• Anomalies and Strong Interactions •

There are two aspects of the Physics of Strong Interactions for which anomalies play a crucial role. The first is the $U(1)_A - g - g$ anomaly, which explains the absence of a further light meson, the second is the "$SU(2)_A$" $-\gamma-\gamma$ anomaly, which explains the large rate of $\pi \to \gamma\gamma$ decay. Let us discuss these two subjects in turn.

Consider two-flavor QCD, we have studied the implications of the $SU(2)_L \times SU(2)_R$ global symmetry which is present if the up and down quarks are massless, and is broken spontaneously to $SU(2)_V$ by the VEV of operators in the $(2,2)$ like

$$X^{ij} \sim \bar{q}_L^{\,i} \; q_R^{\,j}$$

Actually, we have completely ignored in our former discussion that the symmetry group is instead $U(2)_L \times U(2)_R$, where the two additional $U(1)_L \times U(1)_R$ generators can be recast as $U(1)_B \times U(1)_A$, where, in the usual "chiral" notation which is useful to compute the anomalies:

$$T_B = \begin{pmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \\ & & -\mathbb{1} & \\ & & & -\mathbb{1} \end{pmatrix} = 3 \cdot B \; ; \; T_A = \begin{pmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \\ & & \mathbb{1} & \\ & & & \mathbb{1} \end{pmatrix}$$

where "$\mathbb{1}$" is the 3×3 identity, due to the fact that the quarks are in the color $\underset{\sim}{3}$ and $\underset{\sim}{\bar{3}}$. In this notation, the $SU(3)_c$ ~~th~~ color generators are

$$T^a = \begin{pmatrix} \lambda^a & & & \\ & \lambda^a & & \\ & & -\lambda^{a\,t} & \\ & & & -\lambda^{a\,t} \end{pmatrix}$$

from which we immediately see that while $U(1)_B$ has no anomalies with the color group, $\underline{U(1)_A}$ $\underline{\text{has}}$. This tells us that $U(1)_A$ is $\underline{\text{not}}$ $\underline{\text{really}}$ $\underline{\text{a}}$ $\underline{\text{symmetry}}$ at the quantum level, and this is reassuring. Was **it** a symmetry, the VEV of X^{ij} $\underline{\text{would}}$ $\underline{\text{break}}$ $\underline{\text{it}}$ $\underline{\text{spontaneously}}$, and this would imply the existence of a $\underline{\text{new}}$ $\underline{\text{Goldstone}}$ $\underline{\text{Boson}}$, approximately massless,

i.e. with similar mass as the pions.
No such a state exist, the only plausible candidate would be the η', which has however a very high mass, of around 980 MeV. It is because of the anomaly that it gets such a big mass, even though the mechanism through which this happens is rather complicated, it involves instantons, we will perhaps discuss it in the following lectures.

The second anomaly we are interested in is a violation of the axial $T_L - T_R$ generators, and in particular to the ones associated with the π^0 neutral pion. Let us consider QCD with two flavors, the normalized T_L and T_R generators are:

$$\vec{T}_L = \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & 0 \end{pmatrix} \; ; \quad \vec{T}_R = \begin{pmatrix} 0 & 0 \\ 0 & -\vec{\sigma}^t/2 \end{pmatrix}$$

on top of this, there is the $U(1)_B$ we already mentioned:

$$T_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where we omitted to indicate the 3x3 blocks associated to the 3 colors. From now on, let us generalize the number of colors from 3 to a generic N_c, we will see that by the $\pi \to \gamma\gamma$ rate and the anomaly we will be able to <u>measure</u> $\underline{N_c}$.

The electric charge is

$$Q = T_L^3 + T_R^3 + \frac{1}{6} T_B =$$

$$= \begin{pmatrix} 2/3 & & & 0 \\ & -1/3 & & \\ & & -2/3 & \\ 0 & & & 1/3 \end{pmatrix}$$

so that we have an anomaly:

$$D^{T_L^3 QQ} = tr\left[Q^2 T_L^3\right] = N_c \times \left[\frac{4}{9} - \frac{1}{9}\right] \times \frac{1}{2}$$

$$= \frac{N_c}{3} \times \frac{1}{2}$$

$$D^{T_R^3 QQ} = tr\left[Q^2 T_R^3\right] = -\frac{N_c}{3} \times \frac{1}{2}$$

$$\Rightarrow D^{T_V^3 QQ} = 0 \;, \quad D^{T_A^3 QQ} = \frac{N_c}{3} \;; \quad T_A^3 = T_L^3 - T_R^3$$

Notice that the above anomoly is not due to an anomoly within the chiral $SU(2)_L \times SU(2)_R$ group. The latter is indeed absent, it instead comes from

$$D^{T_L^3 T_L^3 T_B} = \frac{1}{2} N_c$$

$$D^{T_L^3 QQ} = \frac{2}{6} D^{T_L^3 T_L^3 T_B} = \frac{1}{2} \frac{N_c}{3}$$

In the 3-flavor case, the same anomoly is found in a rather different way: there the electric charge is a combination of generators which are inside the $SU(3)_L \times SU(3)_R$ chiral group. The $T_A^3 QQ$ anomoly therefore comes from an anomoly within the $SU(3)$. This latter anomoly is called the "Adler-Bardeen Anomoly", and it contains other anomalous variations than the one which we need here. You will check as an exercise that even starting from the $SU(3)$ chiral theory the anomoly is exactly the same. The deep reason for that is what we showed about "potentially massive" spinors, that do not contribute to the anomaly. This is why adding the strange quark does not change the anomaly, the same for the c, b, t... the result agrees with the intuitive notion that heavy particles are "hidden", and do not contribute (or better, weakly contribute) to the low-energy physics.

Our axial anomaly reads:

$$\left[\partial_\mu \langle J_A^\mu J_{em}^\nu J_{em}^\rho \rangle\right]_{FT} = \frac{1}{4\pi^2} \varepsilon^{\kappa\nu\lambda\rho} K_{1\kappa} K_{2\lambda} \cdot \frac{N_c}{3}$$

where of course we have chosen the regulator such as the e.m. current is conserved.

In order to draw easily and rigorously all the consequences of the presence of an anomaly, and here the discussion becomes general again, it is very useful to use the method of introducing sources for the global currents. Suppose indeed that you define

$$Z[A] = \int D\psi \, e^{iS[\psi] + i\int J_\mu^a A_a^\mu}$$

where $A^\mu_\alpha(x,t)$ are non-dynamical "sources" for the global currents, you can think to them as external background fields that perturb the theory. As a function of the sources, Z is the generating functional of current correlators, in the sense that taking functional derivatives and putting A to zero gives the correlators of the original (unperturbed) theory. If there is no anomaly, Z is invariant if A transforms as if it was the gauge field associated to the T^α symmetry:

$$A^\alpha_\mu \to A^\alpha_\mu + (D_\mu \sigma)^\alpha \Rightarrow Z[A] \to Z[A]$$

If there is an anomaly, this is not the case and we have, in general

$$Z[A] \to e^{i\int \alpha(x) \mathcal{A}(x,\Lambda)} Z[A]$$

Let us see how this work for an abelian symmetry. Remember that

$$Z[A] = e^{-i E[A]}$$

where E is the generating functional of the connected Green functions:

$$E[A] = i(i)^2 \iint dx \langle JJ \rangle AA + i(i)^3 \iiint \langle JJJ \rangle AAA + \dots$$

$$E[A+\partial\sigma] = -i(i)^3 \iiint \langle \partial_\mu J^\mu \tilde{J}\tilde{J} \rangle \sigma \tilde{A}\tilde{A} + E[A]$$

where $\tilde{J}$ is in general a different current than J. We assumed that the anomaly only appears in the 3-point correlator that we have computed. This gives:

$$E[A+\partial\sigma] = E[A] - \frac{1}{4\pi^2}\frac{N_c}{3}\iiint dx \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\sigma(z)\, e^{ik_1x} e^{ik_2y} e^{-i(k_1+k_2)z} \varepsilon^{\mu\nu\lambda\rho} k_{1\mu} k_{2\lambda} \tilde{A}_\nu(x)\tilde{A}_\rho(y)$$

$$= E[A] + \frac{(N_c/3)}{4\pi^2}\int d^4z\; \varepsilon^{\mu\nu\lambda\rho}\partial_\mu \tilde{A}_\nu \partial_\lambda \tilde{A}_\rho \times \sigma(z)$$

where the last step is left to the reader as an exercise. This gives:

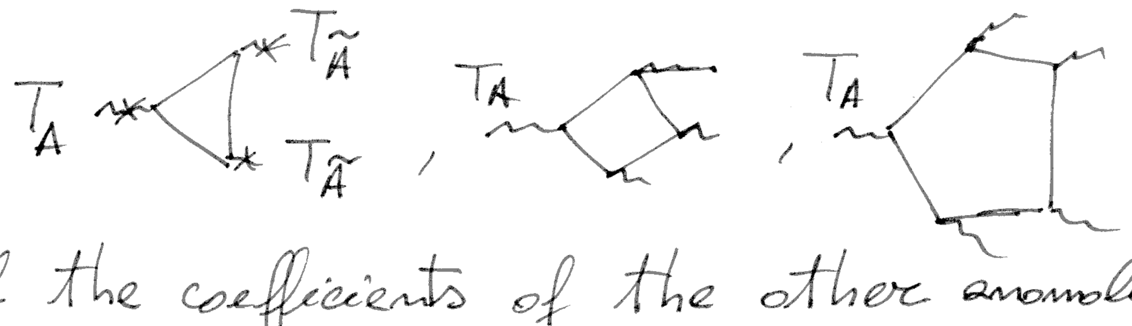
$$Z[A+\partial\sigma] = e^{-i\frac{N_c}{48\pi^2}\int d^4x\, \sigma(x)\times\left[\varepsilon^{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}\tilde{F}_{\rho\sigma}\right]}$$

where $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$. Notice however, that in our derivation we did not assume that $\tilde{A}$ was associated to an abelian symmetry, we have assumed it for the A source. But if $\tilde{A}$ is non-abelian, gauge invariance under the $\tilde{A}$ group (which we assume is not anomalous) tells us that we must find, instead,

$$A = -i \cdot c \cdot \mathrm{tr}\left[\varepsilon^{\mu\nu\rho\sigma} \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma}\right]$$

with $\tilde{F} = \partial\tilde{A} - \partial\tilde{A} + i[\tilde{A}, \tilde{A}]$

This obliges other correlators to be anomalous:



all the coefficients of the other anomalies are related by gauge invariance to the one we have already computed.

Let us go back to the problem of the π^0 decay in QCD. We have seen how the generating functional of current correlators should transform. This we derived in the QCD theory, by computing a diagram with quark loops, we now want to translate this information in the effective theory of the Goldstone Bosons. The prescription is very simple: given that the current correlators must coincide, the generating functional must also coincide and therefore have the same anomalous variation in the effective as well as in the fundamental theory. This is called the "Anomaly Matching" prescription. Of course, the generating functionals are only identical if we managed to compute them with infinite precision in both theories. We can not, but what we can instead compute exactly is the variation, i.e. the anomaly. In the chiral theory, no fermions are present and the anomaly cannot emerge from fermion loops as in QCD.

It must emerge from an explicitely symmetry-breaking term in the action (better, in the effective action in the presence of sources for the gauge fields). This term for the π^0 anomaly is

$$- \frac{N_c}{48\pi^2} \frac{1}{F_\pi} \pi_0(x) F_{\mu\nu} \tilde{F}^{\mu\nu}$$

You can check that indeed the variation of the π^0 is

$$\delta \pi^0 = \sigma F_\pi$$

and the Anomaly is perfectly reproduced

The term above becomes relevant, of course, only when the photon field A_μ is not a static source, but rather it becomes dynamical to describe the physical photon. This dynamics is introduced by adding a kinetic term

$$- \frac{1}{4e^2} F_{\mu\nu} F^{\mu}$$

After rescaling with "e", the "anomalous term" becomes

$$-\frac{N_c}{48\pi^2}\frac{e^2}{F_\pi}\pi_0 \; F_{\mu\nu}F^{\mu\nu}$$

You are invited, at this point, to study by yourselves section 22.1 of the Weinberg, in order to understand the phenomenological impact of this term. You will check as an exercise that it gives the correct rate for $\pi^0 \to \gamma\gamma$, provided of course N_c is correct. From this process, a measure of N_c is possible!