WEINBERG
22.3

• The Anomaly in 4d •

Let us move now to the more realistic and useful case of 4 space-time dimensions. The general lessons learned in the simple 2d example will apply to this case as well, but the technical details will be rather different, and significantly more involved. We will sketch here the calculation of the 4d anomaly in the completely general case of arbitrary conserved currents, you are referred to the Weinberg for the few missing details. Just for completeness, be aware that other anomalies (gravitational, discrete) exist in 4d. The ones we will treat are however the most relevant ones for Particle Physics applications.

Let us first settle our framework. Consider a generic theory (such as QCD, or the EW one) with a generic content of chiral fermions ψ_L and ψ_R, and a set of global symmetry generators t_L^α, t_R^α acting on, respectively, ψ_L and ψ_R. By charge conjugation, our L- and R-handed degrees of freedom can be collected in a single multiplet χ_L of fixed chirality.

Having chosen χ to be L-handed, we have

$$\chi_L = \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix}$$

where $(\psi_R)^c \equiv C\psi_R^*$ is a L-handed spinor. Due to the conjugation, the action of the generators on χ_L is given by the matrices

$$T_\alpha = \begin{pmatrix} t_L^\alpha & 0 \\ 0 & -(t_R^\alpha)^* \end{pmatrix}$$

Correspondingly the global conserved currents are

$$J_\alpha^\mu = \bar{\chi}_L T_\alpha \gamma^\mu \chi_L = \bar{\psi}_L t_L^\alpha \gamma^\mu \psi_L - \overline{\psi_R^c} (t_R^\alpha)^* \gamma^\mu (\psi_R)^c$$
$$= \bar{\psi}_L t_L^\alpha \gamma^\mu \psi_R + \bar{\psi}_R t_R^\alpha \gamma^\mu \psi_R$$

Using that the generators are hermitian

In our general theory, the fermions could be massive, and the global group could be broken by some tiny effects (like the quark mass in QCD). Moreover, the fermions typically interact among themselves and with other fields. To compute the Anomaly, however, we will merely work in the <u>free</u> <u>massless</u> theory of our fermions. The calculation of the Anomaly that we will perform under this assumption, however, is useful also in more realistic cases. First of all, concerning the <u>interactions</u>, it is possible to show that they <u>do not change</u> the result, provided they respect the symmetries under consideration. The reason is that the Anomaly comes from the <u>very specific structure</u> of the leading order diagram, and <u>this is the same</u> in the free and in the interacting theory. The corrections coming from the insertion of interaction vertices in the diagram <u>do not</u> have this peculiar structure, so they cannot lead to Anomalies. In QCD, this leads to the very important result that the <u>chiral anomaly</u> can be computed at order g^0_{STRONG}, <u>and it is exact</u>. The anomaly is the only object of QCD that we can compute exactly, even in the IR where the theory is non-perturbative and nothing else can be computed in perturbation theory! Second, the effect of <u>masses</u>. We will show that any pair of fermions whose mass <u>is allowed</u> by the symmetry does not contribute to the Anomaly. So also in this case, the total Anomaly is unchanged if the symmetry is preserved. If a mass (like in QCD) or even an interaction were to break the symmetry, we can still use the massless calculation for the anomaly (because, again, insertions of any interaction or even a mass in the diagram ruins the peculiarity of the loop integral that leads to the Anomaly), only you have to pay attention that it is the "naive" Ward Identity itself that changes. Typical example is a mass, that breaks some axial symmetry, the naive Ward identity gets changed into

$$\partial_\mu A^\mu = 0 \longrightarrow \partial_\mu A^\mu = m\bar{\Psi}\gamma^5\Psi$$

While the Anomaly changes this operatorial identity into:

$$\partial_\mu A^\mu = \mathcal{A} \longrightarrow \partial_\mu A^\mu = m\bar{\Psi}\gamma^5\Psi + \mathcal{A}$$

where $\mathcal{A}$ is simply the one computed in the massless theory.

Let us start with the calculation. In 4d it is not the 2-point correlator, but the 4-point one that one has to compute.

Let us define:

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \langle T\left[J^\mu_\alpha(x) J^\nu_\beta(y) J^\rho_\gamma(z)\right]\rangle$$

In Fourier space:

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\, e^{-i(k_1+k_2)x}\, e^{ik_1y} e^{ik_2z}\, \tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(k_1,k_2)$$

we will be interested in computing

$$\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \left[-i(k_1+k_2)_\mu \tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(k_1,k_2)\right]\Big|_{F.T.}$$

Before computing the Anomaly, let us recapitulate what would the "naive" result be. As in the 2d case, we proceed as follows. Perform a field redefinition in the path integral which is given by an x-dependent global transformation:

$$\chi_L \longrightarrow \chi_L^{(\sigma)} = e^{i\sigma_\alpha(x)T^\alpha}\chi_L = \chi_L + i\sigma_\alpha(x)T^\alpha\chi_L$$

by definition, the current is such that

$$S[\chi_L^{(\sigma)}] = S[\chi_L] + \int d^4x\, \sigma_\alpha(x)\partial_\mu J^\mu_\alpha(x)$$

From the identity:

$$\int D\chi_L^{(\sigma)} e^{iS[\chi_L^{(\sigma)}]} J^{\nu,(\sigma)}_\beta(y) J^{\rho(\sigma)}_\gamma(z) = \int D\chi_L e^{iS[\chi_L]} J^\nu_\beta J^\rho_\gamma$$

and expanding in σ_α, we will find the Ward identity. Differently from the case of last lesson (where we were dealing with abelian symmetries), we now have contributions from the variations of the currents as well:

$$J^{\nu\,(\sigma)}_\beta(y) \simeq J^\nu_\beta(y) + i\,\sigma_\alpha(y)\,\bar{\chi}_L \gamma^\nu [T_\beta, T_\alpha] \chi_L =$$

$$= J^\nu_\beta(y) - \sigma_\alpha(y) f_{\alpha\beta\delta}\, J^\nu_\delta(y)$$

$$= J^\nu_\beta(y) - f_{\alpha\beta\delta} \int d^4x\, \delta^4(x-y)\, \sigma_\alpha(x)\, J^\nu_\delta(y)$$

Putting all the terms toghether leads to the "naive" Ward Identity:

$$i\,\partial_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \delta^4(x-y)\, f_{\alpha\beta\delta} \langle J^\nu_\delta(y)\, J^\rho_\gamma(z)\rangle +$$

$$+ \delta^4(x-z)\, f_{\alpha\gamma\delta} \langle J^\nu_\beta(y)\, J^\rho_\delta(z)\rangle$$

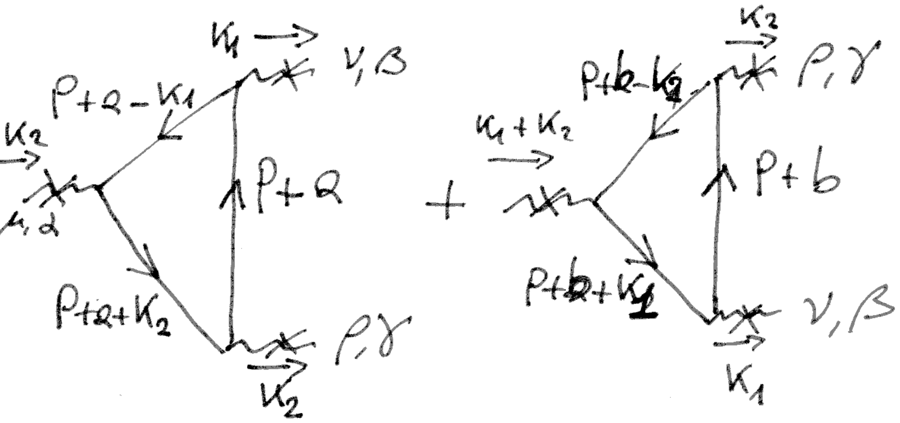
where the structure constants f have been defined, as usual, as

$$[T_\alpha, T_\beta] = i\, f_{\alpha\beta\gamma}\, T_\gamma$$

the r.h.s. terms in the Ward identity are called "contact terms" because they only show up when one of the currents "touches" the position 'x' where $\partial_\mu J^\mu$ is computed.

With the aim of computing the amount of violation of the naive identity, let us write down the diagrammatic contributions to $\tilde{\Gamma}$. There are two:

$$\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(k_1,k_2) =$$



$$= + i \int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{tr}\left[\frac{\not p + \not a - \not k_1}{(p-k_1+a)^2 - i\varepsilon} \gamma^\nu P_L \frac{\not p + \not a}{(p+a)^2 - i\varepsilon} \cdot \gamma^\rho P_L \frac{\not p + \not a + \not k_2}{(p+a+k_2)^2 - i\varepsilon} \gamma^\mu P_L \right] \mathrm{tr}\left[T_\alpha T_\beta T_\gamma\right] + \mathrm{tr}\left[\frac{\not p + \not b - \not k_2}{(p-k_2+b)^2} \gamma^\rho P_L \frac{\not p + \not b}{(p+b)^2} \gamma^\nu P_L \frac{\not p + \not b + \not k_1}{(p+b+k_1)^2} \gamma^\mu P_L \right] \cdot \mathrm{tr}\left[T_\alpha T_\gamma T_\beta\right] \right\}$$

Remember the fermionic loop

where $P_L = \dfrac{1+\gamma_5}{2}$.

Several comments are in order. First, about the presence of P_L in all the vertices of the currents. This comes from the fact that $\chi = \chi_L$ is a chiral fermion, not a Dirac one. In order to compute in the theory with such a chiral state, I should go back to the fermion field's quantization, solve the EOM, work out the propagator etc. The shortcut we are using is to modify the theory by adding χ_R, which combines with χ_L to make one massless Dirac field, for which we can use the standard Feynman rules. Of course, we want that the χ_R we have added does not change our result, so we must pay attention that it decouples completely. This explains the P_L factor, that forbids the coupling of χ_R to the currents.

The second point is the constants shifts "a" and "b" that we have made in the loop integral. This shift reflects our ambiguity in the choice of the regulator, and this can be understood as follows. Suppose you have introduced your PV regulator and written

$$\tilde{\Gamma}_{reg} = \tilde{\Gamma}_0 - \tilde{\Gamma}_{PV} = \int dp\, f(p) - \int dp\, f(p, M_{PV})$$

this is finite because for $p \to \infty$ the two terms in the integrand cancel each other. The same would happens if we shift the first integral and define a new regulated correlator as

$$\tilde{\Gamma}'_{reg} = \int dp\, f(P+a) - \int dp\, f(P, M_{PV})$$

changing "a" (and "b") is then like changing the regulator. We will now manipulate the correlator (or better, its divergence) in order to obtain the anomaly, in a way that the subtracted PV term will not matter, this is why we can ignore it and just put "a" and "b" as a remainder of the ambiguity in the regulator.

We can compute the divergence of $\tilde{\Gamma}$, it is convenient to use the identity

$$\not{K}_1 + \not{K}_2 = (\not{P} + \not{K}_2 + \not{a}) - (\not{P} - \not{K}_1 + \not{a}) =$$
$$= (\not{P} + \not{K}_1 + \not{b}) - (\not{P} - \not{K}_2 + \not{b})$$

which leads to

$$(-i)(K_1+K_2)_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(K_1, K_2) = \int \frac{d^4 P}{(2\pi)^4} \left\{ tr\left[T_\alpha T_\beta T_\gamma\right] \cdot \right.$$
$$\left[tr\left[\frac{\not{P}+\not{a}-\not{K}_1}{(P-K_1+a)^2} \gamma^\nu \frac{\not{P}+\not{a}}{(P+a)^2} \gamma^\rho \frac{1+\gamma_5}{2} \right] + \right.$$
$$\left. \left. - tr\left[\frac{\not{P}+\not{a}}{(P+a)^2} \gamma^\rho \frac{\not{P}+\not{a}+\not{K}_2}{(P+a+K_2)^2} \gamma^\nu \frac{1+\gamma_5}{2} \right] \right] \right\} + tr\left[T_\alpha T_\gamma T_\beta\right] \cdot \left[\begin{array}{c} a \to b \\ K_1 \leftrightarrow K_2 \\ \nu \leftrightarrow \rho \end{array} \right]$$

At this point, we rewrite

$$\mathrm{tr}[T_\alpha T_\beta T_\gamma] = \frac{1}{2}\mathrm{tr}[\{T_\alpha, T_\beta\} T_\gamma] + \frac{1}{2}\mathrm{tr}[[T_\alpha, T_\beta] T_\gamma]$$

$$= D_{\alpha\beta\gamma} + i N f_{\alpha\beta\gamma}$$

where $\mathrm{tr}[T_\alpha T_\beta] = N \delta_{\alpha\beta}$. It is easy to check that D (the so-called "D-symbol") is completely symmetric.

Using the formula above, we get a part of the divergence which is proportional to "D", and one which is proportional to "f". You will check as an exercise that the term proportional to f exactly reproduces the contact terms in the Naive Ward identity.

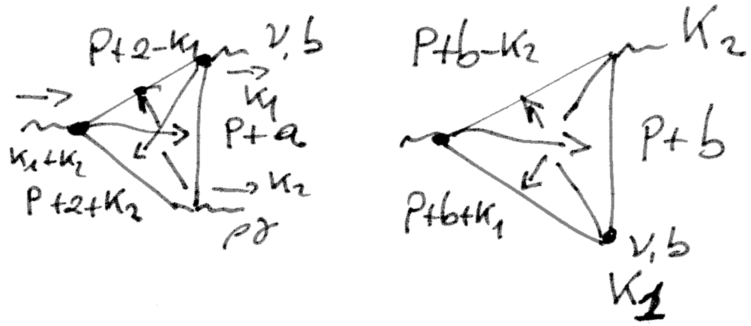
To compute the anomaly, we therefore only need the "D" part. This immediately tells us a very important thing: "The Anomaly is proportional to D", where

$$D_{\alpha\beta\gamma} \equiv \frac{1}{2}\mathrm{tr}[\{T_\alpha, T_\beta\} T_\gamma]$$

In particular, the anomaly cancels if $D = 0$.

The D-term in the divergence is finite, and it can be computed in a rather straightforward way, you should read the Weinberg for the details, here we will just discuss the results. But before doing that, let us discuss another aspect that will be useful soon. What if I had wanted to compute the divergence of $J_\nu^\beta(y)$ instead? We would have started from the same $\tilde{\Gamma}$, with the same "a" and "b" shifts (for each correlator there is only one choice of regulator to be made), but we would have contracted with $(i K_1)^\nu$. Given that the labeling of the momenta stays the same, the difference that we will encounter in this new calculation with respect to what we saw before is that before we were contracting with that external momentum, $(K_1 + K_2)$, that was opposite to the "$p+q$" leg, while now we have the one opposite to the "$p + q + K_2$" leg. This means that in the calculation of $\partial_\nu^y \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}$ it will be like replacing $a \to a + K_2$, and similarly for $\partial_\rho^z \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}$ with $a \to a - K_1$. For b, analogously, the correspondence is $b \to b - K_2$ and $b \to b + K_1$

Pictorially:



Let us now go back to our calculation and describe the result. First of all, there is one piece of the anomaly that can be cancelled by one choice of the regulator. This is the one associated to the $tr[\gamma\gamma\gamma\gamma]$ you get in the calculation, remember that we had

$$tr[\gamma\gamma\gamma\gamma(1+\gamma_5)] \cong tr[\gamma\gamma\gamma\gamma] + i\,\varepsilon^{\cdots}$$

the first term can be made to vanish by the choice:

$$a = -b$$

By this choice, the anomalous term cancels in $\partial^{\alpha}_{\mu} \Pi^{\mu\nu\rho}_{\alpha\beta\gamma}$, but also in $\partial^{\nu}_{\mu} \Pi^{\mu\nu\rho}_{\alpha\beta\gamma}$ etc. This is because what we saw before: computing the divergence of another current is like shifting "a" and "b", but the shift is such that it leaves "a+b" invariant. The $a=-b$ condition is therefore invariant!

Given that we can cancel it, the anomalous term in $tr[\gamma\gamma\gamma\gamma]$ is not an anomaly. It is just due to a "bad choice" of the regulator. If we make such a choice, we add an additional violation of the symmetry, that however we might have avoided by simply choosing $a=-b$.

Let us choose $a=-b$, then, the final result is rather simple, and reads

$$\left[(-i)(K_1+K_2)_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(K_1,K_2)\right]\Big|_{AN} = \frac{2 D_{\alpha\beta\gamma}\pi^2}{(2\pi)^4} \varepsilon^{\kappa\nu\lambda\rho} a_\kappa (K_1+K_2)_\lambda$$

we immediately see that also this one we can eliminate it is enough to choose

$$a_\mu \propto (K_1+K_2)_\mu$$

It is not that easy, however, because the above choice would not cancel the Anomaly in the other currents as well. The above condition for the cancellation can be stated that the momentum one contracts with the amplitude (the one associated to the field you differentiate) must be parallel to "a", defined as before as the shift in p of the line opposite to the current you are differentiating. To cancel $\partial^\mu_\nu \Gamma$ we would therefore need

$$a + K_2 \propto K_1$$

while for $\partial^\sigma_\rho \Gamma$:

$$a - K_1 \propto K_2$$

Only two of these 3 conditions can be solved simultaneously. Suppose for instance that you want to preserve the conservation of $J^\beta_\nu, J^\gamma_\rho$. In this case you must have:

$$\left.\begin{array}{l} a = \delta_1 K_1 - K_2 \\ a = \delta_2 K_2 + K_1 \end{array}\right\} \Rightarrow \quad \begin{array}{c} (1-\delta_1) K_1 = -(1+\delta_2) K_2 \\ \\ \delta_1 = 1, \quad \delta_2 = -1 \end{array}$$

$$\Downarrow$$

$$\boxed{a = K_1 - K_2}$$

With this choice, the anomaly is

$$[\cdots] = \frac{1}{4\pi^2} D_{\alpha\beta\gamma}\, \varepsilon^{\mu\nu\kappa\rho} K_{1\kappa} K_{2\lambda}$$

It should be clear that the choice above is not, a priori, the "correct one". Normally, it is up to us the choice of which currents should be conserved. The only case in which there is no freedom is if J_β and J_γ are currents of some local group, meaning that in the theory there exist also gauge fields which are coupled to these currents. The anomaly for these currents must cancel for the consistency of the theory. If all the 3 currents were gauged, hoping that D cancels is all we can do. If instead two currents are gauged and the third is not, we can (we must, for the theory to make sense) choose the regulator such as to move all the anomaly on the global current's leg.

The choice of "a" determines what is called "the form" of the Anomaly. When there is not any special reason for choosing one or another, one typically uses the so-called "symmetric form" of the Anomaly, which is one in which all the 3 currents are not conserved, but the symmetry under the exchange of the external legs is maintained. This choice turns out to correspond to

$$a = \frac{1}{3}\left(K_1 - K_2\right)$$

which of course leads to a result for the anomaly which is one third of the previous one