is the hypercharge of the doublet $\chi_L = (\nu_L, e_L)$. The third component of the weak current defined in eq.(4.8) is

$$J^3_\mu = \bar{\chi}_L \gamma_\mu \frac{\sigma_3}{2} \chi_L = \frac{1}{2} \bar{\nu}_L \gamma_\mu \nu_L - \frac{1}{2} \bar{e}_L \gamma_\mu e_L, \tag{4.15}$$

and the QED current reads

$$J_\mu^{\mathrm{QED}} = Q_e \left(\bar{e}_R \gamma_\mu e_R + \bar{e}_L \gamma_\mu e_L\right) = J^3_\mu + \frac{1}{2} J^Y_\mu . \tag{4.16}$$

Matching both sides in Eq. (4.14), we get

$$\begin{aligned} Y_{R,e} &= 2Q_e, & Y_{R,\nu} &= 0 \\ Y_{L,e} &= 2Q_e + 1, & Y_{L,\nu} &= -1. \end{aligned}$$

We conclude that the hypercharge Y, as suggested in eq.(4.14), has the following form:

$$\boxed{Y = 2(Q_e^{(\mathrm{QED})} - T_3).} \tag{4.17}$$

This can immediately be verified using

$$\begin{aligned} T_3(e_R) = T_3(\nu_R) &= 0 \qquad \text{(singlet)} \\ \left.\begin{aligned} T_3(\nu_L) &= +\frac{1}{2} \\ T_3(e_L) &= -\frac{1}{2} \end{aligned}\right\} &\text{(doublet).} \end{aligned}$$

We can thus write down the following $SU(2)_\text{L} \times U(1)_\text{Y}$-invariant Lagrangian:

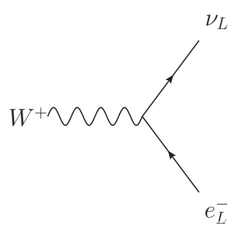
$$\mathcal{L}_\text{int.}^\text{electroweak} = -igJ_\mu^i W^{\mu,i} - i\frac{g'}{2}J_\mu^Y B^\mu \qquad (i = 1, 2, 3) \tag{4.18}$$

with the triplet W_μ^i of weak gauge bosons related to $SU(2)_L$, the vector boson B_μ related to $U(1)_\text{Y}$. The corresponding couplings are g and g', respectively. The complete Lagrangian $\mathcal{L}_\text{int.} + \mathcal{L}_\text{free-fermion}$ can be decomposed into left- and right-handed components:

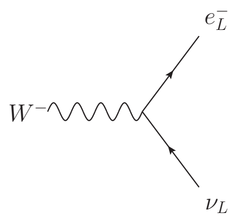
$$\sum_{\chi_L} \bar{\chi}_L \gamma^\mu \left(i\partial_\mu + g\frac{\boldsymbol{\sigma}}{2}\boldsymbol{W}_\mu + g'\frac{Y_L}{2}B_\mu \right) \chi_L + \sum \bar{\Psi}_R \gamma^\mu \left(i\partial_\mu + 0 + g'\frac{Y_R}{2}B_\mu \right) \Psi_R, \tag{4.19}$$

The particle spectrum that arises from the charged and neutral currents of this theory looks as follows:

- Charged bosons responsible for charged interactions: $W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right)$. The "+"-current is $J_\mu^+ = \bar{\chi}_L \gamma_\mu \sigma_+ \chi_L = \bar{\nu}_L \gamma_\mu e_L$ where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The interaction gives rise to a vertex



The current $J_\mu^- = \bar{\chi}_L \gamma_\mu \sigma_- \chi_L = \bar{e}_L \gamma_\mu \nu_L$ gives rise to an interaction



- Neutral fields W^3_μ and B_μ responsible for the neutral interactions. The neutral vector bosons A_μ and Z_μ corresponding to the photon and the Z^0 (mass eigenstates) are linear combinations of W^3_μ and B_μ:

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} \tag{4.20}$$

where θ_W is called the **Weinberg angle** that was first introduced by Glashow (1961). We can thus write the neutral part of the interaction Lagrangian:

$$\begin{aligned} \mathcal{L}^{(\text{neutral})}_{\text{int.}} &= -igJ^3_\mu W^{3\mu} - i\frac{g'}{2}J^Y_\mu B^\mu \\ &= -i\underbrace{\left(g\sin\theta_W J^3_\mu + \frac{g'}{2}\cos\theta_W J^Y_\mu\right)}_{\equiv\alpha_\mu} A^\mu - i\left(g\cos\theta_W J^3_\mu - \frac{g'}{2}\sin\theta_W J^Y_\mu\right) Z^\mu. \end{aligned} \tag{4.21}$$

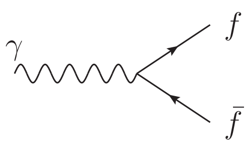
Since $\alpha_m u$ has to be related to the electric charge, i.e.

$$\alpha_m u \overset{!}{=} eJ^{\text{e.m.}}_\mu = e\left(J^3_\mu + \frac{1}{2}J^Y_\mu\right), \tag{4.22}$$

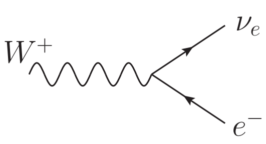
we infer

$$g\sin\theta_W = g'\cos\theta_W = e \qquad \text{and} \qquad \tan\theta_W = \frac{g'}{g}. \tag{4.23}$$

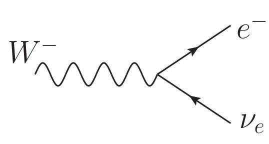
The Feynman rules for vertices are as follows:



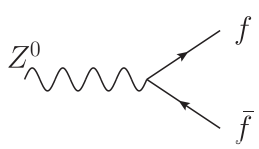
$= -ieq_f\gamma^\mu$



and



$= -\frac{ig}{\sqrt{2}}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)$



$= -\frac{ig}{\cos\theta_W}\gamma^\mu\frac{1}{2}(c^f_V - c^f_A\gamma_5)$

with the vector and axial couplings

$$c_V^f = T_f^3 - 2\sin^2\theta_W\, q_f \tag{4.24}$$
$$c_A^f = T_f^3. \tag{4.25}$$

Therefore, $\mathcal{L}$ constructed with is invariant under

Note that so far W^+, W^-, Z^0 and A^μ are all massless. This is in contradiction with experiments which show that the masses are $m_{W,Z} \sim 100$ GeV. How can we add mass terms for the electroweak gauge bosons Z and W? We cannot just introduce a mass term of the type $\mathcal{L}_M = -\frac{m^2}{2} W_\mu W^\mu$ to the $SU(2)_{\mathrm{L}} \times U(1)$ gauge invariant Lagrangian because this introduction would break gauge invariance. Since we want to maintain gauge invariance, we need another method. The next section deals with a way to do this through the Higgs mechanism using spontaneous symmetry breaking.

4.2 Spontaneous Symmetry Breaking

4.2.1 Discrete Symmetries (2 Examples)

Some physical systems have symmetries which the ground state (the state of minimal energy) does not have.

Example 1: Falling Needle

Consider the example of a balanced needle compressed by a force F. As long as the force is smaller than some critical value F_c, the needle stays in the configuration $x = y = 0$ and the system is rotationally symmetric with respect to rotations around the z-axis. This is the ground state. However, if $F > F_c$, the needle bends into one particular position. The rotational symmetry is then broken and we have an infinite number of ground states, all of which are equivalent and related by rotations around the z-axis. The situation is sketched in fig. (4.1).

We can characterize spontaneous symmetry breaking as follows:

- A parameter of the system assumes a critical value.
- Beyond that value, the symmetric physical state becomes unstable.
- The new ground state is chosen arbitrarily amongst all equivalent ground states. The ground state chosen is not invariant under the original symmetry of the system.

Example 2: Real Scalar Field

As a second example consider a real scalar field ϕ in ϕ^4-theory:

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu\phi)^2 - \left(\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right). \tag{4.26}$$

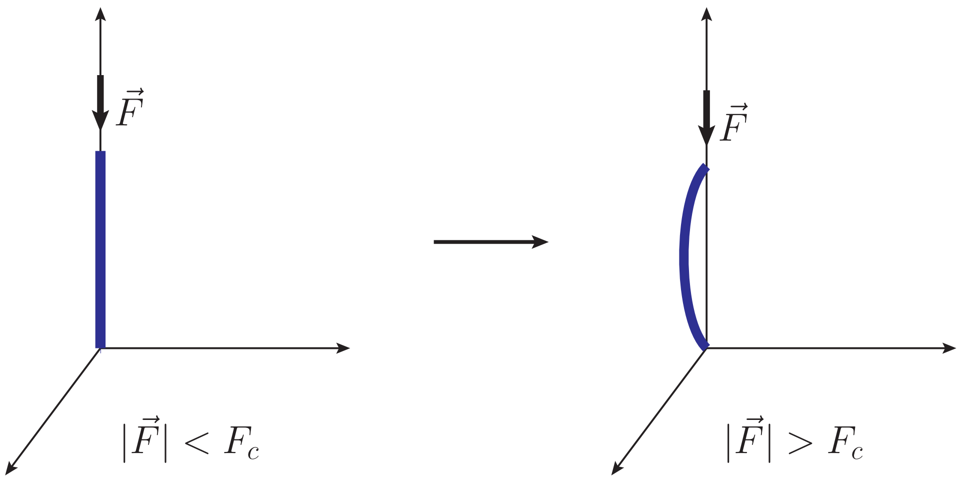


Figure 4.1: Needle standing in an unstable position.

This Lagrangian is invariant under $\phi \to -\phi$. There are essentially two very different forms of the potential V which are sketched in fig. (4.2): In the first case, the vacuum characterized by $\frac{\partial V}{\partial \phi} = 0$ is given by $\phi = 0$ and it is obviously stable. In the second case, $\frac{\partial V}{\partial \phi} = 0$ leads to two possible solutions:

1. $\phi = 0$ (local maximum),
2. $\phi = \pm\sqrt{-\mu^2/\lambda} \equiv v$ (two vacua, two local minima).

We analyze the physics close to one of the minima by considering the series expansion around the minimum, i.e.

$$\phi(x) = v + \eta(x) \tag{4.27}$$

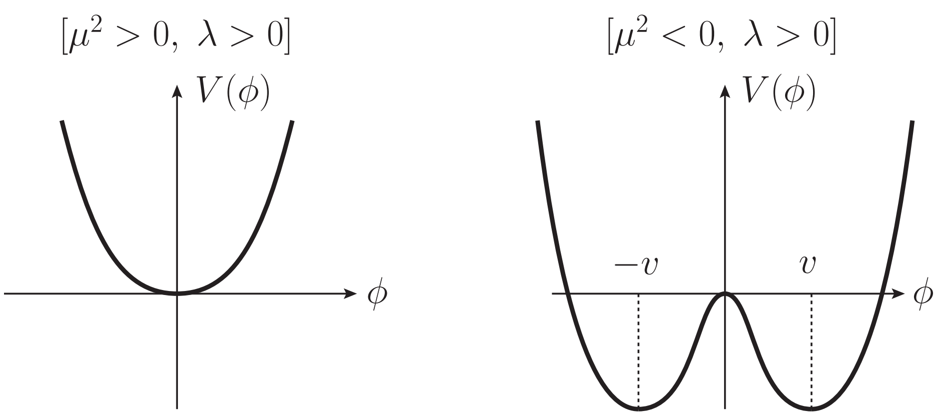


Figure 4.2: The two different regimes of the quartic potential.

where $\phi(x)$ is the classical degree of freedom and $\eta(x)$ is the quantum fluctuation of the field configuration close to v (perturbative degree of freedom). In general, we are not able to use ϕ and solve the theory generally. Instead we do perturbation theory and calculate fluctuations close to a stable minimum energy. For that we use a Lagrangian in terms of the perturbative degrees of freedom.

Choosing $\phi(x) = v + \eta(x)$ (i.e. expanding around the minimum on the positive ϕ-axis) breaks the symmetry of $\mathcal{L}$. Performing this replacement in the Lagrangian yields a transformed Lagrangian $\mathcal{L}'[\eta]$ which is not invariant under $\eta \to -\eta$ anymore. Thus the symmetry is broken by the special choice $\phi_0 = v$. The new Lagrangian reads explicitly

$$\mathcal{L}'[\eta] = \frac{1}{2}(\partial_\mu \eta)^2 - \lambda v^2 \eta^2 + \mathcal{O}(\eta^3, \eta^4) \tag{4.28}$$

where the second term is a mass term (with the correct sign) giving a mass $m_\eta = \sqrt{2\lambda v^2}$ to η. The $\mathcal{O}(\eta^3, \eta^4)$ terms describe η self-interactions. The conclusion we can draw here is that spontaneous symmetry breaking generates a mass term for η.

4.2.2 Spontaneous Symmetry Breaking of a Global Gauge Symmetry

Consider scalar ϕ^4-theory again. This time, we consider a complex scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$. The Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* \underbrace{-\mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2}_{=-V(\phi,\phi^*)} \tag{4.29}$$

is invariant under global $U(1)$ gauge transformations

$$\phi(x) \longrightarrow \phi'(x) = e^{i\Lambda}\phi(x), \qquad \Lambda = \text{const.} \tag{4.30}$$

Consider again the case of $\mu^2 < 0$, $\lambda > 0$ where $\mathcal{L}$ has a mass term with positive sign: the ground state is obtained as the minimum of $V(\phi)$:

$$\frac{\partial V}{\partial \phi} = \mu^2 \phi^* + 2\lambda \phi^* (\phi^* \phi) = 0. \tag{4.31}$$

If $\mu^2 < 0$, then the Lagrangian has a mass term with the "wrong" sign. The minimum is at $|\phi|^2 = -\frac{\mu^2}{2\lambda}$. In terms of ϕ_1, ϕ_2 the vaucum is a circle of radius v in the ϕ_1-ϕ_2-plane such that

$$\phi_1^2 + \phi_2^2 = v^2 \qquad \text{with } v^2 = -\frac{\mu^2}{\lambda}. \tag{4.32}$$

The situation is sketched in fig. (4.3). The tangent to the circle of vacua is called ξ. The

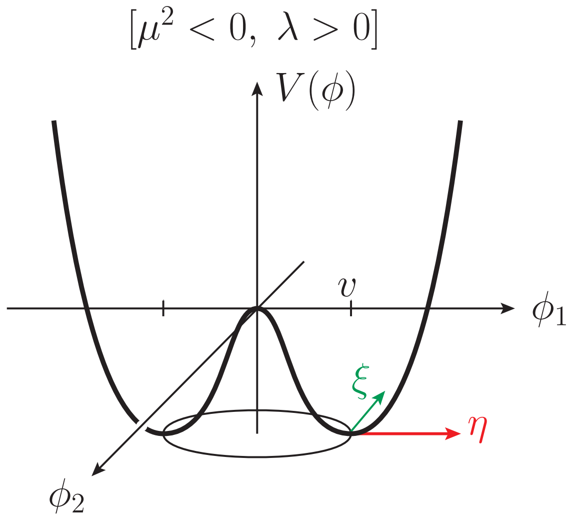


Figure 4.3: The ϕ^4-potential for a complex scalar field ϕ.

radial is denoted by η. We consider the minimum given by $\phi_1 = v$, $\phi_2 = 0$ and expand ϕ around this minimum:

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x)) \tag{4.33}$$

where we think of $\eta(x) + i\xi(x)$ as being the quantum fluctuation close to the minimum (the perturbative degree of freedom). The Lagrangian in terms of η, ξ reads

$$\mathcal{L}[\eta, \xi] = \underbrace{\frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \eta)^2}_{\text{kinetic terms for } \xi,\, \eta} \underbrace{-\frac{1}{2}m_\eta^2}_{\mu^2} \eta^2 + \text{const.} + \mathcal{O}(\eta^3, \xi^2) \tag{4.34}$$

where the higher order terms are again interaction terms that we do not care about for the moment (we are interested in the generated mass term).

The particle spectrum is as follows: we have a mass term for η which is $-\frac{1}{2}m_\eta^2\eta^2$ with $m_\eta^2 = -2\mu^2$. For both η and ξ we have kinetic terms, so both fields are dynamical degrees of freedom. But there is no mass term for ξ. This can be interpreted as follows: the potential V is tangential in ξ-direction implying a massless mode. It costs instead energy to displace η because the potential is not flat in the η direction, so this is a massive mode. This is an example of the general

Goldstone Theorem:
Massless particles ("Goldstone bosons") occur whenever a continuous symmetry is spontaneously broken.

The particular case of a global $U(1)$ symmetry being spontaneously broken gives rise to one massless Goldstone boson ξ (here: a scalar).

Proof of Goldstone's Theorem

We prove the Goldstone theorem in the classical field theory approach. Consider a theory involving several fields $\phi^a(x)$ described by

$$\mathcal{L} = \mathcal{L}_{\text{kin.}} - V(\phi^a) \tag{4.35}$$

and let ϕ_0^a be a constant field that minimizes V. For each a we have

$$\frac{\partial}{\partial \phi_a} V(\phi^a)\bigg|_{\phi^a(x)=\phi_0^a} = 0. \tag{4.36}$$

Expanding V about the minimum ϕ_0^a, we get

$$V(\phi^a) = V(\phi_0^a) + \frac{1}{2}\underbrace{\left(\phi^a - \phi_0^a\right)\left(\phi^b - \phi_0^b\right)}_{\text{quadratic term}}\left(\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\right)_{\phi_0^a} + \ldots \tag{4.37}$$

The coefficient of the quadratic term is the **mass matrix**

$$\left(\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\right)_{\phi_0^a} = m_{ab}^2. \tag{4.38}$$

It is a symmetric matrix whose eigenvalues are the masses of the fields. If ϕ_0^a is a minimum, then all eigenvalues of m_{ab}^2 are ≥ 0.

In order to prove Goldstone's theorem, we must show that every continuous symmetry of $\mathcal{L}$ that is not a symmetry of ϕ_0^a (which is the stable minimum) gives rise to a zero eigenvalue of this mass matrix. We consider a general continuous symmetry transformation of $\mathcal{L}$ given by

$$\phi^a \longrightarrow \phi^a + \alpha\Delta^a(\phi) \tag{4.39}$$

where α is an infinitesimal parameter and $\Delta^a(\phi)$ is a function of the ϕ's. If we specialize to constant fields, then the derivative terms in $\mathcal{L}$ vanish and only the potential V must be invariant under the transformation (4.39). We have then

$$V(\phi^a) = V\left(\phi^a + \alpha\Delta^a(\phi)\right). \tag{4.40}$$

By means of the expansion of the right-hand side around $\alpha = 0$, this is equivalent to

$$\Delta^a(\phi)\frac{\partial}{\partial\phi^a}V(\phi^a) = 0. \tag{4.41}$$

Differentiating this equation with respect to ϕ^b and choosing $\phi^a = \phi^b = \phi_0$, we find

$$0 = \left(\frac{\partial\Delta^a(\phi)}{\partial\phi^b}\right)_{\phi_0}\underbrace{\left(\frac{\partial}{\partial\phi^a}V(\phi^a)\right)_{\phi^a=\phi_0}}_{=0 \text{ since } \phi_0 \text{ minimum}} + \Delta^a(\phi_0)\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V(\phi^a)\right)_{\phi_0}. \tag{4.42}$$

There are thus two possibilities:

1. If $\Delta^a(\phi_0) = 0$ we are in the situation where the transformation leaves ϕ_0 unchanged, i.e. the symmetry is also respected by the ground state. This case is trivial (no symmetry breaking).

2. If $\Delta^a(\phi_0) \neq 0$ we have spontaneous symmetry breaking. In this case we have

$$\frac{\partial^2}{\partial\phi^a\partial\phi^b}V(\phi^a)\bigg|_{\phi^a=\phi^0}\Delta^a(\phi_0) = 0 \tag{4.43}$$

   which is equivalent to $m^2_{ab}\Delta^a(\phi^0) = 0$. Thus $\Delta^a(\phi^0)$ is our desired vector with (squared) mass eigenvalue 0. In this case, $\Delta^a(\phi^0)$ are our massless Goldstone boson candidates.

This proves the theorem at the classical level.

4.2.3 Spontaneous Symmetry Breaking of a Local Gauge Symmetry and the Abelian Higgs Mechanism

Consider a complex scalar field ϕ and a local $U(1)$ gauge transformation

$$\phi \longrightarrow \phi' = e^{i\alpha(x)}\phi. \tag{4.44}$$

For the corresponding Lagrangian $\mathcal{L}$ to be invariant under this $U(1)$ transformation, we know from QED that we have to impose two conditions. The partial derivative has to be replaced by a covariant derivative and the gauge field has to satisfy a particular transformation behaviour:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA_\mu \qquad \text{and} \qquad A_\mu \longrightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x). \tag{4.45}$$

Starting from the Lagrangian

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2, \tag{4.46}$$

we obtain the following $U(1)$-invariant Lagrangian:

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.47}$$

Consider the case $\mu^2 < 0$, $\lambda > 0$ and a small perturbation close to the ground state vacuum $v^2 = -\frac{\mu^2}{\lambda}$:

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right] \tag{4.48}$$

where $\eta(x)$ and $i\xi(x)$ are the quantum fluctuations around the ground state v. The Lagrangian becomes

$$\mathcal{L}[\eta,\xi] = \frac{1}{2}(\partial_\mu\xi)^2 + \frac{1}{2}(\partial_\mu\eta)^2 - \frac{1}{2}(2v^2\lambda^2)\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu - \\ - evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{ interactions.} \tag{4.49}$$

First of all, we observe the presence of kinetic terms for both η and ξ, a mass term for η, and a mass term for A_μ. This means that we have the following a priori particle spectrum:

- a massless Goldstone boson ξ,
- a massive scalar particle η with $m_\eta = \sqrt{2}\lambda v$,
- a massive $U(1)$ vector field A_μ with $m_A = ev$.

Furthermore, an off-diagonal term of the form $A_\mu\partial^\mu\xi$ appears in the Lagrangian (4.49). By giving a mass to A_μ, we raised the number of degrees of freedom from 2 to 3 since a massive A_μ has an additional longitudinal degree of freedom. This is not satisfactory: it cannot be right that just by writing $\phi(x) = v + \eta(x) + i\xi(x)$ (i.e. by changing the parametrization), we can create a new degree of freedom. The interpretation thus has to be as follows: in $\mathcal{L}[v,\eta,\xi]$ the fields do not correspond to distinct physical particles. Some of the fields are unphysical. If this interpretation is correct, then we should be able to find a particular gauge transformation which eliminates this unphysical field. Indeed, this can be achieved. To this end, we note that $\phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\xi(x)\right)$ is the first order in ξ of the expansion of

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x)\right)e^{i\xi(x)/v}. \tag{4.50}$$

This suggests that the field ξ is actually a gauge parameter of the form $\alpha(x)$. To show that this is indeed correct, we perform a gauge transformation on the initial fields to obtain a different set of real fields $h(x)$, $\theta(x)$, $\widetilde{A}_\mu(x)$ (with $\xi(x)$ given as $\alpha(x) = \theta(x)$ here):

$$\phi(x) \longrightarrow \frac{1}{\sqrt{2}}(v + h(x))e^{i\theta(x)/v} \tag{4.51}$$

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{ev}\partial_\mu\theta(x) = \widetilde{A}_\mu \tag{4.52}$$

where $\widetilde{A}_\mu$ is a field which is gauge equivalent to A_μ. Because $\theta(x)$ is just a gauge parameter now, it should completely drop out if we write the Lagrangian in terms of these fields. We can check that this is indeed the case:

$$\mathcal{L}[h,\widetilde{A}_\mu] = \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2h^2 + \frac{1}{2}e^2v^2\widetilde{A}_\mu\widetilde{A}^\mu - \lambda vh^3 - \\ - \frac{1}{4}\lambda h^4 + \frac{1}{2}e^2\widetilde{A}_\mu^2h^2 + ve^2\widetilde{A}_\mu^2h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.53}$$

This is the Lagrangian which has been proposed by Peter Higgs (Englert, Brout) in 1964. The particle spectrum of this Lagrangian is as follows:

- no $\theta(x)$ field (the Goldstone field has been "eaten" by A_μ),
- a massive, scalar field h ("Higgs field") with $m_h = \sqrt{2\lambda v^2}$,
- a massive $U(1)$ vector field $\widetilde{A}_\mu$ with $m_{\widetilde{A}} = ev$.

We have a conservation of the degrees of freedom between the field content $[\phi, A]$ in the original $\mathcal{L}$ and the field content $[h, \widetilde{A}_\mu]$ in the Lagrangian (4.53):

$\mathcal{L}[\phi, A_\mu]$:	complex scalar field ϕ :	$\longrightarrow$ 2 degrees of freedom,
	massless vector field A_μ :	$\longrightarrow$ 2 degrees of freedom,
$\mathcal{L}[h, \widetilde{A}_\mu]$:	real scalar field h :	$\longrightarrow$ 1 degree of freedom,
	massive vector field $\widetilde{A}_\mu$:	$\longrightarrow$ 3 degrees of freedom.

The Goldstone boson in (4.49) was therefore just a spurious degree of freedom. It has given a longitudinal degree of freedom to A_μ. Now A_μ has "eaten" the Goldstone boson and became the massive field $\widetilde{A}_\mu$.

In this mechanism, called the Abelian Higgs mechanism, the Goldstone bosons are not independent fields and can be gauged away. The Goldstone fields which are at the same time the gauge parameters $\theta(x)$, are chosen such that $\phi(x)$ is real valued at every point x. Furthermore, they do not appear any more in $\mathcal{L}$, when we rewrite $\mathcal{L}$ in terms of the transformed fields using Eq. (4.53).

This gauge choice is called **unitary gauge**. In this gauge, $\mathcal{L}$ describes just h and $\widetilde{A}_\mu$ where $\widetilde{A}_\mu$ is a massive vector boson and h a massive scalar.

This picture will be clarified by studying the quantization of theories with spontaneously broken symmetries.

4.2.4 Spontaneous Symmetry Breaking of a Local $SU(2) \times U(1)$ Gauge Symmetry: Non-Abelian Higgs Mechanism

We want to find a mechanism which gives masses to the $W^\pm$ and Z bosons. This mechanism should also ensure that the photon γ is massless. Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \tag{4.54}$$

with ϕ being a doublet of scalar fields:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.55}$$

with $Y = 1$ (choice by Weinberg, 1967). The Lagrangian is invariant if we replace ∂_μ by the $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$ covariant derivative D_μ, i.e.

$$\partial_\mu \phi \longrightarrow D_\mu \phi = \left(\partial_\mu - ig\frac{\boldsymbol{\sigma}}{2}\boldsymbol{W}_\mu - ig'\frac{Y}{2}B_\mu \right) \phi \tag{4.56}$$

where $\boldsymbol{W}_\mu$ are the $SU(2)_\mathrm{L}$ gauge bosons related to weak isospin (T_3) and B_μ is the $U(1)_\mathrm{Y}$ gauge boson related to hypercharge.

The $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$ transformation acts on ϕ as follows:

$$\phi \longrightarrow e^{i\alpha_a T^a} e^{i\beta/2} \phi \tag{4.57}$$

where α_a and β are the group parameters for $SU(2)_\mathrm{L}$ and $U(1)_\mathrm{Y}$, respectively. To generate gauge boson masses, we break the symmetry spontaneously. Choosing

$$\mu^2 < 0, \quad \lambda > 0, \quad v^2 = -\frac{\mu^2}{\lambda} \quad \text{and} \quad \phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \quad \phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.58}$$

where $\phi(x)$ is written as an expansion around the vacuum v, the symmetry under $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$ is spontaneously broken. The obtained theory will contain

- one massless gauge boson associated with $Q = T_3 + \frac{Y}{2}$,
- three massive gauge bosons (Higgs mechanism),
- four generators, three of which will independently break the symmetry and generate three massless Goldstone modes. The Goldstone modes are eaten by the gauge bosons in order for the latter to aquire a mass (additional degree of freedom).

Let us show that although the choice of vacuum (4.58) breaks $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$, the electromagnetic $U(1)_Q$ symmetry is preserved by this vacuum. To this end observe that the chosen vacuum expectation value ϕ_0 satisfies $Q\phi_0 = 0$ since ϕ_0 has $Y = 1$ and $T_3 = -\frac{1}{2}$ and

$$\phi_0' = e^{iQ\alpha(x)} \phi_0 = e^{i0} \phi_0 = \phi_0 \tag{4.59}$$

for any value of $\alpha(x)$ generating the local $U(1)_\mathrm{Q}$ symmetry of electromagnetic interactions given by

$$\phi(x) \longrightarrow \phi'(x) = e^{iQ\alpha(x)} \phi(x). \tag{4.60}$$

The particle spectrum of the theory can be studied by inserting the vacuum value $\phi_0 = \frac{1}{\sqrt{2}}(0, v)$ from (4.58) into the kinetic term $(D_\mu\phi)^\dagger(D^\mu\phi)$. This term will generate a mass term for the weak gauge bosons and it will leave the photon massless as shown below. Using $\boldsymbol{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$, we obtain

$$\begin{aligned} (D_\mu\phi)^\dagger(D^\mu\phi) &= \left| \left(-ig\frac{\boldsymbol{\sigma}}{2}\boldsymbol{W}_\mu - i\frac{g'}{2}B_\mu \right) \phi \right|^2 \\ &= \left| \frac{1}{8} \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\ &= \frac{1}{8} v^2 g^2 \left(|W_\mu^1|^2 + |W_\mu^2|^2 \right) + \frac{1}{8} v^2 \left(g'B_\mu - gW_\mu^3 \right) \left(g'B^\mu - gW^{3\mu} \right). \end{aligned} \tag{4.61}$$

Using

$$\begin{aligned} Z_\mu &= \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}} \quad \text{for the neutral gauge boson } Z^0, \\ W_\mu^\pm &= \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \quad \text{for the } W^\pm \text{ gauge bosons,} \\ A_\mu &= \frac{g'W_\mu^3 + gB_\mu}{\sqrt{g^2+g'^2}} \quad \text{for the photon field as defined before,} \end{aligned}$$

Eq. (4.61) becomes

$$\begin{aligned} (D_\mu\phi)^\dagger(D^\mu\phi) &= \left(\frac{1}{2}gv\right)^2 W_\mu^+W^{-\mu} + \frac{1}{8}v^2\left(g'B_\mu - gW_\mu^3\right)^2 + \underbrace{0}_{=M_A^2}\left(g'W_\mu^3 + gB_\mu\right)^2 \\ &= M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z Z_\mu Z^\mu \qquad (4.62) \\ \text{with} \quad & \boxed{M_W = \frac{1}{2}gv, \quad M_Z = \frac{1}{2}v\sqrt{g^2+g'^2}}. \qquad (4.63) \end{aligned}$$

As before, we denote

$$\frac{g'}{g} = \tan\theta_W. \qquad (4.64)$$

Note that the masses M_W, M_Z are predicted by the theory. The fact that $M_A = 0$ is a consistency check. One finds

$$M_W \sim 80.4 \text{ GeV}, \qquad M_Z \sim 91.2 \text{ GeV}, \qquad v = 246 \text{ GeV}. \qquad (4.65)$$

4.2.5 The Electroweak Standard Model Lagrangian

We are now in the position to write down the electroweak part of the Standard Model which has been discovered by S. L. Glashow, S. Weinberg and A. Salam in 1966 (Nobel prize 1979). The Lagrangian for this spontaneously broken $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ model reads

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}_{\text{field}} + \mathcal{L}_{\text{matter}} &+ \mathcal{L}_{\text{Higgs, field}} + \mathcal{L}_{\text{Higgs, matter}}, \\
\mathcal{L}_{\text{field}} &= -\frac{1}{4}\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
&\quad \text{where } W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\varepsilon^{ijk}W^j_\mu W^k_\nu \\
&\qquad\qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \\
\mathcal{L}_{\text{matter}} &= \sum_L \bar{L}\gamma^\mu \left(i\partial_\mu + g\frac{\boldsymbol{\sigma}}{2}\boldsymbol{W}_\mu + g'\frac{Y}{2}B_\mu\right) L + \sum_R \bar{R}\gamma^\mu \left(i\partial_\mu + g'\frac{Y}{2}B_\mu\right) R \\
\mathcal{L}_{\text{Higgs, field}} &= \left| \left(i\partial_\mu + g\frac{\boldsymbol{\sigma}}{2}\boldsymbol{W}_\mu + g'\frac{Y}{2}B_\mu\right)\phi \right|^2 - V(\phi) \\
&\quad \text{where } V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \\
\mathcal{L}_{\text{Higgs, matter}} &= -G_1\left(\bar{L}\phi R + \bar{R}\bar{\phi}L\right) - G_2\left(\bar{L}\phi_c R + \bar{R}\bar{\phi}_c L\right) + \text{h.c.} \\
&\quad \text{where } \phi_c = i\sigma^2\phi^*
\end{aligned}
$$

where L and R in $\mathcal{L}_{\text{matter}}$ are the usual left-handed fermion doublet and right-handed fermion singlet, respectively. So $\mathcal{L}_{\text{matter}}$ contains kinetic terms for leptons and quarks and their interactions with the gauge bosons. The $\mathcal{L}_{\text{Higgs, field}}$-part is responsible for the spontaneous symmetry breaking. It contains $W^\pm$, Z^0, γ and Higgs masses and their couplings with the potential $V(\phi)$ as anticipated in the previous section. The last part, $\mathcal{L}_{\text{Higgs, matter}}$, contains lepton and quark masses and their couplings to the Higgs. G_1 is the Yukawa coupling for $T_3 = -\frac{1}{2}$ and G_2 is the Yukawa coupling for $T_3 = +\frac{1}{2}$. For example, one finds for the leptons

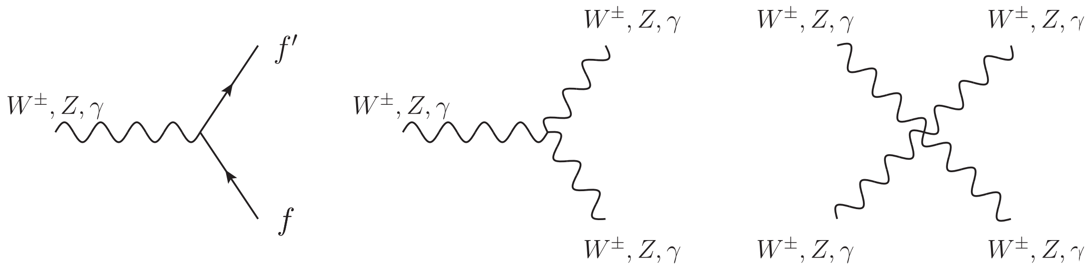
$$M_e = \frac{G_e v}{\sqrt{2}} \tag{4.66}$$

where G_e is arbitrary. The Higgs mass

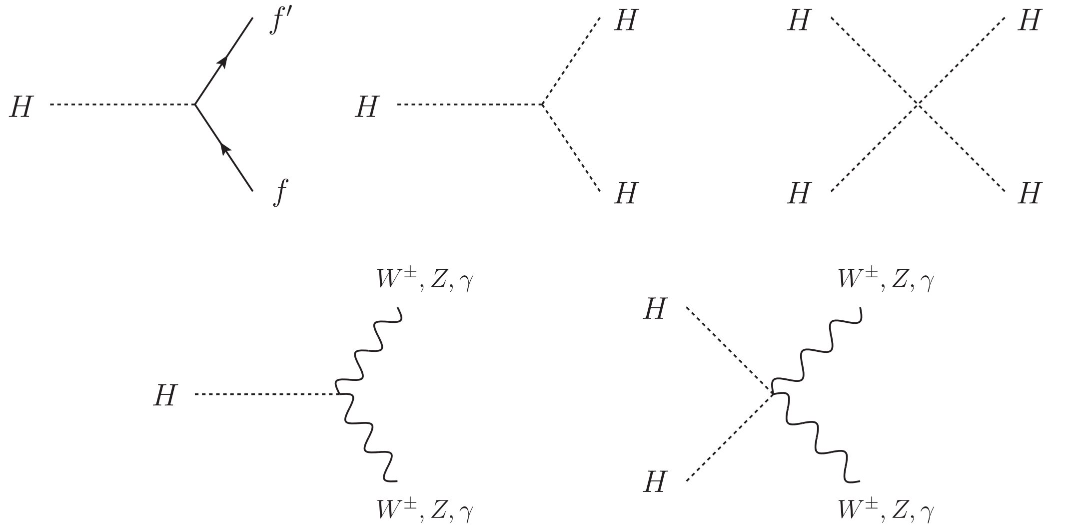
$$M_h = v\sqrt{2\lambda} \tag{4.67}$$

is also arbitrary. Fermion and Higgs masses are arbitrary parameters of the model. The following couplings are possible (→ exercise):

Without Higgs:



Including Higgs:



Note that not every arbitrary combination of gauge bosons in these vertices is allowed. For example, the photon can only interact with electrically charged fields.

This concludes our discussion of spontaneous symmetry breaking at the classical level. We will now turn to questions that deal with the quantization of such theories.

Chapter 5

Quantization of Spontaneously Broken Gauge Theories

In the last chapter we have seen how the gauge bosons aquire a mass via the Higgs mechanism with the classical field approach. We argued that by imposing a special gauge (the *unitary gauge* or *physical gauge*), because only physical fields appear, i.e. h and $\widetilde{A}$ in the case of $U(1)$ symmetry, the Goldstone bosons are "eliminated" while the gauge bosons acquire a mass. We ask now whether the unitary gauge also works at higher orders. We would also like to quantize these spontaneously broken theories in gauges where the Goldstone bosons are present, so we can study their effects.

For this purpose we will consider the functional approach with the Faddeev-Popov gauge fixing method for theories with spontaneously broken gauge symmetries. We will define a class of gauges which are called $\mathbf{R}_\xi$ **gauges** (R stands for renormalizability) and which contain Goldstone bosons explicitly. The R_ξ gauges will finally be linked to the renormalizability of spontaneously broken gauge theories.

5.1 The Abelian Model

We start by considering a spontaneously broken $U(1)$ gauge theory described by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - V(\phi) \qquad (5.1)$$
$$\text{with } D_\mu = \partial_\mu + ieA_\mu .$$

Here $\phi(x)$ is a complex scalar field which we write as $\phi(x) = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$. We want to analyze $\mathcal{L}$ in terms of ϕ^1, ϕ^2. We consider a local $U(1)$ transformation on (ϕ^1, ϕ^2),

$$\delta\phi^1 = -\alpha(x)\phi^2, \quad \delta\phi^2 = \alpha(x)\phi^1, \quad \delta A_\mu = -\frac{1}{e}\partial_\mu\alpha(x). \qquad (5.2)$$

The potential $V(\phi) = \mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2$ generates the spontaneous breaking of the symmetry (choose $\mu^2 < 0$, $\lambda > 0$). We write the vacuum as $\phi^1 = v$, $\phi^2 = 0$ with $v = (-\mu^2/\lambda)^{1/2}$, which gives the following parametrization of the fields close to the vacuum:

$$\phi^1(x) = v + h(x), \qquad \phi^2(x) = \varphi(x). \tag{5.3}$$

Here, $h(x)$ is the Higgs field and $\varphi(x)$ is the field corresponding to the Goldstone boson. Eq. (5.1) becomes

$$\mathcal{L}[A, h, \varphi] = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu h - eA_\mu\varphi)^2 + \frac{1}{2}\left(\partial_\mu\varphi + eA_\mu(v + h)\right)^2 - V(\phi). \tag{5.4}$$

We observe that $\mathcal{L}$ contains a Goldstone boson φ and it has an off-diagonal term $\partial_\mu\varphi A^\mu$. But $\mathcal{L}$ is still invariant under the local $U(1)$ symmetry, where the fields h, φ, A_μ transform as

$$\delta h = -\alpha(x)\varphi, \quad \delta\varphi = \alpha(x)(v + h), \quad \delta A_\mu = -\frac{1}{e}\partial_\mu\alpha(x). \tag{5.5}$$

For $\mathcal{L}$ given by Eq. (5.4), we consider the following path integral generating functional:

$$Z = \int \mathcal{D}A\mathcal{D}h\mathcal{D}\varphi\; e^{i\int d^4x\, \mathcal{L}[A,h,\varphi]}. \tag{5.6}$$

To make sense of Z, we must introduce a gauge fixing condition $G(A)$ leading to

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A \int \mathcal{D}h \int \mathcal{D}\varphi\; e^{i\int d^4x\, \mathcal{L}[A,h,\varphi]}\delta\left(G(A^\mu_\alpha)\right) \det\left(\frac{\delta G(A^\mu_\alpha)}{\delta\alpha}\right) \tag{5.7}$$

with A^μ_α being the gauge transformed field

$$A^\mu_\alpha(x) = A^\mu(x) - \frac{1}{e}\partial^\mu\alpha(x). \tag{5.8}$$

Eq. (5.7) is obtained by inserting

$$1 = \int \mathcal{D}\alpha\, \delta\left(G(A^\mu_\alpha)\right) \det\left(\frac{\delta G(A^\mu_\alpha)}{\delta\alpha}\right) \tag{5.9}$$

into the path integral (5.6). The gauge fixing condition is

$$G(A^\mu_\alpha) = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu(x) - w(x) - \xi e v\varphi\right). \tag{5.10}$$

Compared to the gauge fixing condition we saw for the quantization of Abelian gauge theories in section 2.2, this condition has a new term proportional to $\xi ev\varphi$ that appears only for spontaneously broken Abelian gauge theories. This term has the convenient form such that after taking the square of G, it will cancel the $A^\mu\partial_\mu\phi$ non-diagonal term in (5.4).

After integrating over w with a Gaussian weight, i.e. by introducing the integral $\int \mathcal{D}w \exp\left[-i\int d^4x \frac{w^2}{2\xi}\right]$ (exactly as in chapter 2), one finds

$$Z = \int \mathcal{D}\alpha \mathcal{D}A\mathcal{D}h\mathcal{D}\varphi \; e^{i\int d^4x\left(\mathcal{L}-\frac{1}{2}G^2\right)} \det\left(\frac{\delta G(A^\mu_\alpha)}{\delta\alpha}\right) \tag{5.11}$$

$$\text{with } G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu - \xi e v\varphi\right). \tag{5.12}$$

Forming G^2, the term quadratic in A^μ provides the gauge dependent term $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ (as in chapter 2). As expected, the term $\propto v\varphi\partial_\mu A^\mu$ in G^2 cancels the term of the form $\partial_\mu\varphi A^\mu$ in the original Lagrangian (5.4). The quadratic terms in the gauge fixed Lagrangian ($\mathcal{L}-\frac{1}{2}G^2$) are given by

$$\begin{aligned}\mathcal{L}_{(q^2)} - \frac{1}{2}G^2 = &-\frac{1}{2}A_\mu\left(-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu - (ev)^2 g^{\mu\nu}\right)A_\nu + \frac{1}{2}(\partial_\mu h)^2 \\ &-\frac{1}{2}m_h^2 h^2 + \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\xi}{2}(ev)^2\varphi^2.\end{aligned} \tag{5.13}$$

where $\mathcal{L}_{(q^2)}$ contains only the quadratic terms in $\mathcal{L}$ given in (5.4). Furthermore, by applying the transformation (5.5) to (5.12), we find

$$\det\left(\frac{\delta G(A^\mu_\alpha)}{\delta\alpha}\right) = \det\left[-\frac{1}{e}\partial_\mu\partial^\mu - \xi e v(v+h)\right]\cdot\frac{1}{\sqrt{\xi}} \tag{5.14}$$

which is independent of A_μ and α but depends on h. Therefore it cannot be pulled out of the functional integral. We use the path integral formulation of the determinant in terms of ghost fields as we did in Eq. (2.90):

$$\det\left(\frac{\delta G(A^\mu_\alpha)}{\delta\alpha}\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\; \exp\left[i\int d^4x\; \mathcal{L}_{\text{ghost}}\right] \tag{5.15}$$

$$\text{with } \mathcal{L}_{\text{ghost}} = \bar{c}\left[-\partial^2 - \xi m_A^2\left(1+\frac{h}{v}\right)\right]c \tag{5.16}$$

where a factor $\frac{1}{e\xi}$ has been absorbed into the ghost fields. In this theory, the ghosts do not couple directly to the gauge fields (the theory is Abelian). However, they do couple to the Higgs field (so they cannot be "ignored" as in QED).

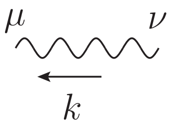
The particle spectrum is as follows:

- one massive gauge field A_μ with $m_A^2 = e^2v^2$,
- no φ-A mixing,
- one massive Goldstone boson field, whose mass depends on the gauge field: $m_\varphi^2 = \xi(ev)^2 = \xi m_A^2$. Since m_φ is gauge dependent ($m_\varphi \propto \xi$), the Goldstone bosons are fictitious fields which will not be produced in physical processes.

- an unphysical massive ghost field with the same gauge-dependent mass as the Goldstone, $m^2_{\text{ghost}} = \xi m_A^2$.

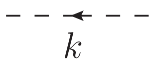
Using $\mathcal{L} - \frac{1}{2}G^2 + \mathcal{L}_{\text{ghost}}$ instead of $\mathcal{L}$ in the generating functional Z (given in Eq. (5.6)), one can derive the propagators for (A_μ, h, φ, c) in R_ξ gauge (ξ unfixed). The propagators in the Abelian model with spontaneous symmetry breaking in R_ξ gauge read:

A_μ : (Photon)



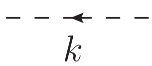
$$= \frac{-i}{k^2 - m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m_A^2}(1-\xi)\right)$$

h : (Higgs)



$$= \frac{i}{k^2 - m_h^2}$$

φ : (Goldstone)



$$= \frac{i}{k^2 - \xi m_A^2}$$

c : (Ghost)

$$= \frac{i}{k^2 - \xi m_A^2}$$

5.1.1 ξ-dependence in Physical Processes

In physical processes, we expect the ξ-dependence to cancel and the Goldstone bosons not to be present since they have gauge-dependent mass terms $m_\varphi^2 \propto \xi m_A^2$.

We illustrate this with an example where this cancellation can be seen to work at tree level. The cancellation of ξ at all orders can be proven using the BRST symmetry of the gauge fixed Lagrangian (we will not do this here).

Example: Fermion-Fermion Scattering

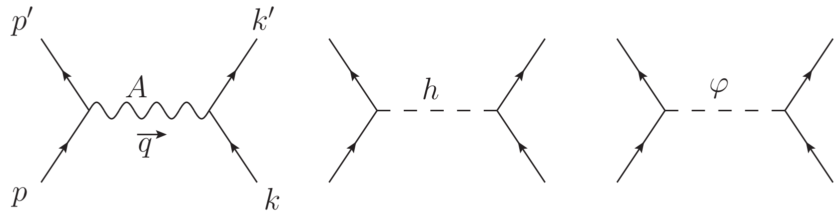
Consider the Abelian model with spontaneous symmetry breaking and couple to it a fermion through

$$\mathcal{L}_{\text{matter}} = \bar{\Psi}_L(i\not{D})\Psi_L + \bar{\Psi}_R(i\not{\partial})\Psi_R - \lambda_f(\bar{\Psi}_L\phi\Psi_R + \bar{\Psi}_R\phi^*\Psi_L) \tag{5.17}$$

where $\Psi_{L,R}$ are left- and right-handed fermions and ϕ is a complex scalar field with $\phi^1(x) = v + h(x)$ and $\phi^2(x) = \varphi(x)$. We have introduced the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ as before. The fermions Ψ have a mass

$$m_f = \lambda_f \frac{v}{\sqrt{2}} \tag{5.18}$$

due to spontaneous symmetry breaking. The relevant diagrams for tree level fermion-fermion scattering are



They include interactions with a

- gauge boson A (($V-A$) type),
- Higgs field h,
- Goldstone boson φ.

By calculating these amplitudes, we will show that the ξ-dependence and the unphysical polarisations drop out in the physical amplitude at tree level.

The amplitude for the φ-exchange reads

$$iM_\varphi = \frac{(\lambda_f)^2}{2}\bar{u}(p')\gamma_5 u(p)\frac{i}{q^2-\xi m_A^2}\bar{u}(k')\gamma_5 u(k). \tag{5.19}$$

To obtain this amplitude, we have replaced Ψ_L, Ψ_R in terms of Ψ yielding $\Psi_{\substack{L\\R}} = (1\mp\gamma_5)\Psi$ and we expressed ϕ in terms of ϕ_1, ϕ_2 in

$$-\lambda_f\left(\bar{\Psi}_L\phi\Psi_R + \bar{\Psi}_R\phi^*\Psi_L\right). \tag{5.20}$$

The amplitude for the gauge boson exchange reads

$$\begin{aligned} iM_A =&(-ie^2)^2\bar{u}(p')\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u(p)\underbrace{\left[\frac{-i}{q^2-m_A^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2-\xi m_A^2}(1-\xi)\right)\right]}_{[\alpha]}\times \\ &\times\bar{u}(k')\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)u(k). \end{aligned} \tag{5.21}$$

The bracket $[\alpha]$ can be rewritten as

$$[\alpha] = \frac{-i}{q^2-m_A^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{m_A^2}\right)+\frac{-i}{q^2-\xi m_A^2}\left(\frac{q_\mu q_\nu}{m_A^2}\right). \tag{5.22}$$

Using the identity

$$
\begin{aligned}
q^\mu \bar{u}(p')\gamma_\mu \left(\frac{1-\gamma_5}{2}\right) u(p) &= \frac{1}{2}\bar{u}(p')[(\not{p}-\not{p}') - (\not{p}-\not{p}')\gamma_5]u(p) \\
&= \frac{1}{2}\bar{u}(p')[\not{p}'\gamma_5 + \gamma_5\not{p}]u(p) \\
&= m_f \bar{u}(p')\gamma^5 u(p)
\end{aligned} \tag{5.23}
$$

and an analogous identity for the other fermion line ($m_f = \lambda_f \frac{v}{\sqrt{2}}$, $m_A = ev$), one finds

$$
\begin{aligned}
iM_A =& (-ie)^2 \bar{u}(p')\gamma_\mu \left(\frac{1-\gamma_5}{2}\right) u(p) \frac{i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) \bar{u}(k')\gamma_\nu \left(\frac{1-\gamma_5}{2}\right) u(k) - \\
& \underbrace{- \frac{(\lambda_f)^2}{2}\bar{u}(p')\gamma_5 u(p) \frac{i}{q^2 - \xi m_A^2} \bar{u}(k')\gamma_5 u(k)}_{=-M_\varphi}
\end{aligned} \tag{5.24}
$$

We see that everything worked out as we expected it to do:

- Goldstone boson diagrams cancel the unphysical (scalar) contributions from the polarisation state of the gauge boson in iM_A.
- Only the three physical polarisation states of A^μ contribute to the physical process.
- The physical process is ξ-independent. Note that the third diagram (containing the virtual h-exchange) cannot give further ξ-dependencies because the Higgs field propagator does not depend on ξ.

Looking only at the first term in M_A (since the second term is canceled by the amplitude M_φ), one can define an effective propagator for A_μ acting in the physical process:

$$
iD^{\mu\nu}_{A,\text{eff.}} = \frac{-i}{k^2 - m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}\right) \tag{5.25}
$$

which contains only physical polarisations of A_μ. Note that for an on-shell vector field A_μ of mass m_A, one has the polarisation sum

$$
\sum_{\substack{\varepsilon_\mu q^\mu = 0 \\ (\lambda=1,2,3)}} \varepsilon^{\mu(\lambda)}(q)\varepsilon^{*\nu(\lambda)}(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) \tag{5.26}
$$

which is just the numerator of $D^{\mu\nu}_{A,\text{eff.}}$. So the tensor structure of the effective vector boson propagator (5.25) indicates a polarisation sum just over the three physical polarisations, the unphysical timelike polarisation being absent (in fact, it is cancelled by the diagram involving Goldstone boson exchange).

5.2 Quantization of Spontaneously Broken non-Abelian Gauge Theories

Consider a general Yang-Mills theory with a gauge group G which is spontaneously broken by the vacuum expectation value of a scalar field ϕ. The system is invariant under the infinitesimal symmetry transformations induced by G:

$$\phi_i \longrightarrow (1 + i\alpha^a(x)t^a)_{ij}\phi_j \tag{5.27}$$

where ϕ_i are the real-valued components of ϕ. The generators t^a thus have to be purely imaginary:

$$t^a_{ij} = iT^a_{ij} \tag{5.28}$$

with real, hermitian, antisymmetric generators T^a. The gauge transformation reads

$$\phi_i \longrightarrow \phi_i + \delta\phi_i = \phi_i - \alpha^a T^a_{ij}\phi_j, \tag{5.29}$$

$$A^a_\mu \longrightarrow A^a_\mu + \delta A^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A^c_\mu = A^a_\mu + \frac{1}{g}D_\mu\alpha^a. \tag{5.30}$$

For simplicity, assume that the group G is simple such that we have the same coupling g for each parameter a. Then the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi)^2 - V(\phi) \tag{5.31}$$

$$\text{with } D_\mu\phi_i = \partial_\mu\phi_i + gA^a_\mu T^a_{ij}\phi_j. \tag{5.32}$$

For spontaneous symmetry breaking to occur, we assume that for some indices i we have a vacuum expectation value

$$\langle\phi_i\rangle = \langle 0|\phi_i(x)|0\rangle = v_i. \tag{5.33}$$

We expand the ϕ_i close to this value:

$$\phi_i(x) = v_i + \chi_i(x). \tag{5.34}$$

The values of χ_i are divided into two orthogonal subspaces

- related to the Goldstone bosons, and
- related to the Higgs fields.

In terms of generators these subspaces are characterized as follows:

- T^a is **broken** if $(T^a)_{ij}v_j \neq 0$,
- T^a is **unbroken** if $(T^a)_{ij}v_j = 0$,

and to each broken generator, a massless Goldstone boson is associated. These statements can be proven as follows:

We note that the potential $V(\phi)$ must be invariant under a global transformation[1]:

$$V[(1-\alpha_a T^a)(\phi)] = V(\phi) \qquad \Leftrightarrow \qquad \frac{\partial V}{\partial \phi_j}(T^a)_{jk}\phi_k = 0. \tag{5.35}$$

Differentiating Eq. (5.35) with respect to ϕ_i, we get

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(T^a)_{jk}\phi_k + \frac{\partial V}{\partial \phi_j}(T^a)_{ji} = 0. \tag{5.36}$$

In this equation we have to insert

$$\phi_i = v_i, \qquad \left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i = v_i} = 0, \qquad \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\phi_i = v_i} = (m^2)_{ij} \tag{5.37}$$

where the last identification contains the mass matrix for the scalars after symmetry breaking according to Goldstone's theorem. Eq. (5.36) thus reads

$$(m^2)_{ij}\,(T^a v)_j = 0. \tag{5.38}$$

If $(T^a v) \neq 0$, then $(T^a v)$ is an eigenvector of $(m^2)_{ij}$ with eigenvalue 0 corresponding to a massless Goldstone boson. As a consequence each broken generator gives rise to a massless Goldstone boson since it provides a $T^a v \neq 0$. The subspace for the Goldstone bosons is the space spanned by $(T^a)_{ij}v_j$ for each v_j.

We consider

$$\phi_i(x) = v_i + \chi_i(x) \tag{5.39}$$
$$(D_\mu \phi)_i = \partial_\mu \chi_i + g A^a_\mu T^a_{ij}(v+\chi)_j. \tag{5.40}$$

It is convenient to define a real rectangular matrix

$$F^a_i := T^a_{ij} v_j \tag{5.41}$$

such that

$$(D_\mu \phi)_i = \partial_\mu \chi_i + g A^a_\mu (F^a + T^a \chi)_i. \tag{5.42}$$

Observing that the index a in T^a_{ij} labels a set of $(N \times N)$ real, antisymmetric matrices, we conclude that $F^a_i = T^a_{ij} v_j$ are rectangular (not necessarily square) matrices with one row for each generator (labelled by a) and one column for each component ϕ_i (labelled by i). The part of the Lagrangian which contains quadratic terms only ($\mathcal{L}_{(q^2)}$) reads

$$\mathcal{L}_{(q^2)} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \underbrace{\frac{1}{2}(\partial_\mu \chi)^2}_{\substack{\text{kinetic term} \\ \text{(Goldstone bosons)}}} + \underbrace{g\partial_\mu \chi_i A^\mu_a F^i_a}_{\substack{\text{off-diag.} \\ \text{term}}} + \frac{1}{2}\underbrace{\left(g^2 F^a_j F^b_j\right)}_{=(m^2_A)^{ab}} A^a_\mu A^{\mu b} + \ldots \tag{5.43}$$

We see that in this Lagrangian we have

[1] It is sufficient to consider a global transformation here and not a local one, to ensure that $\mathcal{L}$ is invariant under the same transformation.

- an off-diagonal term,
- a kinetic term for the Goldstone bosons χ_i,
- gauge bosons A^a_μ with mass squared matrix $(m_A^2)^{ab} = g^2 F^a_j F^b_j$.

The F^a_i will only be non-zero for the components of ϕ_i that are Goldstone bosons which come with a broken generator. These non-zero F^a_i elements will be related to the gauge boson masses, as the mass matrix m_A^2 is composed of them.

As an example, consider the matrix F^a_j in the GWS electroweak theory where we use the following parametrization:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i(\phi^1 - i\phi^2) \\ v + (h + i\phi^3) \end{pmatrix}. \tag{5.44}$$

Here ϕ^i are the Goldstone bosons and h is the Higgs field. The vacuum expectation value of ϕ is

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{5.45}$$

The generators T^a_{ij} ($a = 1, 2, 3$,Y and $i, j = 1, 2, 3$) are given by

$$T^a = -i\frac{\sigma^a}{2}, \qquad T^Y = -iY = -\frac{i}{2}, \tag{5.46}$$

such that we have 3 generators T^a for $SU(2)$ and 1 generator for $U(1)_\mathrm{Y}$. This yields, for example,

$$F^1_1 = T^1\phi_0 = \frac{v}{2} \times (\text{unit vector in the } \phi^1 \text{ direction}). \tag{5.47}$$

In general, F^a_i is given by

$$g_\alpha F^a_i = \frac{v}{2} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & g' \end{pmatrix} \tag{5.48}$$

which is a matrix with indices $a = 1, 2, 3, 4$ and $i = 1, 2, 3$. The number g_α is g for the first two columns and it is g' for the last column.

Before we return to the example of GWS theory, we continue to quantize the general non-Abelian, spontaneously broken gauge theory. To study the quantum theory, we consider

$$Z = \int \mathcal{D}A\mathcal{D}\chi \; e^{i\int d^4x \; \mathcal{L}[A,\chi]} \tag{5.49}$$

where $\mathcal{L}$ is given in Eq. (5.31) and has the quadratic part $\mathcal{L}_{(q^2)}$ given by Eq. (5.43). We "modify" $\mathcal{L}$ to make Z finite, by imposing a gauge fixing procedure (Faddeev-Popov method):

$$Z \longrightarrow C' \int \mathcal{D}A\mathcal{D}\chi \; \exp\left[i \int d^4x \; \left(\mathcal{L} - \frac{1}{2}(G^2)\right)\right] \det\left(\frac{\delta G}{\delta\alpha^{(a)}}\right) \tag{5.50}$$

where $G(A,\chi)$ is given in the R_ξ gauge (ξ undetermined) for each a by

$$G^a = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^{\mu a} - \xi g F_i^a \chi_i\right) \tag{5.51}$$

where the first term is as for the unbroken system and the second term will compensate the off-diagonal term in $\mathcal{L}$. The second term should be compared with the analogous term $-ev\varphi\xi$ in the Abelian case (c.f. Eq. (5.10)). The gauge fixing condition G involves only components of χ that lie in the subspace of the Goldstone bosons because χ_i is multiplied by F_i^a which is non-zero only in this case. The quadratic terms which will appear in the gauge fixed Lagrangian read

$$\left(-\frac{1}{2}G^2\right)_{(q^2)} = \frac{1}{2}A_\mu^a\left(\frac{1}{\xi}\partial^\mu\partial^\nu\right)A_\nu^a + g\partial_\mu A^{\mu a}F_i^a\chi_i - \frac{1}{2}\xi g^2\left[F_i^a\chi_i\right]^2 \tag{5.52}$$

where the second term on the right-hand side is engineered such that it cancels the off-diagonal term in $\mathcal{L}$. The quadratic terms in the complete Lagrangian thus read

$$\begin{aligned}\left(\mathcal{L} - \frac{1}{2}G^2\right)_{(q^2)} = &-\frac{1}{2}A_\mu^a\Bigg(\left[-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]\delta^{ab} - \underbrace{g^2F_i^aF_i^b}_{\substack{=g^2(FF^{\mathrm{T}})^{ab}\\ =(m_A^2)^{ab}}}\; g^{\mu\nu}\Bigg)A_\nu^b \\ &+\frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}\underbrace{\xi g^2F_i^aF_j^a}_{\substack{=\xi g^2(F^{\mathrm{T}}F)_{ij}\\ =(m^2_{\mathrm{Goldst.}})_{ij}}}\;\chi_i\chi_j.\end{aligned} \tag{5.53}$$

We note that in this Lagrangian, the mass square matrices of $A_\mu^{(a)}$ and of the Goldstone bosons appear as indicated. Both of them are essentially determined by the matrix F. The Goldstone square mass matrix only is proportional to the gauge parameter ξ.

We can now construct the ghost Lagrangian:

$$\det\left(\frac{\delta G^{(a)}}{\delta\alpha^{(b)}}\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\; \exp\left[i\int d^4x\; \mathcal{L}_{\text{ghost}}\right] \tag{5.54}$$

$$\text{where } G^{(a)} = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^{\mu(a)} - \xi g F_i^a\chi_i\right). \tag{5.55}$$

In order to find $\frac{\delta G^{(a)}}{\delta\alpha^{(b)}}$ we consider the variations of $A_\mu^{(a)}$ and χ_i under the gauge transformation with gauge parameter $\alpha^{(a)}(x)$:

$$A_\mu^a \longrightarrow A_\mu^a + \delta A_\mu^a = A_\mu^a + \frac{1}{g}(D_\mu \alpha^a) \tag{5.56}$$

$$\phi_i \longrightarrow \phi_i + \delta\phi_i \tag{5.57}$$

$$\chi_i \longrightarrow \chi_i - \alpha^a(x) T^a_{ij}\phi_j \qquad (\phi_j = v_j + \chi_j). \tag{5.58}$$

This yields

$$\begin{aligned}
\frac{\delta G^{(a)}}{\delta\alpha^{(b)}} &= \frac{1}{\sqrt{\xi}}\left(\frac{1}{g}\partial_\mu D^\mu\right)^{ab} + \xi g F^a_j T^b_{jk}(v_k + \chi_k) \\
&= \frac{1}{\sqrt{\xi}}\left(\frac{1}{g}\partial_\mu D^\mu\right)^{ab} + \xi g F^a_j F^b_j + \xi g F^a_j T^b_{jk}\chi_k \\
&= \underbrace{\frac{1}{\sqrt{\xi}}\left(\frac{1}{g}\partial_\mu D^\mu\right)^{ab}}_{\substack{\text{as in non-Abelian case} \\ \text{without symmetry breaking}}} + \underbrace{\frac{1}{g}\xi g^2 F^a_j F^b_j + \frac{1}{g}\xi g^2 F^a_j T^b_{jk}\chi_k}_{\substack{\text{new for spontaneously broken,} \\ \text{non-Abelian case}}}\,.
\end{aligned} \tag{5.59}$$

This gives the following ghost Lagrangian (absorbing $\frac{1}{g}$ in the definition of c):

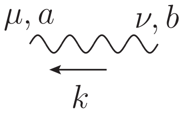
$$\mathcal{L}_{\text{ghost}} = \bar{c}^a\left[(\partial_\mu D^\mu)^{ab} + \underbrace{\xi g^2 F^a_j F^b_j}_{\substack{=(m^2_{\text{ghost}})^{ab} \\ =\xi m_A^2}} + \xi g^2 F^a_j T^b_{jk}\chi_k\right] c^a \tag{5.60}$$

where the second term gives the mass term for the ghosts c. Using the definition $Z = \int \mathcal{D}A\mathcal{D}\chi\; e^{i\int d^4x\, \mathcal{L}}$ with

$$\mathcal{L} = \mathcal{L}[A_\mu, \chi] + \mathcal{L}_{\text{gauge-fix}} + \mathcal{L}_{\text{ghost}} \qquad \text{where } \mathcal{L}_{\text{gauge-fix}} = -\frac{1}{2}G^2, \tag{5.61}$$

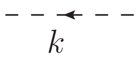
we can compute the propagators for the gauge field $A_\mu^{(a)}$, the scalar fields χ and h, and the ghosts c, $\bar{c}$ in an R_ξ gauge. For propagators we find:

$A_\mu^{(a)}$: (Gauge)



$$= \left(\frac{-i}{k^2 - g^2 (FF^{\mathrm{T}})} \left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi g^2 FF^{\mathrm{T}}} (1-\xi) \right] \right)^{ab}$$

h : (Higgs)



$$= \frac{i}{k^2 - m_h^2}$$

χ : (Goldstone)

$$= \left(\frac{i}{k^2 - \xi g^2 F^{\mathrm{T}} F} \right)_{ij}$$

c : (Ghost)

$$= \left(\frac{i}{k^2 - \xi g^2 FF^{\mathrm{T}}} \right)^{ab}$$

where the Higgs propagator comes from the potential $V(\phi)$ and is independent of ξ. The matrix F appears in all these equations. When appearing in the denominator, it should be interpreted as an inverse matrix.

Specializing again these general results for a non-Abelian, spontaneously broken gauge theory to the electroweak theory, let us see what are the mass matrices FF^{T} and $F^{\mathrm{T}}F$ in the GWS electroweak theory. We have gF_i^a given by the (4×3) matrix in Eq. (5.48) such that the (4×4) mass matrix for the gauge bosons reads

$$g^2 FF^{\mathrm{T}} = \frac{v^2}{4} \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix}. \tag{5.62}$$

This matrix acts on (W^1, W^2, W^3, B_μ). If we diagonalize this matrix, we obtain relations for the masses of the physical fields (W^+, W^-, Z^0, γ) given by

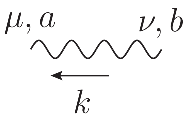
$$m_W = \frac{1}{2} v g, \tag{5.63}$$

$$m_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}, \tag{5.64}$$

$$m_A = 0, \tag{5.65}$$

and the obtained matrix for $g^2 FF^{\mathrm{T}}$ acts on the physical fields (W^+, W^-, Z^0, γ). In the mass eigenstate basis, the four gauge boson propagators decouple to give simply in an R_ξ gauge

$A_\mu^{(a)}$:



$$\frac{-i}{k^2 - m^2} \left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m^2} (1-\xi) \right]^{ab}$$

with $m \in \{m_{W^+}, m_{W^-}, m_Z, m_\gamma = 0\}$.

The mass matrix for the Goldstone bosons reads

$$\xi g^2 F^{\mathrm{T}} F = \xi \frac{v^2}{4} \begin{pmatrix} g^2 & & \\ & g^2 & \\ & & g^2 + g'^2 \end{pmatrix}. \tag{5.66}$$

This matrix acts on (ϕ^1, ϕ^2, ϕ^3), so that we find that the Goldstone boson propagator is given by

$$\underset{\text{(Goldstone)}}{\chi :} \quad \overset{i}{-}\,-\,-\!\!\leftarrow\!\!-\,-\,\overset{j}{-}_{\;k} \quad = \left(\frac{i}{k^2 - \xi m^2}\right)_{ij}$$

with $m^2 = m_W^2$ for ϕ^1, ϕ^2 (the Goldstone bosons which give $W^\pm$ their masses) and $m^2 = m_Z^2$ for ϕ^3 (the Goldstone boson which is eaten by Z^0). Finally we have four ghost fields with propagators

$$\underset{\text{(Ghost)}}{c :} \quad \overset{a}{\cdots}\cdots\!\leftarrow\!\cdots\overset{b}{\cdots}_{\;k} \quad = \left(\frac{i}{k^2 - \xi m^2}\right)^{ab}$$

with the same values m^2 as masses as for the gauge bosons $(A_\mu, W_\mu^\pm, Z_\mu^0)$.

5.3 R_ξ Gauge Dependence in Perturbation Theory

The aim of this section is to analyze qualitatively the renormalizability and unitarity of quantum field theories with spontaneous symmetry breaking.

Remember that renormalization has to "handle" divergences at higher orders. For loop momenta k, there are UV divergences ($k \to \infty$) and IR divergences ($k \to 0$). In this section we will only deal with UV divergences. For renormalizable theories, UV divergences are removed at each order by a finite number of counterterms.

As seen in QFT I, in order to decide whether a theory is renormalizable, it is useful to define the superficial degree of divergence D where D is a function of the number of loops, the number of legs and the number of internal and external propagators.

A theory is **renormalizable** if D is independent of n (the number of vertices) and if D is independent of the order in perturbation theory. One can then identify the class of divergent graphs and determine their ultraviolet behaviour. If $k \to \infty$ in the graph, then the integral $\int^\infty dk\ k^{D-1}$ is convergent if $D < 1$. A theory is renormalizable if there is only a finite set of divergent gaphs and if a finite set of counterterm graphs is enough to absorb all divergences present in these divergent graphs. The infinities are absorbed into redefinitions of parameters (order by order). We shall review these properties in the next chapter.

Unitarity is the property that ensures $S^\dagger S = 1$. If $|n\rangle$ denotes a final state and $\langle m|$ denotes an initial state, then the probability amplitude for the transition $m \to n$ is just $\langle m|S|n\rangle$. The conservation of probability demands that the sum over all probability amplitudes must be unity,

$$\sum_n |\langle m|S|n\rangle|^2 = 1 \qquad \Rightarrow \qquad S^\dagger S = 1, \tag{5.67}$$

i.e. S is unitary.

Furthermore, since S is a physical quantity, it should be independent of the gauge parameter ξ. Unitarity thus also demands that there are no contributions from unphysical states (like Goldstone bosons) to observables.

To summarize, we want any QFT to be renormalizable and unitary. Let us see the "a priori" situation for R_ξ theories.

5.3.1 QFT with Spontaneous Symmetry Breaking and finite ξ

Concerning renormalizability, what do we expect from the naïve power counting argument?

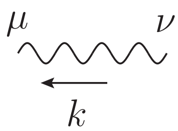
In an R_ξ gauge, the propagators of gauge bosons and Goldstone bosons fall off as $\frac{1}{k^2}$. Therefore, by power counting we expect theories in R_ξ gauge to be renormalizable.

However, theories in R_ξ gauge are not manifestly unitary because they have unphysical degrees of freedom. The cancellations of these unphysical contributions is not trivial.

5.3.2 QFT with Spontaneous Symmetry Breaking for $\xi \to \infty$

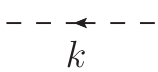
If the gauge parameter $\xi \to \infty$, we have a completely different physical picture. In this limit the unphysical bosons which have masses $m^2 \propto \xi$ disappear. The propagators become

A_μ : (Gauge b.)



$$= \frac{-i}{k^2-m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}\right)$$

φ : (Goldstone)



$$= 0$$

Note that the gauge boson propagator contains exactly and only the three physical spacelike polarisations

$$g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2} = \sum_{\substack{\text{transv.,}\\ \text{long.}}} \varepsilon_\mu^{(\lambda)*}(k)\varepsilon_\nu^{(\lambda)}(k). \tag{5.68}$$

Remember from chapter 4 that by constructing theories with spontaneous symmetry breaking, in the case of the Abelian Higgs mechanism (see section 4.2.3), we saw that in a unitary gauge, the Goldstone field could be eliminated by a gauge transformation and the Higgs field was real. We can now view the $\xi \to \infty$ limit of R_ξ gauges as a quantum realization of the unitary gauge. Unitarity is then manifest for such a theory with $\xi \to \infty$: there are no unphysical states. However, the renormalizability is non-trivial. By naïve power counting, the gauge boson propagator falls off more slowly than $\frac{1}{k^2}$. A QFT with $\xi \to \infty$ is manifestly unitary but not necessarily renormalizable.

These considerations motivate the next chapter: renormalizability of broken and unbroken gauge theories.

Chapter 6

Renormalizability of Broken and Unbroken Gauge Theories: Main Criteria

The aim of this chapter is to present the main criteria leading to a proof of renormalizability. Most of the results in this chapter will not be derived in detail and not all statements will be proven. We just want to give an overview over the methods since calculations beyond leading order are rather complicated.

6.1 A Renormalization Program (UV Divergences only)

Given the requirements presented in section 5.3 related to a renormalizable QFT, we want to give the renormalization conditions of a proof by induction of the renormalizability of a QFT. We will consider a procedure in four steps:

1. Define the superficial degree of divergence and identify the divergent graphs.

2. Evaluate the divergent graphs by a regularization procedure (dimensional regularization in $d = 4 - 2\varepsilon$ dimensions). Divergences appear as poles in ε. The advantage of dimensional regularization is that gauge invariance is preserved.

3. Construction of the renormalization counterterms. In this step we have to add new terms to the Lagrangian of the theory in order to subtract divergent graphs. We have to perform the renormalization (redefinition) of the parameters in $\mathcal{L}$.

4. The last step is inductive. Provided that steps 1,2,3 are done, we assume that the theory is renormalizable at n^{th} order and show that it is renormalizable at $(n+1)^{\text{th}}$ order by using a recursion relation which enables us to construct graphs of $(n+1)^{\text{th}}$ order from n^{th} order graphs.
   All inductive proofs are based on the

Weinberg Theorem:
A Feynman graph is convergent if the superficial degrees of divergence of the diagram itself and all its subgraphs are negative.

Some general remarks concerning this program:

- We only need to worry about 1PI diagrams: as we have seen in QFT I, any diagram can be written in terms of products of 1PI diagrams.
- Our considerations are purely formal, so we only consider UV divergences ($k \to \infty$) and ignore IR divergences ($k \to 0$). However, we should keep in mind, that in QED and QCD with massless gauge bosons also IR divergences do occur.
- Unitarity and gauge invariance should be kept throughout the renormalization procedure.
- For step 4, we will at best present the necessary recursion relation, but we will not give any formal proof.

In the following, let us perform these steps for QED and QCD.

6.2 Overall Renormalization of QED

We recall what has already been seen in QFT I.

First step: Definition of superficial degree of divergence D. The first step is to define the superficial degree of divergence D by counting the powers of momenta in vertices, loops and propagators appearing in the theory:

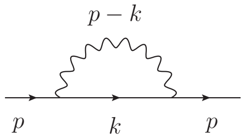
$$D = 4L - 2P_i - E_i \tag{6.1}$$

where L is the number of loops, P_i the number of internal photon lines, and E_i is the number of internal electron lines. A priori, D depends on internal properties of the diagram (L, E_i, P_i). This is not what we want for a renormalizable theory. Therefore, in order to obtain the final form of D, we rewrite the internal properties in terms of external ones (external fermion lines E_e and external photon lines P_e) using momentum conservation at each vertex and overall momentum conservation in the diagram. As we have seen in QFT I, this yields

$$D = 4 - \frac{3}{2}E_e - P_e. \tag{6.2}$$

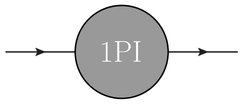
Written in this form, D is independent of the number of vertices n and independent of the number of loops, L. We conclude that QED is in principle renormalizable according to this criterion.

Second step: Identify the divergent diagrams. The diagram



$$\sim (-ie)^2\mu^{2-d/2}\int \frac{d^dk}{(2\pi)^d}\,\gamma_\mu\frac{i(\not{k}+m)}{(k^2-m^2+i\varepsilon)}\gamma^\mu\frac{-i}{(p-k)^2+i\varepsilon}$$

is a 1-loop contribution to the 1PI diagram related to the electron self-energy,



$$= -i\Sigma(p).$$

One finds at second order in perturbation theory (cf. QFT I) that[1]

$$\Sigma_2(p)\Big|_{\text{div.}} = \frac{\alpha}{4\pi\varepsilon}\underbrace{\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\Gamma(1+\varepsilon)}_{[\beta]}(-\not{p}+4m) \tag{6.3}$$

in $d = 4-2\varepsilon$ dimensions with $\Gamma(1+\varepsilon) = e^{-\varepsilon\gamma}+\mathcal{O}(\varepsilon)$. The divergence in this graph appears as $\frac{1}{\varepsilon}$ and $[\beta]$ is kept unexpanded. The presence of μ is directly related to dimensional regularization: since the action

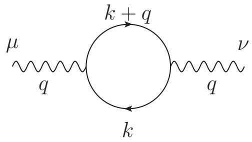
$$S = \int d^dx\,\mathcal{L} \tag{6.4}$$

has to have mass dimension zero, the Lagrangian $\mathcal{L}$ has mass dimension d. Therefore, one has to replace the coupling e as follows:

$$e \longrightarrow e\mu^{2-d/2}. \tag{6.5}$$

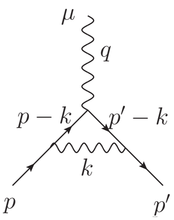
In order for the coupling e appearing on the right-hand side to be dimensionless, the original coupling is multiplied by an appropriate power of μ, where μ is some arbitrary mass scale. The term $[\beta]$ in Eq. (6.3) comes for "free" in the calculation, so we keep it unexpanded.

There are two more divergent graphs in QED. On the one hand we have the photon vacuum polarisation



$$= i\Pi_2^{\mu\nu}(q)$$

which provides a correction to the photon propagator. Finally, the electron-photon vertex is corrected at 1-loop level by the diagram



$$= ie\mu^{\varepsilon}\Lambda_2^{\mu}(p,q,p')$$

The expressions for these 1-loop divergent graphs were given in QFT I.

[1]The subscript in $\Sigma_2(p)$ denotes the order in perturbation theory.

Third step: Construction of counterterms. Starting from a bare Lagrangian

$$\mathcal{L}_{\text{QED}}^{\text{bare}} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \mu^\varepsilon e\bar{\Psi}\gamma_\mu\Psi A^\mu, \tag{6.6}$$

we have to write down a quantized theory in d dimensions. The fields and parameters are redefined such that $\mathcal{L}_{\text{QED}}^{\text{bare}}$ written in terms of renormalized quantities (which we denote by a subscript r) looks reasonably simple. We define four renormalization constants by the following relations:

$$\Psi = \sqrt{Z_2}\Psi_r \tag{6.7}$$

$$A^\mu = \sqrt{Z_3}A_r^\mu \tag{6.8}$$

$$m_e = m_r + \delta m \tag{6.9}$$

$$e = \frac{Z_1}{Z_2\sqrt{Z_3}}e_r. \tag{6.10}$$

We consider the bare Lagrangian $\mathcal{L}_{\text{QED}}^{\text{bare}}$ written in terms of a renormalized Lagrangian $\mathcal{L}_{\text{QED}}^{\text{ren.}}$ and a counterterm Lagrangian $\mathcal{L}_{\text{QED}}^{\text{counter.}}$, all three written in terms of the renormalized quantities:

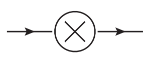
$$\mathcal{L}_{\text{QED}}^{\text{bare}}[\Psi_r, A_r^\mu, e_r, m_r](Z_i) = \mathcal{L}_{\text{QED}}^{\text{ren.}}(Z_i = 1) + \mathcal{L}_{\text{QED}}^{\text{counter.}}(\delta_i = 1 - Z_i) \tag{6.11}$$

$$\text{with } \mathcal{L}_{\text{QED}}^{\text{counter.}} = -\frac{1}{4}\delta_3 F_{\mu\nu,r}F_r^{\mu\nu} - \frac{1}{2\xi}\delta_3(\partial_\mu A_r^\mu)^2 + \bar{\Psi}_r(i\delta_2\partial\!\!\!/ - \delta m - m\delta_2)\Psi_r$$
$$- \mu^\varepsilon e_r\delta_1\bar{\Psi}_r\gamma_\mu\Psi_r A_r^\mu. \tag{6.12}$$

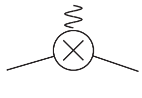
From $\mathcal{L}_{\text{QED}}^{\text{counter.}}$ one can obtain the Feynman rules for the counterterm diagrams which have to be such that the sum over all divergent diagrams plus the counterterm diagrams is finite. From $\mathcal{L}_{\text{QED}}^{\text{counter.}}$ we find which parameters δ_i are associated with which divergent diagram. We have the following counterterm rules:

δ_3 is associated to the photon propagator. The counterterm is 

$\rightarrow$ δ_3 will absorb the divergences in the photon vacuum polarization graph: $\delta_3 \propto \Pi^{\mu\nu}(q)\big|_{\text{div.}}$

$\delta_2, \delta m$ are associated to the fermion propagator. The counterterm is 

$\rightarrow$ $\delta_2, \delta m$ will absorb the divergences in the electron self energy graph: $\delta_2, \delta m \propto \Sigma_e(p)\big|_{\text{div.}}$

δ_1 is associated to the vertex: 

$\rightarrow$ δ_1 will absorb the divergences in the vertex graph: $\delta_1 \propto \Lambda_2(p, q, p')\big|_{\text{div.}}$

The sum of divergent diagrams plus their respective counterterms in the $\overline{\text{MS}}$-scheme is performed as follows. The counterterms contain a purely divergent part ($\frac{1}{\varepsilon}$). They also contain a finite part $(4\pi)^\varepsilon e^{-\varepsilon\gamma}$ and they have a dependence on the renormalization constant μ_R (which is introduced in the $\overline{\text{MS}}$ scheme) to compensate the unphysical μ-dependence coming from dimensional regularization in divergent graphs. Any divergent graph is thus proportional to

$$\frac{1}{\varepsilon}\left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon e^{-\varepsilon\gamma} \tag{6.13}$$

where m is the physical scale. And the counterterm graph is proportional to

$$-\frac{1}{\varepsilon}\left(\frac{4\pi\mu^2}{\mu_{\text{R}}^2}\right)^\varepsilon e^{-\varepsilon\gamma} \tag{6.14}$$

such that the sum of divergent graph plus counterterm is proportional to $\log(\mu_R^2/m^2)$ and is thus related to the physical observable. This will then depend on μ_R (called the renormalization scale) and the physical scale m.

As we are able to find a finite number of counterterms which absorb all divergences of divergent graphs at 1-loop order, we conclude that QED is renormalizable at one loop.

Fourth step: Induction. Is QED renormalizable at all loops? From QFT I we know a recurrence relation called the Ward-Takahashi identity which provides a relation between on-shell amplitudes with n and with $(n+1)$ external photons. This relation guarantees gauge invariance at all orders. As a consequence of the Ward-Takahashi identity, we have seen that

$$Z_1 = Z_2. \tag{6.15}$$

Therefore, the Ward-Takahashi identity helps to reduce the number of independent renormalization constants and to keep the number of counterterms finite. It is a crucial ingredient to prove the renormalizability of QED at all orders in perturbation theory.

6.3 Renormalizability of QCD

In this section, we will go over the four steps of the inductive proof of renormalizability for the gauge theory of QCD.

First step: Define superficial degree of divergence D. We use the quantized version of the QCD Lagrangian given in Eq. (2.96),

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{gauge}} \\ &= \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gauge fix}} + \mathcal{L}_{\text{ghost}}.\end{aligned} \tag{6.16}$$

The fields appearing in this Lagrangian are the fermions Ψ (a triplet of quarks), gauge bosons $A_\mu^{(a)}$ (vectors) and ghost fields $c^{(a)}$ (scalars). To define D, we denote

E_Ψ : number of external fermion lines,
E_A : number of external vector lines,
E_G : number of external ghost lines,
I_Ψ : number of internal fermion lines,
I_A : number of internal gauge boson lines,
I_G : number of internal ghost lines,
V_A^3 : number of 3-gauge boson vertices,
V_A^4 : number of 4-gauge boson vertices,
V_Ψ : number of fermion-vector vertices,
V_G : number of ghost vertices.

If QCD is renormalizable, we should find that D is independent of V_i and the number of loops. By power counting of k in the propagators and vertices, one finds in four dimensions

$$D = 4L - 2I_A - 2I_\Psi - 2I_G + V_G^3 + V_G^4 \tag{6.17}$$

which depends a priori on internal propagators. As in QED we have that the number of fermion-vector vertices is related to the number of propagators and the number of external lines by

$$V_\Psi = I_\Psi + \frac{1}{2}E_\Psi. \tag{6.18}$$

Since there are no external ghost lines, one finds

$$E_A + 2I_A = 4V_A^4 + 3V_A^3 + V_G + V_\Psi. \tag{6.19}$$

Each ghost propagator is connected to one end of a ghost vertex:

$$V_G = I_G. \tag{6.20}$$

In order to define the number of loops, we observe that each internal line (I_A, I_G, I_Ψ) is associated to constrained momenta. Imposing momentum conservation at vertices, one finds

$$L = I_A + I_\Psi + I_G - V_A^3 - V_A^4 - V_G + 1 \tag{6.21}$$

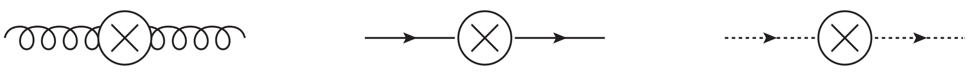
such that

$$\boxed{D_{\text{QCD}} = 4 - E_A - \frac{3}{2}E_\Psi} \tag{6.22}$$

exactly as in QED. Therefore, according to the power counting argument, QCD is in principle renormalizable.

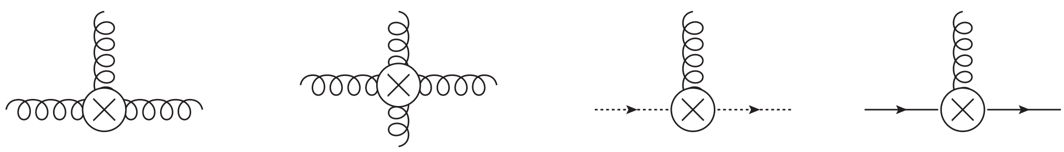
Second step: Identify divergent diagrams. In QCD there are seven divergent diagrams with non-negative superficial degree of divergences:

- Three divergent self-energy diagrams associated to the propagators. Therefore, we need three propagator counterterms associated respectively to the gluon, quark and ghost propagators:



$(D=2)$ $(D=1)$ $(D=1)$

- Four vertex counterterms related respectively to the 3-gluon, 4-gluon, ghost-gluon and quark-gluon vertices:



$(D=1)$ $(D=0)$ $(D=0)$ $(D=0)$

Third step: Construction of the counterterm diagrams. We want to evaluate the divergent and counterterm graphs at 1-loop order together such that their sum is finite. In order to do so, we need the counterterm Lagrangian $\mathcal{L}_{\mathrm{QCD}}^{\mathrm{counter.}}$ from which we can deduce the Feynman rules for the counterterm graphs.

We start with the quantized form of $\mathcal{L}_{\mathrm{QCD}}$ in d dimensions written in terms of a free part and an interaction part:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} = & -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 \\
& + i(\partial_\mu c^a)^*(\partial^\mu c^a) + \bar{\Psi}^i(i\partial\!\!\!/ - m)\Psi^i \\
& - \frac{g}{2}\mu^\varepsilon f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A^{\mu b}A^{\nu c} - \frac{g^2}{4}\mu^{2\varepsilon} f^{abc} f^{cde} A_\mu^a A_\nu^b A^{\mu d} A^{\nu e} \\
& - ig\mu^2 f^{abc}(\partial^\mu c^a)^* c^b A_\mu^c + \mu^\varepsilon g \bar{\Psi}^i T_{ij}^a \Psi^j A_\mu^a.
\end{aligned} \tag{6.23}
$$

To define the renormalized Lagrangian $\mathcal{L}_{\mathrm{QCD}}^{\mathrm{ren.}}$, fields and parameters need to be redefined such that the bare Lagrangian $\mathcal{L}_{\mathrm{QCD}}^{\mathrm{bare}}$ contains simple factors of Z_i. This can be achieved by defining

$$A^a_\mu = Z_3^{1/2} A^a_{r\mu} \tag{6.24}$$
$$c^a = \widetilde{Z}_3^{1/2} c^a_r \tag{6.25}$$
$$\Psi = Z_2^{1/2} \Psi_r \tag{6.26}$$
$$g = Z_g g_r \tag{6.27}$$
$$\xi = Z_3 \xi_r \tag{6.28}$$
$$m = Z_m m_r \tag{6.29}$$

where the renormalization of α has been chosen such that the gauge fixing term in $\mathcal{L}$ is kept unchanged between $\mathcal{L}^{\text{ren.}}_{\text{QCD}}$ and $\mathcal{L}^{\text{bare}}_{\text{QCD}}$. This yields

$$\mathcal{L}^{\text{bare}}_{\text{QCD}} = \mathcal{L}^{\text{ren.}}_{\text{QCD}} + \mathcal{L}^{\text{counter.}}_{\text{QCD}} \tag{6.30}$$

where $\mathcal{L}^{\text{counter.}}_{\text{QCD}}$ is the counterterm Lagrangian and $\mathcal{L}^{\text{ren.}}_{\text{QCD}}$ is the renormalized Lagrangian. Written in terms of renormalized quantities ($A_{r\mu}$, Ψ_r, c^a_r, g_r, m_r), we have

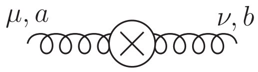
$$\begin{cases} \mathcal{L}^{\text{ren.}}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}[A_{r\mu}, \Psi_r, c^a_r](Z_i = 1) \\ \mathcal{L}^{\text{bare}}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}[A_{r\mu}, \Psi_r, c^a_r](Z_i) \\ \mathcal{L}^{\text{counter.}}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}(-1 + Z_i) \end{cases} \tag{6.31}$$

$$\text{and} \qquad \mathcal{L}^{\text{bare}}_{\text{QCD}} = \mathcal{L}^{\text{ren.}}_{\text{QCD}} + \mathcal{L}^{\text{counter.}}_{\text{QCD}}. \tag{6.32}$$

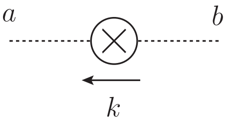
To find $\mathcal{L}^{\text{counter.}}_{\text{QCD}}$, we use the fact that in the action there appears actually an integral over $\mathcal{L}$ such that we can apply integration by parts and remove surface terms. In this way $\mathcal{L}$ can be written in a form which is quadratic in the fields and from which the Feynman rules can be easily found:

$$\begin{aligned}
\mathcal{L}^{\text{counter.}}_{\text{QCD}} = {} & (Z_3 - 1)\frac{1}{2} A^a_{\mu,r} \delta_{ab} (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) A^b_{\nu,r} \\
& + (\widetilde{Z}_3 - 1) c^a_r \delta_{ab} (-i\partial^2) c^b_r \\
& + (Z_2 - 1)\Psi^i_r (i\partial\!\!\!/) \bar{\Psi}^i_r - (Z_2 Z_m - 1) m_r \bar{\Psi}^i_r \Psi^i_r \\
& + \underbrace{(Z_g^2 Z_3^{1/2} - 1)}_{\equiv Z_1 - 1} \frac{1}{2} g \mu^\varepsilon f^{abc} (\partial_\mu A^a_{\nu,r} - \partial_\nu A^a_{\mu,r}) A^{\mu b}_r A^{\nu c}_r \\
& - \underbrace{(Z_g^2 Z_3^2 - 1)}_{\equiv Z_4 - 1} \frac{1}{4} g_r^2 \mu^{2\varepsilon} f^{abc} f^{cde} (A^a_{\mu,r} A^b_{\nu,r} A^{\mu d}_r A^{\nu e}_r) \\
& - \underbrace{(Z_g \widetilde{Z}_r^{1/2} - 1)}_{\equiv \widetilde{Z}_1 - 1} i g_r \mu^\varepsilon f^{abc} (\partial^\mu c^a_r) c^{b*}_r A^c_{\mu,r} \\
& + \underbrace{(Z_g Z_2 Z_3^{1/2} - 1)}_{\equiv Z_{1F} - 1} g \mu^\varepsilon \bar{\Psi}^i_r T^a_{ij} \gamma^\mu \Psi^j_r A^a_{\mu,r}.
\end{aligned} \tag{6.33}$$

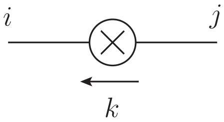
This yields the Feynman rules for the counterterms



$$= (Z_3 - 1)\delta_{ab}(k_\mu k_\nu - k^2 g_{\mu\nu})$$

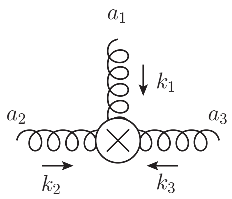


$$= (\widetilde{Z}_3 - 1)\delta_{ab} k^2$$

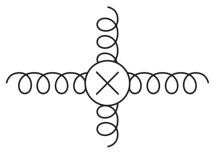


$$= \left[(Z_2 - 1)\not{k} - (Z_2 Z_m - 1)m_r\right]\delta_{ij}$$

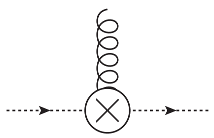
which correct the propagators by absorbing the divergences in self-energy diagrams. Furtheremore, we get the following vertex counterterms:



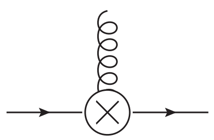
$$= -ig_r f^{a_1 a_2 a_3} V_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)$$



$$= (Z_4 - 1)(-1)g_r^2 W_{\mu_1\mu_2\mu_3\mu_4} f^{a_1 a_2 b} f^{b a_3 a_4}$$



$$= (\widetilde{Z}_1 - 1)(-ig_r) f^{abc} k_\mu$$



$$= (Z_{1\mathrm{F}} - 1) g_r T^a_{ij} \gamma_\mu$$

where the vertex functions $V_{\mu_1\mu_2\mu_3}$ and $W_{\mu_1\mu_2\mu_3\mu_4}$ have been defined in chapter 2.

From these rules, we then also know which counterterm diagram has to be associated with which divergent diagram and how the renormalization constants (Z_i) are associated with the counterterm diagrams.

Remark: Universality of the coupling constant g_r: A priori, we have four different ways of extracting g_r (and therefore the renormalization constant Z_g) from the vertex counterterm diagrams. For example, the quark-gluon vertex corresponds to a term

$$\bar{\Psi}^i_r \left[(Z_{1\text{F}} - 1) g_r T^a_{ij} \gamma_\mu\right] \Psi^j_r A^a_{r\mu} \tag{6.34}$$

in the Lagrangian. According to the definition in Eq. (6.33),

$$(Z_{1\text{F}} - 1) = (Z_g Z_2 Z_3^{1/2} - 1). \tag{6.35}$$

We infer that

$$Z_g = \frac{Z_{1\text{F}}}{Z_2 \sqrt{Z_3}} \tag{6.36}$$

similar as in QED. By gauge symmetry of $\mathcal{L}$, all different ways of extracting g_r and Z_g are the same, leading to the universality of g_r (not proven here). The universality of g_r is a consequence of the generalized Ward identity in the non-Abelian case (the equivalent for $SU(N)$ of the Ward identity for QED). The so-called **Slavnov-Taylor** identity ensures gauge invariance through all orders in QCD. If Z_g is always the same, independent of which counterterm we use to extract it, then not all renormalization constants (Z_i) can be independent. One finds the relations

$$\frac{Z_1}{Z_3} = \frac{\widetilde{Z}_1}{\widetilde{Z}_3} = \frac{Z_{1\text{F}}}{Z_2} = \frac{Z_4}{Z_1}. \tag{6.37}$$

This result is the analog to the identity $Z_1 = Z_2$ in QED (consequence of Ward identity).

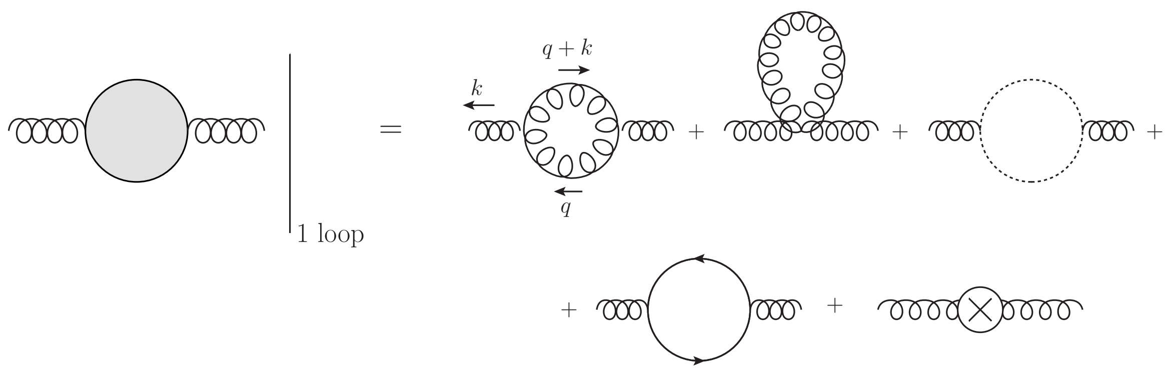
The generalized Ward identities help to reduce the number of renormalization constants and helps to prove the renormalizability of QCD.

6.3.1 One Loop Renormalization of QCD

We need to identify the divergent graphs and the required counterterms in order to extract the renormalization constants. In the $\overline{\text{MS}}$-scheme, we

- adjust the renormalization constants with renormalization scale dependence μ_R, and
- add a finite term $(4\pi)^\varepsilon e^{-\varepsilon\gamma}$.

As an example, consider the gluon self-energy. We have the following corrections to the gluon propagator:



$$= \Pi^{ab}_{\mu} = \delta^{ab}(k_\mu k_\nu - k^2 g_{\mu\nu})\Pi(k^2)$$

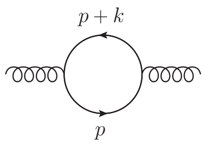
$$\text{with } \Pi(k^2) = \frac{g_r^2}{(4\pi)^2} e^{-\varepsilon\gamma} \left(\frac{4\pi\mu^2}{k^2}\right)^{\varepsilon} \left[\frac{4}{3}T_R N_f - \frac{1}{2}C_G\left(\frac{13}{3} - \alpha_r\right)\right] \cdot \frac{1}{\varepsilon} + (Z_3 - 1) + \text{finite} \tag{6.38}$$

where $T_R N_f$ comes from the quark loop contribution (N_f is the number of quark flavours) and C_G comes from ghost and gluon loops. The color factors T_R and C_G are defined by ($\to$ exercise)

$$\mathrm{tr}(T^a T^b) = \delta^{ab} T_R, \tag{6.39}$$

$$f^{acd} f^{bcd} = \delta^{ab} C_G. \tag{6.40}$$

As an example, the quark loop contribution to Π^{ab} can be calculated as



$$= -N_f \int \frac{d^d p}{(2\pi)^d} \,\mathrm{tr}\left[g\gamma_\mu T^a \frac{1}{[\not p + \not k - m]} g\gamma_\nu T^b \frac{1}{[\not p - m]}\right]$$

where the minus sign comes from the fermion loop and the trace runs over Dirac matrices and T^a generators.

Similarly one can calculate the other 1-loop corrections to the gluon propagator. The 1-loop contributions sum up to give

$$Z_3 = 1 - \frac{g_r^2}{4\pi^2} e^{-\varepsilon\gamma} \left(\frac{4\pi\mu^2}{\mu_R^2}\right)^{\varepsilon} \cdot \left[\frac{4}{3}T_R N_f - \frac{1}{2}C_G\left(\frac{13}{3} - \alpha_r\right)\right] \tag{6.41}$$

such that $\Pi(k^2)$ is finite and contains $\log\left(\frac{\mu_R^2}{k^2}\right)$ dependent terms.

Performing the calculation of all divergent graphs and their associated counterterms, we can show that their sum is finite (at one loop) and therefore conclude that QCD is renormalizable at the 1-loop level, and all Z_i are determined such that the associated counterterm contributions cancel the 1-loop divergences present in the divergent graphs.

6.4 Renormalizability of Spontaneously Broken Gauge Theories

The aim of this section is to show on an explicit example (linear σ-model) that the spontaneous breaking of a renormalizable unbroken QFT does not affect its renormalizability. In particular, the spontaneous symmetry breaking of a renormalizable QFT does not spoil the cancellation of UV divergences between divergent and counterterm graphs. We shall see this for the linear σ-model at 1-loop level.

6.4.1 The Linear σ-Model

Consider N real scalar fields $\phi^i(x)$ described by the Lagrangian

$$\mathcal{L}_\sigma = \sum_i \left[\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 \right] \tag{6.42}$$

This Lagrangian is invariant under the orthogonal transformation

$$\phi^i \longrightarrow R^{ij}\phi^j, \qquad R^{ij} : \text{orthogonal } (N \times N) \text{ matrix} \tag{6.43}$$

which is described by the rotation group in N dimensions, $O(N)$. The potential in $\mathcal{L}_\sigma = T - V$ is

$$V(\phi^i) = -\frac{1}{2}\mu^2|\boldsymbol{\phi}|^2 + \frac{\lambda}{4}|\boldsymbol{\phi}|^4. \tag{6.44}$$

This potential which is symmetric under rotations of $\boldsymbol{\phi}$ is minimized by any constant field configuration ϕ_0^i which satisfies

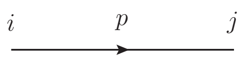
$$\boldsymbol{\phi_0} = \mathbf{0} \qquad \text{or} \qquad |\boldsymbol{\phi_0}| = \sqrt{\frac{\mu^2}{\lambda}} \equiv v, \tag{6.45}$$

depending on the value of μ^2. The situation is analogous to the one in section 4.2.1: For $\mu^2 \leq 0$ we have only one minimum at $|\boldsymbol{\phi}_0| = 0$ and so there is only one vacuum which is $O(N)$-symmetric. For $\mu^2 \geq 0$, all values $\boldsymbol{\phi}_0$ with $|\boldsymbol{\phi_0}| = v$ are minima. This condition only fixes the absolute value of $\boldsymbol{\phi}$ but not its direction. By choosing a particular vacuum, the $O(N)$ symmetry is spontaneously broken.

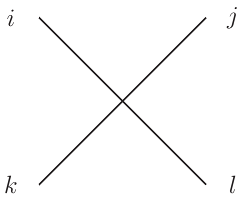
Our next aim is to prove the renormalizability of the unbroken ($\mu^2 \leq 0$) σ-model at 1-loop order. This will not be particularly difficult. But afterwards, we want to show that the broken theory ($\mu^2 > 0$) is still renormalizable at 1-loop order despite having much more divergent diagrams.

Unbroken σ-Model ($\mu^2 \leq 0$)

For $\mu^2 \leq 0$, the symmetry is unbroken and the $\phi_i(x)$ satisfy a Klein-Gordon equation for $m_\phi = m$. The Feynman rules at tree level in this case are



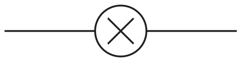
$= \delta^{ij}\frac{i}{p^2-m^2}$

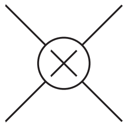


$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$

Note that for two identical ϕ^i, the vertex is just $-2i\lambda$. If all four ϕ^i are identical, then the vertex reads $-6i\lambda$. The factor 6 is related to ϕ^4-theory where the quartic interaction term reads $-\frac{\lambda}{4!}\phi^4$ which gives a vertex $-i\lambda$. In the Lagrangian (6.54), the analogous term has a prefactor $\frac{\lambda}{4}$ instead, so the vertex reads $-6i\lambda$.

At loop level there will appear UV divergences. An amplitude with N_e external legs has superficial degree of divergence $D = 4 - N_e$, so the σ-model is renormalizable according to power counting arguments. One has to include counterterms for the following divergent diagrams:

Self-energy of $\phi^i \to$ correction to propagator: 

correction to the vertex: 

We need to redefine ϕ^j, m and λ. We renormalize ϕ^j with Z, m with Z_m, and λ with Z_λ and consider

$$\begin{aligned} \delta_Z &= Z - 1, \\ \delta_m &= m^2 Z_m - m_r^2, \\ \delta_\lambda &= \lambda Z_\lambda^2 - \lambda_r. \end{aligned} \tag{6.46}$$

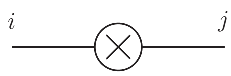
Writing

$$\mathcal{L}_\sigma^{\text{bare}}(Z_i) = \mathcal{L}_\sigma^{\text{ren.}}(Z_i = 1) + \mathcal{L}_\sigma^{\text{counter}}(\delta_i) \tag{6.47}$$

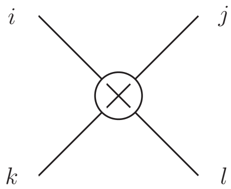
in terms of renormalized quantities $(\boldsymbol{\phi}_r, m_r, \lambda_r)$, we obtain

$$\mathcal{L}_\sigma^{\text{counter}}(\delta_i) = \sum_i \left[\frac{1}{2}\delta_Z(\partial_\mu \phi_r^i)^2 - \frac{1}{2}\delta_m(\phi_r^i)^2 - \frac{\delta_\lambda}{4}\left((\phi_r^i)^2\right)^2 \right]. \tag{6.48}$$

This Lagrangian depends on three renormalization constants $(\delta_Z, \delta_m, \delta_\lambda)$. The Feynman rules for the counterterm diagrams are



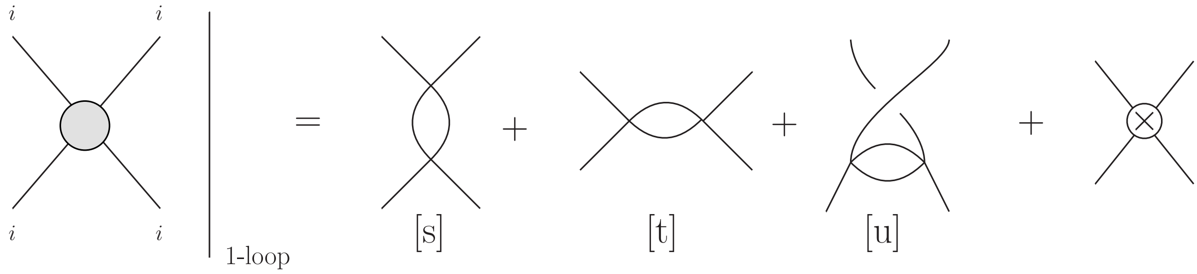
$= i\delta^{ij}(p^2\delta_Z - \delta_m)$



$= -2i\delta_\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$

To obtain one of the δ_i at 1-loop order, we need to compute the sum of the relevant divergent graphs plus the respective counterterm graphs.

For example, to obtain δ_λ, consider the 1-loop corrections to the vertex with four identical ϕ^i:



The s-channel contribution reads

$$\begin{aligned}[s] &\propto (-6i\lambda)^2 \cdot \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \\ &= (18i\lambda^2) \cdot V(p^2, m^2) \end{aligned} \tag{6.49}$$

with $V(p^2, m^2) = \frac{1}{(4\pi)^2}[\Gamma(2 - d/2) + f(p^2, m^2)]$ where f is finite as $\varepsilon \to 0$ and $\Gamma(2 - d/2) = \Gamma(\varepsilon) = \frac{1}{\varepsilon} + \text{finite}$. This yields the amplitude

$$iM = (18i\lambda^2) \underbrace{\frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^2}}_{\equiv L(\varepsilon)} (V'(s) + V'(t) + V'(u)) - 6i\delta_\lambda + \text{finite} \tag{6.50}$$

which fixes

$$\delta_\lambda = \frac{\lambda^2}{(4\pi)^2} \frac{9}{\varepsilon} + \text{finite}. \tag{6.51}$$

In a similar manner, one can calculate the 1-loop corrections to the propagator and thus determine δ_m and δ_Z. Having fixed δ_λ, δ_m and δ_Z, we conclude that the unbroken σ-model is renormalizable since the UV divergences can be removed by a finite number of counterterms.

Broken σ-Model ($\mu^2 > 0$)

For $\mu^2 > 0$, the $O(N)$ symmetry is spontaneously broken. We want to show that the theory is nevertheless renormalizable (at one loop).

The potential $V(\boldsymbol{\phi})$ is minimized for any ϕ_0^i which has $\sum_i (\phi_0^i)^2 = v = \frac{\mu^2}{\lambda}$. Note that $\boldsymbol{\phi}_0$ is fixed in norm, not in direction. We choose ϕ_0^i such that $\boldsymbol{\phi}_0$ points in the N-th direction:

$$\boldsymbol{\phi}_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}, \qquad v = \frac{\mu}{\sqrt{\lambda}}. \tag{6.52}$$

We consider a perturbation close to this minimum ϕ_0^i for ϕ^i parametrized as follows:

$$\phi^i(x) = \left(\Pi^k(x), v + \sigma(x)\right) \qquad (k = 1, ..., N-1). \tag{6.53}$$

Here, $\Pi^k(x)$ corresponds to the massless Goldstone bosons and $\sigma(x)$ is the massive real scalar field. In terms of these fields the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_\sigma[\Pi^k, \sigma] = &\frac{1}{2}(\partial_\mu \Pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \\ &- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\Pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\Pi^k)^2\sigma^2 - \frac{\lambda}{4}\left((\Pi^k)^2\right)^2. \end{aligned} \tag{6.54}$$

This Lagrangian contains

- a massive scalar field σ with $m_\sigma^2 = 2\mu^2$,
- a set of $(N-1)$ massless Π fields (scalars).

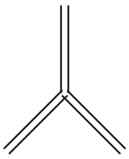
The $O(N)$ symmetry is broken, but there is still an $O(N-1)$ symmetry present allowing the Π-fields to rotate among themselves.

Using a double line for σ-fields and a single line for Π-fields, the Lagrangian (6.54) yields the following propagators:

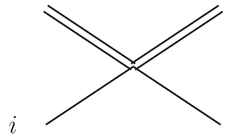
 $= \frac{i}{p^2 - 2\mu^2}$

$= \frac{i\delta^{ij}}{p^2}$

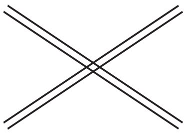
We can also immediately read off the tree level vertex rules (with μ replaced by $[v\sqrt{\lambda}]$):



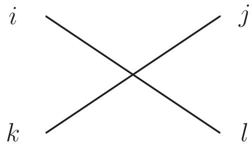
$= -6i\lambda v$



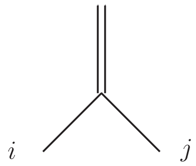
$= -2i\lambda\delta_{ij}$



$= -6i\lambda$



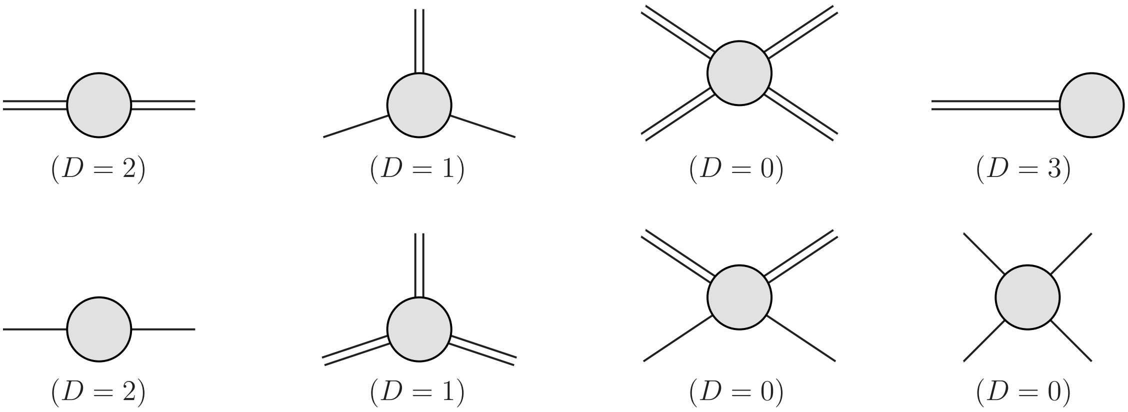
$= -2i\lambda(\delta_{ij}\delta_{kl}+$
$+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$



$= -2i\delta_{ij}\lambda v$

From these Feynman rules, one can compute tree level amplitudes.

What happens at 1-loop level? We still have $D = 4 - N_e$ as for the unbroken σ-model, so that a priori the broken linear σ-model is expected to be renormalizable from the superficial degree of divergence argument. In the case of the unbroken σ-model we had 2 superficially divergent graphs whose divergences could be absorbed into the 3 counterterms. However, in the broken case we need a priori 8 counterterms, one for each of the following divergent graphs:



How to define these counterterm diagrams? We start with the counterterm Lagrangian $\mathcal{L}_\sigma^{\text{counter.}}[\phi^i]$ in terms of the fields ϕ^i. This Lagrangian depends on three renormalization constants $(\delta_Z, \delta_m, \delta_\lambda)$. We can then insert the fields (Π, σ). Writing

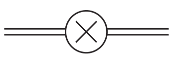
$$\phi_r^i = \left(\Pi_r^k(x), v + \sigma_r(x)\right), \tag{6.55}$$

we obtain

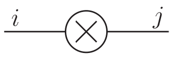
$$\begin{aligned}\mathcal{L}_\sigma^{\text{counter.}}[\Pi_r,\sigma_r](\delta_i) =& \frac{\delta_Z}{2}(\partial_\mu \Pi^k)^2 - \frac{1}{2}(\delta_\mu + \delta_\lambda v^2)(\Pi^k)^2 - (\delta_\mu v + \delta_\lambda v^3)\sigma - \\ &- \delta_\lambda v\sigma(\Pi^k)^2 - \delta_\lambda v\sigma^3 + \frac{\delta_Z}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\sigma^2 \\ &- \frac{\delta_\lambda}{4}\left((\Pi^k)^2\right)^2 - \frac{\delta_\lambda}{2}\sigma^2(\Pi^k)^2 - \frac{\delta_\lambda}{4}\sigma^4 \end{aligned} \tag{6.56}$$

where we have dropped the intex r. The associated Feynman rules for the counterterm diagrams still depend on three renormalization constants only and they read

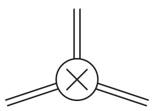
$= -i(\delta_\mu v + \delta_\lambda v^3)$



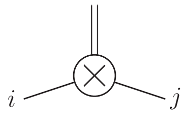
$= i(\delta_Z p^2 - \delta_\mu - 3\delta_\lambda v^2)$



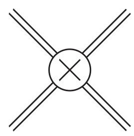
$= i\delta_{ij}(\delta_Z p^2 - \delta_\mu - \delta_\lambda v^2)$



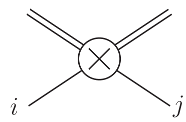
$= -6i\delta_\lambda v$



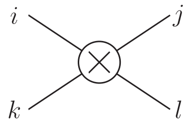
$= -2i\delta^{ij}\delta_\lambda v$



$= -6i\delta_\lambda$

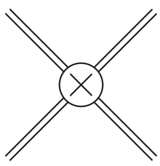


$= -2i\delta_{ij}\delta_\lambda$



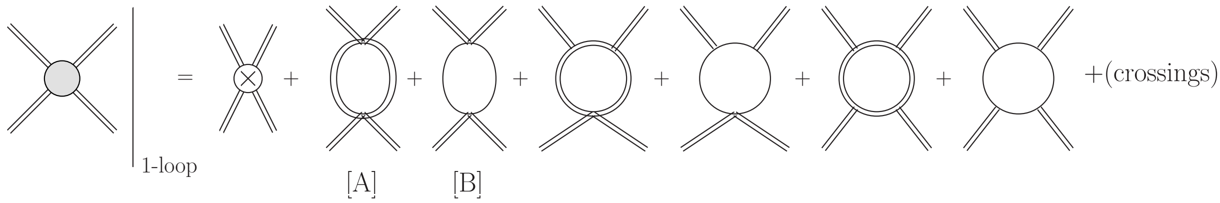
$= -2i\delta_\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$

As in the unbroken theory, the counterterms only depend on three renormalization constants. The question is, whether these are sufficient to cancel all divergences (8 diagrams!) of the broken theory. The answer will turn out to be yes. We will prove this for the vertex diagrams: evaluate δ_λ from (4σ)-diagrams, see that it cancels the divergence in $(2\sigma)(2\Pi)$-diagrams and therefore deduce that the δ_λ found can be used for all divergent vertex diagrams. We obtain δ_λ by computing the 1-loop correction to the (4σ)-amplitude given at tree level by the (4σ)-vertex



$= -6i\delta_\lambda$

At one loop, σ and Π fields can propagate in the loop. There are three types of diagrams with σ and Π in the loop and three 4-point vertices:



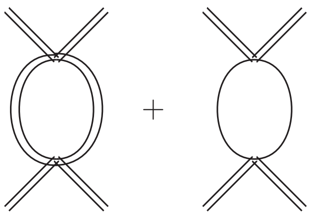
where the last four indicated contributions (and their crossings) are convergent by power counting arguments. Indeed, as soon as there are three or more propagators in the loop, the loop integral converges. The other two diagrams can be calculated, too. We find as an example,

$$\begin{aligned}[\mathrm{A}] &= \frac{1}{2}(-6i\lambda)^2 \int \frac{d^d k}{(2\pi)^d} \frac{i}{[k^2 - (2\mu^2)]} \frac{i}{[(k+p)^2 - (2\mu^2)]} \\ &= 18i\lambda^2 L(\varepsilon) + \text{ finite} \end{aligned} \tag{6.57}$$

where $L(\varepsilon) = \Gamma(2 - d/2)/(4\pi)^2$ as in Eq. (6.50). Similarly, one finds

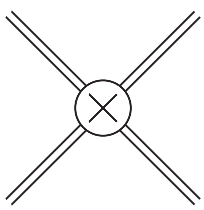
$$[\mathrm{B}] = 2i\lambda^2 (N-1) L(\varepsilon) + \text{ finite} \tag{6.58}$$

where $(N-1)$ is the number of Π-fields. Since the divergent parts of these contributions are independent of momenta, the crossed diagrams give exactly the same contributions. Counting the crossed diagrams, the infinite results of $[A]$ and $[B]$ need to be multiplied by 3, respectively. This way we get the (4σ)-contribution (at one loop) given by



+ (crossings) $= 6i\lambda^2(N+8)L(\varepsilon) + \text{ finite}$

Comparing this to the (4σ)-counterterm diagram



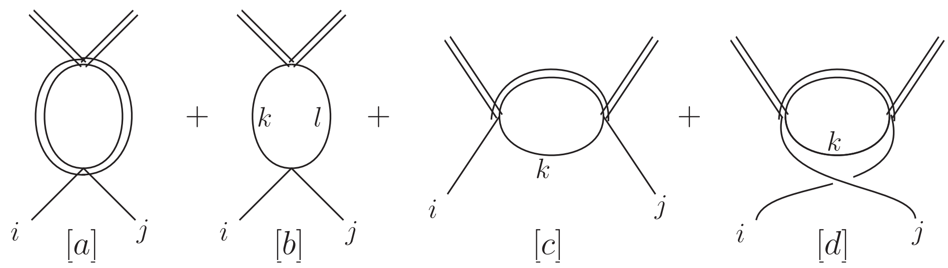
$= -6i\delta_\lambda$

we conclude that

$$\delta_\lambda = \lambda^2(N+8)L(\varepsilon) = \frac{(N+8)\lambda^2}{(4\pi)^2}\frac{1}{\varepsilon}. \tag{6.59}$$

Remember that in the unbroken theory we had $\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2}\frac{1}{\varepsilon}$. This corresponds just to the case where the four ϕ^i-legs are all equal and therefore $N = 1$.

We have now fixed the δ_λ counterterm from the 1-loop counterterm to the (4σ)-vertex. Does it also fix the divergences of other vertex corrections? Consider, for example, the 1-loop correction of the amplitude for $(2\sigma)(2\Pi)$ which is given by



(again, there are further corrections which, however, are finite having more than two loop propagators). Each of these loop integrals has an equal infinite part which is proportional to

$$-i\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} = -iL(\varepsilon). \tag{6.60}$$

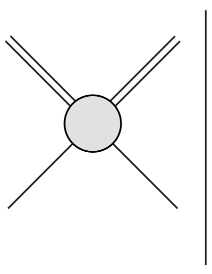
The only differences in these divergent diagrams come from the vertices. One finds that the infinite contributions to the four diagrams are

$$[a] \sim -iL(\varepsilon)\cdot\frac{1}{2}\cdot(-6i\lambda)(-2i\delta^{ij}\lambda) \tag{6.61}$$

$$[b] \sim -iL(\varepsilon)\cdot\frac{1}{2}\cdot(-2i\delta^{kl}\lambda)(-2i\lambda(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})) \tag{6.62}$$

$$[c] = [d] \sim -iL(\varepsilon)\cdot\frac{1}{2}\cdot(-2i\delta^{ik}\lambda)(-2i\delta^{jk}\lambda) \tag{6.63}$$

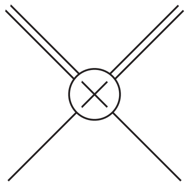
The sum of these four diagrams yields



$$= 2i\lambda^2\delta^{ij}L(\varepsilon)(N+8)$$

1-loop

which is indeed cancelled by

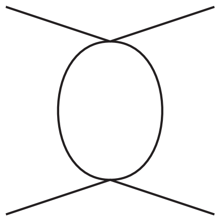


$$= -2i\delta^{ij}\delta_\lambda$$

with δ_λ obtained in Eq. (6.59) from the cancellation of (4σ)-divergences.

Similar results can be obtained for the other combinations (e.g. ΠΠΠΠ) such that δ_λ found from one vertex correction (4σ) is sufficient to cancel all the UV divergences present in all divergent graphs of the broken theory. Similar conclusions hold for δ_m and δ_Z.

In the particular case of the vertex diagrams, we can understand this effect as follows. All the diagrams are manifestations of the same basic diagram



If the $O(N)$ symmetry is broken, this diagram manifests itself in different ways with loops containing Π- or σ-fields. However, the divergent part of this diagram is unaffected by the symmetry breaking (i.e. the appearence of other fields in the loop). Three renormalization constants in the broken theory are thus sufficient to cancel all (UV-) divergences as in the unbroken theory.

We have thus shown in the particular example of the linear σ-model that if a QFT is renormalizable, then the spontaneous breaking of the theory does not affect the cancellation of UV-divergences and therefore the spontaneously broken theory is renormalizable, as well.

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